A Numerical Study of Viscoelastic Flow Through an Array of Cylinders

by

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Department of Mechanical and Industrial Engineering
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Abstract

This thesis is a study on creeping flow of an ideal viscoelastic fluid through square arrays of cylinders to predict the pressure drop. Numerical simulations were completed for arrays of three different solid volume fractions: 2.5%, 5%, and 10%. Substantial amounts of elastic stresses were found beyond a critical flow rate, up to six times that of the highest Newtonian stresses. An increase in pressure drop caused by elasticity was found, in contrast to many other numerical studies which find a decrease or no change. This pressure drop was, however, considerably smaller than what was found experimentally by James, Yip, & Currie (2012). The absence of elastic extensional stresses downstream of the cylinder for the 10% array supports the argument that the increase in pressure drop is caused by elastic stresses due to shear.
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Chapter 1  Introduction

Flows through porous media were initially studied by Henry Darcy, who investigated the flow of water through sand. From studying these flows, he found a linear relationship between the flow rate and the pressure drop for Newtonian fluids, which became Darcy’s law,

\[ Q = \frac{KA\Delta p}{\eta l} \]  \hspace{1cm} (1.1)

where \( Q \) is the flow rate, \( \Delta p \) is the pressure drop, \( A \) is the cross-sectional area, \( l \) is the length through the porous medium over which the pressure drop is taking place, \( \eta \) is the viscosity of the fluid, and \( K \) is a proportionality constant based on the geometry of the medium, known as the intrinsic permeability.

Darcy’s law is the foundation for studying aquifers and groundwater flows. This relationship is valid for a Reynolds number up to about 10, and for porous media of any particles, and not just for grains of sand.

The study of fluids flowing through a bed of fibres is of interest in engineering because of applications such as resin transfer molding in manufacturing and waterproofing breathable fabrics by coating the fabric with polymeric substances. The fluids used in these applications are polymeric liquids which exhibit both elastic and viscous behaviour, thus the necessity to study the viscoelastic fluids through fibrous porous media.

These fibrous beds of materials can be modelled by an array of regularly spaced cylinders called a square array of cylinders (Figure 1.1).
Flows through this idealized geometry have been extensively studied because this geometry is well-defined, two-dimensional, and periodic, allowing for comparisons between studies. The only property which defines an array is the solid volume fraction, \( \phi \), which is the fraction of solid volume to the total volume. For a square array, the solid volume fraction is,

\[
\phi = \frac{\pi D^2}{4L^2}
\]  

(1.2)

where \( D \) is the diameter of a cylinder and \( L \) is the distance between neighbouring cylinders, as illustrated in Figure 1.1.

The flow through a square array is known as a mixed flow, meaning that it is a combination of shear and extension, as opposed to pure shear or pure extension. The flow field is mainly dominated by shear, particularly around the poles of the cylinder. The cylinder surface is a region of pure shear, while there are regions of pure extension directly downstream of the cylinder and at a line parallel to this region half a unit cell length up or down. These regions will stretch the polymers differently, as it will be explained later in background.
The viscoelastic flow through this porous medium has been shown experimentally to exhibit an increase in pressure drop caused by the elasticity of the fluid (Chmielewski, Nichols, & Jayaraman, 1990; James et al., 2012; Khomami & Moreno, 1997; Skartsis, Khomami, & Kardos, 1992). Replicating this increased pressure drop effect with numerical simulations has so far eluded the scientific community. In fact, most numerical studies actually predict a marginal reduction rather than a definite increase in pressure drop.

The reason for this enhanced pressure drop effect has not been established, whether it is caused by elastic effects due to shear, to extension, or to flow instabilities. For example, James (2016) and Yip (2011) have made arguments that the elastic effects are due to shear, while Chmielewski & Jayaraman (1993), Khomami & Moreno (1997), and Liu, Wang, & Hwang (2017), have argued that the effects are caused by extension or the formation of flow instabilities. By numerically simulating the flow of viscoelastic fluid through a square array of cylinders, I hope to shed some light on this debate.
Chapter 2  Background

This chapter first gives a brief introduction to viscoelasticity and polymer solutions. Then an overview of viscoelastic constitutive models is presented, including stresses caused by these types of fluids. Finally, a review of related experimental and numerical research on viscoelastic flow through porous media is presented later in this chapter, proceeding to the primary research objectives.

2.1  What Is Viscoelasticity

Newtonian fluids exhibit a linear relationship between shear stress and shear rate, or in other words, a viscosity independent of shear rate. Most textbooks, therefore, define non-Newtonian fluids as fluids that have a shear rate dependent viscosity, fluids that are shear-thinning, shear thickening fluids, or have a yield behaviour (Cengel & Cimbala, 2013; Potter & Wigger, 2010; White, 1994). The shear stress in a Newtonian fluid under pure shear conditions is calculated as follows, where \( \tau_{xy} \) is the shear stress, \( \eta \) is viscosity, and \( u \) is the velocity in the \( x \) direction.

\[
\tau_{xy} = \eta \frac{du}{dy} \quad (2.1)
\]

There is, however, another category of non-Newtonian fluids are viscoelastic fluids. Although these viscoelastic fluids tend to be shear-thinning, they are also elastic, which is characterized by their stringiness, whereby they may form long filaments which persist. Examples of these fluids include polymer melts, egg whites, and saliva.

Viscoelastic fluids can exhibit some peculiar phenomena not observed by other types of fluids such as the open channel siphon (James, 1966) and rod climbing (Barnes, Hutton, & Walters, 1989). The open siphon effect is a phenomenon where a vessel containing a viscoelastic fluid is tipped so that fluid starts flow over its edge. If the vessel is then straightened, the fluid continues climbing up the side and out of the vessel like a siphon without a tube. The rod
climbing effect is exhibited when a rotating rod is inserted in a pool of viscoelastic fluid. Instead of the fluid being pushed outward from centrifugal forces, the fluid climbs up the rod.

The fluids which exhibit these phenomena are solutions or melts of long-chained polymer molecules. These long chains can stretch and become entangled with each other. In the case of the open siphon effect, the entangled polymers pull each other up and out of the vessel. In rod climbing, the polymer chains stretch like an elastic band, tightening around the rod. This creates a region of higher pressure around the rod, which pushes the fluid up the rod.

2.2 Polymer Solutions

The polymer molecules, at their rest state, are randomly coiled without a preferred direction. Under an applied deformation or strain, however, these coils start to unravel and align in the direction of the strain, which can be caused by either shear or extension (Figure 2.1).

![Figure 2.1. Polymer deformation in shear and extension](image)

As these polymer coils stretch, the Brownian motion of surrounding molecules in the solution randomly bombard the polymer, causing it to return to its unaligned rest state. This recoil makes the polymer molecules act like microscopic springs in the solution. The combined effect of millions of these microscopic springs give these polymer solutions their characteristic “stringiness” or elasticity.
Most polymer solutions, however, are shear-thinning, making it difficult to isolate the effects of elasticity from those caused by shear-thinning. Fortunately, a class of fluids called Boger fluids exhibit high elasticity with little shear-thinning, making them ideal for studying the effects of elasticity. A Boger fluid is a dilute solution of high molecular weight polymer generally dissolved in a highly viscous solvent. By comparing the results of a viscoelastic flow with those of a Newtonian flow field of the same Reynolds number, it is then possible to separate viscous and elastic effects, allowing purely elastic effects can be identified from the difference.

2.3 The Maxwell Model

Polymer molecules create extra stresses when the fluid is being deformed or stretched under shear or extension. This behaviour can be understood using a spring and damper system called, the Maxwell model (Figure 2.2).

![Figure 2.2. The 1-dimensional upper-convected Maxwell model](image)

The Maxwell model is the simplest viscoelastic model, in which most other models are based on. The stress acting on the spring and damper system, \( \tau \), can be derived as follows to obtain the following differential equation. The derivation starts by dividing with strain rate of the system, \( \dot{\gamma} \), into an elastic component, \( \dot{\gamma}_E \), and a viscous component, \( \dot{\gamma}_V \),

\[
\dot{\gamma} = \dot{\gamma}_E + \dot{\gamma}_V.
\]  

(2.2)

Then relating these strain rates to the stress, \( \tau \), using the damper constant, \( \eta \), and the spring constant, \( G \),

\[
\dot{\gamma} = \frac{\dot{\gamma}}{G} + \frac{\tau}{\eta},
\]  

(2.3)

Before finally rearranging the equation to obtain the equation for the Maxwell model
\[ \tau + \lambda \dot{t} = \eta \dot{\gamma}, \quad \lambda = \frac{\eta}{G}. \] (2.4)

To help illustrate the behaviour of the Maxwell model, the stress calculated when a sudden increase in shear rate, \( \dot{\gamma} \), is applied from rest is,

\[ \tau = \eta \dot{\gamma} \left(1 - e^{-\frac{t}{\lambda}}\right). \] (2.5)

Equation 2.5 shows that the stress increases asymptotically to its steady state values. The stress of a fluid element depends not only on the current strain rate, but on past strain rates as well. In other words, the fluid has a ‘memory’ of past stresses and strains. The rate at which the stress reaches its steady-state value depends on the value known as the relaxation time, \( \lambda \), which is a measure of the elasticity of the fluid. In contrast, a Newtonian fluid would instantaneous react to the sudden increase in shear rate.

If the viscoelastic fluid is subjected to a small amplitude oscillatory shearing of the form,

\[ \gamma = \gamma_0 \cos(\omega t), \] (2.6)

where \( \gamma \) is the applied strain, \( \omega \) is the oscillatory rate, \( \gamma_0 \) is the strain amplitude, and \( t \) is time since starting the test, the response of the Maxwell model from this oscillatory shear input is

\[ \tau = \frac{\eta \omega \gamma_0}{1 + \omega^2 (\lambda)^2} (\omega \lambda \cos(\omega t) - \sin(\omega t)). \] (2.7)

This response can be separated into an in-phase component called the storage modulus, \( G' \), and an out-of-phase component called the loss modulus, \( G'' \), as follows,

\[ \tau/\gamma_0 = G' \cos(\omega t) - G'' \sin(\omega t), \]

where

\[ G' = \frac{\eta \lambda \omega^2}{1 + \omega^2 \lambda^2}, \] (2.8)

and

\[ G'' = \frac{\eta \omega}{1 + \omega^2 \lambda^2}. \] (2.9)

The storage modulus is related to the elasticity of the fluid while \( G'' \) is related to the viscous response of the material. Purely elastic materials, such as most metals, have a loss modulus of zero and purely viscous materials would have a zero storage modulus.
In a fluid flow, these springs and dumbbells are rotated and stretched. Mathematically, this is handled using the upper-convected time derivative. This model is known as the upper-convected Maxwell (UCM) model:

\[
\tau + \lambda \dot{\gamma} = \eta \dot{\gamma},
\]

(2.10)

where \( \tau \) represents the stress tensor, \( \dot{\gamma} \) represents the strain rate tensor, \( \eta \) represents the viscosity, and the \( \dot{} \) accent represents the upper-convected time derivative, which is as follows,

\[
\dot{\tau} = \frac{D\tau}{Dt} - ((\nabla \mathbf{u})^T \cdot \tau + \tau \cdot \nabla \mathbf{u}) = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau - \{\tau \cdot \nabla \mathbf{u}\}^T - \{\tau \cdot \nabla \mathbf{u}\},
\]

(2.11)

where the \( \mathbf{T} \) superscript is the transpose of the tensor, \( \mathbf{u} \) is the velocity vector, and \( \frac{D}{Dt} \) is the material time derivative.

Although there are no fluids that behave exactly like a Maxwell fluid, the the Maxwell model is the foundation for other more sophisticated models such as the Oldroyd-B model.

2.4 The Oldroyd-B Model

The Oldroyd-B model, sometimes also known as the Jeffreys model, is an extension to the upper-convected Maxwell model. This model combines a Newtonian solvent with the upper-convected Maxwell model representing the polymer. The Oldroyd-B model is the simplest model that can simulate the behaviour the class of viscoelastic fluid called Boger fluids introduced earlier (James, 2009; Prilutski et al., 1983). For this reason, the Oldroyd-B model was chosen as the constitutive model to find the stresses in the fluid in this the flow of a polymer solution through a square array of cylinders.
The one-dimensional Oldroyd-B model can be represented as as the upper-convected Maxwell model with an additional parallel damper, shown as the mechanical system in Figure 2.3. In the figure, this parallel damper, $\eta_s$, represents the stress contributed by the Newtonian solvent while the spring, $G$, and damper, $\eta_p$, combination in series represents the contribution from the polymer. The stress of the combined system can be calculated as follows to give the Oldroyd-B equation by summing up the solvent ($\tau_s$) and polymer ($\tau_p$) stresses together:

$$\tau = \tau_p + \tau_s.$$  \hspace{1cm} (2.12)

The stresses in the system can be related with the strain rate of the system.

$$\dot{\gamma} = \frac{\dot{\tau}_p}{G} + \frac{\tau_p}{\eta_p} = \frac{\tau_s}{\eta_s}$$  \hspace{1cm} (2.13)

Then rearranging the above equation to solve the UCM polymer stress component, $\tau_p$, and the Newtonian solvent stress component, $\tau_s$,

$$\tau_p = \eta_p \dot{\gamma} - \frac{\eta_p \dot{\tau}_p}{G}$$  \hspace{1cm} (2.14)

$$\tau_s = \eta_s \dot{\gamma}$$  \hspace{1cm} (2.15)

$\tau_p$ is in terms of $\dot{\tau}_p$, so to relate $\dot{\tau}_p$ with the total stress of the system, $\tau$, it can be shown that

$$\dot{\tau}_p = \dot{\tau} - \dot{\tau}_s = \dot{\tau} - \eta_s \dot{\gamma}.$$  

Now substituting $\dot{\tau}_p$ with $\dot{\tau} - \eta_s \dot{\gamma}$ in Equation 2.14,

$$\tau_p = \eta_p \dot{\gamma} - \frac{\eta_p \dot{\tau}}{G} + \frac{\eta_p \eta_s \dot{\gamma}}{G}.$$  \hspace{1cm} (2.16)
Combining Equations 2.12, 2.15, and 2.16 and solving the equation to be in terms of $\tau$ and $\dot{\gamma}$ yields,

$$\tau + \frac{\eta_p \dot{\gamma}}{G} = \eta_p \dot{\gamma} + \frac{\eta_p \eta_s \dot{\gamma}}{G} + \eta_s \dot{\gamma}.$$ \hspace{1cm} (2.17)

Simplifying the above equation with the following constants,

$$\lambda = \frac{\eta_p}{G}, \quad \eta = \eta_s + \eta_p, \quad \lambda_2 = \frac{\eta_s \lambda}{\eta},$$ \hspace{1cm} (2.18)

produces the one-dimensional form of the Oldroyd-B equation

$$\tau + \lambda \dot{\gamma} = \eta (\dot{\gamma} + \lambda_2 \dot{\gamma}).$$ \hspace{1cm} (2.19)

Again, the upper-convected time derivative can be used to model how the stresses are convected throughout a flow field:

$$\tau + \lambda \vec{\gamma} = \eta \left( \dot{\gamma} + \lambda_2 \vec{\gamma} \right).$$ \hspace{1cm} (2.20)

Expanded equations of the upper-convected Maxwell and Oldroyd-B models in Cartesian coordinates are provided in Appendix A.

To understand the behaviour of the Oldroyd-B model, it is instructive to observe how the model acts under pure shear and pure extensional, for which there are simple analytical solutions. The following sections outline these solutions.

### 2.5 The Oldroyd-B Model in Shear Flow

Under a constant shear flow, the polymers are stretched in the shearing direction, $x$, as shown in Figure 2.1. These polymers stretch in the $x$-direction, creating in two-dimensions $\tau_{xx}$ and $\tau_{yy}$ stresses that would otherwise not exist in Newtonian fluids. These stresses cannot be measured individually, so the elastic stresses are normally represented by the first normal stress difference ($N_1$), defined as $\tau_{xx} - \tau_{yy}$. For the Oldroyd-B model in steady shear, the first normal stress difference is found to be,

$$N_1 = 2\eta_p \lambda \dot{\gamma}^2.$$

where $\dot{\gamma}$ is the shear rate. $N_1$ is oftentimes used to refer to the elastic stresses due to shear, and will be referred to as such throughout the paper.
Under an oscillatory shear flow with the Oldroyd-B model, $G'$ and $G''$ are found as follows using an analysis similar to that in section 2.3:

$$G' = \frac{\eta_p \lambda \omega^2}{1 + \omega^2 \lambda^2} \quad (2.22)$$

$$G'' = \frac{\eta_p \omega}{1 + \omega^2 \lambda^2} + \eta_s \omega \quad (2.23)$$

One of the useful results of the oscillatory shear test is the ability to obtain the viscosity contribution of the solvent, $\eta_s$, and thus the polymer viscosity, $\eta_p$. The solvent viscosity may be obtained by measuring high frequency $G''$ plateau and then calculating the solvent viscosity as follows (Prilutski et al., 1983):

$$\eta_s = \lim_{\omega \to \infty} \frac{G''}{\omega} \quad (2.24)$$

The polymer viscosity can then be calculated by subtracting the solvent viscosity from the total viscosity:

$$\eta_p = \eta - \eta_s \quad (2.25)$$

The relaxation time can then be calculated using the low frequency $G'$ plateau as followed:

$$\lambda = \lim_{\omega \to 0} \frac{G'}{\omega^2 \eta_p} \quad (2.26)$$

Additionally, in steady shear flow, the Oldroyd-B model (2.20) predicts that

$$\tau_{xy} = (\eta_s + \eta_p)\dot{\gamma}_{xy} \quad (2.27)$$

thus confirming that the viscosity of the fluid is simply the sum of the solvent and polymer viscosities. This also means that the Oldroyd-B model assumes negligible shear-thinning effects, a reasonable assumption for Boger fluids. Additionally, if solvent viscosity is negligible, then the Oldroyd-B model turns back into the upper-convected Maxwell model. However, in a Boger fluid, $\eta_p < \eta_s$, or even $\eta_p \ll \eta_s$. 
2.6 The Oldroyd-B Model in Extensional Flow

It may be further instructive to examine what happens with the Oldroyd-B model in extensional flow. In such a flow, there is an elongation but no shearing. In a steady planar extensional flow field with a constant extensional rate in the $x$-direction, $\dot{\varepsilon}$,

$$\dot{\varepsilon} = \frac{\partial u}{\partial x}$$  \hspace{1cm} (2.28)

and from continuity,

$$\dot{\varepsilon} = -\frac{\partial v}{\partial y}$$  \hspace{1cm} (2.29)

The only way to satisfy the above equations is

$$u = \dot{\varepsilon} x, \quad v = -\dot{\varepsilon} y,$$  \hspace{1cm} (2.30)

where $u$ is the fluid velocity in the $x$-direction and $v$ is the fluid velocity in the $y$-direction.

Using this extensional rate, the Oldroyd-B model (2.20) gives for stretching in the $x$-direction (Bird, Armstrong, & Hassager, 1987),

$$\tau_{xx} + \lambda \frac{d\tau_{xx}}{d\dot{\varepsilon}} - 2\lambda \tau_{xx} \dot{\varepsilon} = \eta \left[ 2\dot{\varepsilon} \left( 1 - 2\lambda_2 \dot{\varepsilon} \right) + 2\lambda_2 \frac{d\dot{\varepsilon}}{dx} \right]$$  \hspace{1cm} (2.31)

yielding the stress

$$\tau_{xx} = 2\eta \dot{\varepsilon} \frac{1 - 2\lambda_2 \dot{\varepsilon}}{1 - 2\lambda \dot{\varepsilon}} + 2\eta \dot{\varepsilon} \left( 1 - \frac{\lambda_2}{\lambda} \right) e^{-\frac{\lambda_1 - 2\lambda \dot{\varepsilon}}{\lambda}} \quad \text{when } \lambda \dot{\varepsilon} \neq \frac{1}{2}$$  \hspace{1cm} (2.32)

and

$$\tau_{xx} = \frac{\eta}{\lambda} \left( 2\dot{\varepsilon} - 4\lambda_2 \left( \dot{\varepsilon}^2 \right) t + 2\eta_2 \dot{\varepsilon} \right) \quad \text{when } \lambda \dot{\varepsilon} = \frac{1}{2}$$  \hspace{1cm} (2.33)

Notably, when $\lambda \dot{\varepsilon} \geq 1/2$, $\tau_{xx}$ will continuously increase with no steady-state value. In fact, the stress grows exponentially when $\lambda \dot{\varepsilon} > 1/2$ because of the $\lambda \dot{\varepsilon}$ term in the exponent, meaning that the elastic stress can increase rapidly and without limit.

In extension, there is also a $\tau_{yy}$ stress. This stress, however, does not experience such exponential growth, so it is often negligible compared to $\tau_{xx}$. 

2.7 The Deborah Number and Weissenberg Number

The Deborah number, like the Reynolds number, is a dimensionless quantity that helps to characterize the flow field. It is the ratio of the relaxation time of the fluid, $\lambda$, to the characteristic time of the deformation process, $t_c$ (Barnes et al., 1989; Reiner, 1964).

$$De = \frac{\lambda}{t_c} \quad (2.34)$$

The silicone-based material called “Bouncing putty” may be used to illustrate the importance of the ratio of the two time scales (Barnes et al., 1989). When placed in a container, the material will start to level and take the form of the container given sufficient time, therefore, acting like a viscous liquid. When, however, rolled into a ball and dropped from a height, the material bounces like an elastic solid. The difference between the two behaviours is due to the different time scales of the two processes: long time scales make the material act like a viscous liquid, such as when settling in a container, and shorter time scales make the material act like an elastic solid, such as the impact of the fluid on the ground.

In terms of the Deborah number, a value much less than one means that elasticity has little to no influence while a high value means that elasticity dominates the stress field. At high Deborah numbers, flow processes happen too quickly to allow the stresses in the material to relax, allowing for strong elastic effects.

Another dimensionless number used to characterize the effects of elasticity in a flow is the Weissenberg number, which is defined as

$$Wi = \lambda \dot{\gamma} \quad (2.35)$$

In principle, the Weissenberg number is used for flows dominated by shear; however, the Weissenberg and Deborah numbers are often used interchangeably, for they both are measures of the elastic effects of the flow. Both numbers are, however, proportional to the flow rate and the relaxation time of the fluid, often making them interchangeable. In this study, the Deborah number will be used to quantify flow elasticity.
2.8 Experimental Literature Review

One of the earliest studies on the viscoelastic flows through a porous media was carried out while attempting to increase the viscosity of brine solutions in enhanced oil recover. Pye (1964) added small amounts of polyacrylamide polymer to a brine and measured the pressure drop in flows of through porous sandstone rocks. He measured the viscosity of the brine solution, but noticed that the increase in viscosity did not completely account for the increase in pressure drop through the sandstone for a given flow rate. In fact, they measured a pressure drop 4.5 to 15 times greater than what was expected from a purely viscous fluid. They concluded that the effect was caused by the polymer, but they did not discover the mechanism by which these polymers increase the pressure drop.

This enhanced pressure drop effect was later observed in various experiments studying the flow of viscoelastic fluids through packed beds of spheres. The pressure drop was found to be up to two orders of magnitude higher (Dauben & Menzie, 1967; Durst, Haas, & Interthal, 1987; James & McLaren, 1975; Marshall & Metzner, 1967; Rodriguez et al., 1993; Vorwerk & Brunn, 1991). At low flow rates, the pressure drop matched what was expected from a purely viscous fluid. Dauben & Menzie (1967) noted that N1 stresses were often as large or even larger than the viscous stresses. Later studies, however, argued that the increase in pressure drop was caused by extensional stresses due to the polymer (Durst et al., 1987; James & McLaren, 1975; Vorwerk & Brunn, 1991).

Skartsis, Khomami, & Kardos (1992) studied the flow of viscoelastic fluids through both a square and staggered array of cylinders, with a high solid volume fraction of 55%, using various types polymeric solutions, including two Boger fluids. They observed an enhanced pressure drop after increasing the flow rate to a Deborah number above approximately 0.01 for both geometries for all fluids. They concluded that the onset of elastic effects does not strongly depend on the geometry of the bed.
In a later study, Khomami and Moreno (1997) investigated viscoelastic flows through square arrays of cylinders with solid volume fractions of 14% and 55%. They observed that when the Weissenberg number increased above a critical value, the flow resistance increased. They used particle image velocimetry and streak photography to study the velocity field of the flow. They found that the critical Weissenberg number corresponded to a change in flow regime from a steady two-dimensional flow to a steady three-dimensional one, and then, at even higher numbers, to an unsteady three-dimensional one. However, for the 55% array, the flow skipped the steady three-dimensional flow. They attributed the abrupt increase in flow resistance to an elastic flow instability.

Much of the study by Khomami and Moreno confirmed the results of a similar study done earlier by Chmielewski and Jayaraman (1992, 1993). Chmielewski and Jayaraman found an increase in pressure drop above a Deborah number of 1 using a square array with a solid volume fraction of 30%. They, however, found pressure fluctuations above the critical Deborah number and ascribed this effect to elastic instabilities. These elastic instabilities were then observed using laser Doppler velocimetry and streakline photography. Their studies suggested that the enhanced pressure drop was caused by flow instabilities created by elasticity.

Later experiments were published by James et al. (2012) based on the thesis by Yip (2011), who used square arrays with much lower solid volume fractions (2.5%, 5%, and 10%). They observed a steady increase in flow resistance above a critical Deborah number of approximately 0.5, as shown in Figure 2.4. The increase in flow resistance was similar for all three arrays up to a Deborah number of 1.5, above which the flow resistance of the 10% array diverged from the other two.
Figure 2.4. Normalized flow resistance reproduced from James et al. (2012).

James et al. (2012) and Yip (2011) used particle image velocimetry to monitor the flow field, and contrary to the study by Khomami & Moreno (1997) and Chmielewski & Jayaraman (1993), they found that the flow field was steady and two-dimensional well above the critical Deborah number. Some of their particle image velocimetry results shown in Appendix D. Furthermore, no sudden flow field changes were found after the critical Deborah number. This absence of a sudden change in flow field is in direct contrast to what was found by Khomami and Moreno (1997), who observed that the enhanced pressure drop effect was associated with a three-dimensional flow regime transition. The main difference between the two studies, however, were the much higher solid volume fractions used in Khomami and Moreno (1997). This thesis will mainly be based on the work completed by James et al. (2012) and Yip (2011) because their flows were steady, and thus amenable to numerical simulation.
James, Shiau, & Aldridge (2015) studied the viscoelastic flow around an isolated cylinder and found a monotonic increase in pressure drop above an onset Deborah number of 0.6. The pressure drop increased 50% due to elasticity at a Deborah number of 3. Notably, they used the same batch of Boger fluids as James, Yip, & Currie (2012). Particle image velocimetry results show that the flow field remained steady and changed gradually after onset. Though the particle image velocimetry results go up to De = 10.28, substantial changes in the flow field happen above a Deborah number of 1.3.

2.9 Numerical Literature Review

Numerically simulating viscoelastic flows through an array of cylinders is not easy. Talwar and Khomami (1992) attempted to numerically simulate the viscoelastic flow through arrays of high solid volume fractions (55% and 35%), using both the upper-convected Maxwell model and the Oldroyd-B model. Despite finding substantial elastic stresses using both models, they predicted a small monotonic decrease in flow resistance with increasing Weissenberg number, contrary to what was observed in the experiment by (Skartsis et al., 1992) for the 55% array. However, they were limited by the computational power of the day, so their maximum number of degrees of freedom was less than 7,000 (less than 10,000 degrees of freedom is considered small).

Later, Talwar and Khomami attempted the same simulation with 12,000 degrees of freedom (Khomami, Talwar, & Ganpule, 1994). This time, they used a different finite element technique, but they still predicted a small decrease in flow resistance.

Talwar & Khomami (1995) again attempted to simulate the flow through a 45% array using two new models: the Phan-Thien-Tanner and the Giesekus models. Again, they found substantial $N_1$ and extensional stresses, but these stresses did little to affect the pressure drop. They found only a small 10% decrease in flow resistance to a minimum before it increased back to its original value. They suggested that a temporal instability or non-linear flow transition may have caused the increase in flow resistance observed in experiments.
Other numerical studies have been similarly unsuccessful in predicting an enhanced pressure drop. Souvaliotis and Beris (1992) and Hua & Schieber (1998) studied the flow through arrays of cylinders with a solid volume fractions ranging from 12.6% to 55%. Both studies predicted small pressure drop decreases, in contrast to the pressure drop increases found experimentally. Both studies speculated that the discrepancy between numerical and experimental results was caused by flow field changes causing three-dimensional effects, time-dependent effects, or the formation of flow instabilities.

Liu, Wang, & Hwang (2017) were able to predict a sharp flow resistance increase using hexagonally-packed and randomly-packed arrays, but were unsuccessful in predicting the same effect for a square array of cylinders. They associated the flow resistance increase with the extensional region downstream of the cylinders. Therefore, they argued that polymer stretch was limited in a square array of cylinders and thus the pressure drop was limited.

To the author’s knowledge, only the numerical solution by Hemingway, Clarke, Pearson, & Fielding (2018) has shown a pressure drop increase for a square array. Their simulations show a small dip before a sharp increase in flow resistance up to 5%. This decrease to a minimum, however, has not been observed experimentally, although, it would be difficult to measure a dip of only 8% for their highest solid volume fraction of 38.4%. Unfortunately, their results are cut off soon after the sharp increase in flow resistance likely because of problems with convergence.

In summary, all of the experiments studying the the flow of viscoelastic fluid through porous media show a definite increase in flow resistance above a critical Deborah number. In contrast, the numerical simulations have generally predicted a small decrease to a plateau or a small decrease to a minimum before returning back to the original value. Only one numerical study completed by Hemingway, Clarke, Pearson, & Fielding (2018) predicted an increase, though the amounts of increase were small. The cause of these discrepancies is currently unknown,
although they have been speculated to be caused by flow field changes either to a three-dimensional flow field, a time-dependent flow field, or from the formation of flow instabilities.

To investigate this inconsistency, a numerical study of flow using the Oldroyd-B model was carried out. Different from prior attempts, the flow field was assumed to be exactly the same as its Newtonian counterpart with additional elastic stresses. This assumption was made because particle image velocimetry results by Yip (2011) show that the flow field changes only gradually beyond the onset Deborah number and to simplify the simulations because simulations of viscoelastic flows generally take considerable computational resources and are difficult to converge.

2.10 Objectives

As indicated above, the objective of this thesis research as to numerically simulate the flow through a periodic array of cylinders. The solid volume fractions (2.5%, 5%, and 10%) were the same as that used in James et al. (2012) and Yip (2011).

Consequently, the primary objectives of this study are as follows:

- Find the elastic stresses of the flow field and examine how they vary with increasing Deborah number
- Using these elastic stresses, attempt to predict the pressure drop along a square array of cylinders
- To predict the critical Deborah number before which the flow field has only marginal elastic effects
- To shed light on the cause of the observed increase in pressure drop
- To determine whether the increase is caused by elastic stresses in shear or extension
Chapter 3  Methodology

The commercially available software called ANSYS FLUENT™ was used to set up the domain and to simulate the Newtonian flow field. ANSYS FLUENT™ is a solver that uses the finite volume method with a cell-centred formulation. A user-defined function was used with the software to calculate the upper-convected Maxwell stresses for the polymer contribution of the fluid. The Oldroyd-B stresses were then obtained by adding the Newtonian solvent contribution; thus, finding the elastic stresses of the flow field.

3.1  Geometry

![Unit cell diagram](image)

*Figure 3.1. Unit cell to represent a periodic array of cylinders*

The periodic array was treated using a unit cell, as presented in Figure 3.1. The cell has a single cylinder with a diameter $D$ at the centre of a square and the cell length is $L$. Three different solid volume fractions were used in this study, 2.5%, 5%, and 10%, representing the low solid volume fractions used in James et al. (2012) and Yip (2011), enabling possible comparisons with experimental data.
3.2 Mesh Setup

Before any simulation can start, the domain has to be divided into smaller cells or elements in which the equations are solved.

The domain of the square cell was divided into two or three distinct regions. Region A, shown in Figure 3.2, uses a mesh of regular quadrilateral elements mapped to the concentric circles enclosing the region. The mesh of this inner region had a fivefold bias to the cylinder, so finer elements were used near the cylinder.

For regions B and B*, the automatic meshing algorithm provided by ANSYS Meshing created mainly quadrilateral elements of similar size. These outer regions were meshed using ANSYS’s quadrilateral-dominant mesh algorithm. The left and right hand sides of the unit cell were made identical, so that they could be matched perfectly to satisfy the periodic boundary condition. Region B* was separate from region B only for the 2.5% array to locally refine that region. Otherwise, the number of elements would have become too numerous, making simulations unnecessarily computationally expensive.
The meshes comprised approximately 350,000 elements, except for the 2.5% array, which had 1,200,000 elements. The number of elements was considered sufficient using a mesh convergence analysis presented in section 4.4.

To show the shape and the relative size of the elements, Figures 3.3 - 3.4 show the coarsest mesh used in the 5% array. The finest meshes used in this work had mesh elements at the poles of the cylinder with widths less than 0.1% of the cylinder diameter.

![Figure 3.3. 5% array coarsest mesh (23,664 elements)](image)
Figure 3.4. 5% array close-up of coarsest mesh (23,664 elements) around the top pole

3.3 Boundary Conditions

The treatment at the boundaries of the domain must be defined at the outset. In our simulations, three boundaries required different treatment: the left and right boundaries, the top and bottom boundaries, and the surface of the cylinder.

At the left and right hand sides of the unit cell, a periodic boundary condition was specified. A periodic boundary condition imposes the restriction that the two boundaries must have the same distributions of velocity, stress, and pressure gradient. Simulations using up to six cylinders lined up in the flow direction confirm the validity of using a periodic boundary condition. The top and bottom boundaries were required to have zero mass flux through the boundary and zero shear rate. Finally, the boundary conditions at the cylinder surface were the no slip condition and that there was zero mass flux passing through the surface.
Formally, these boundary conditions are:

\[
\bar{u}\left(-\frac{L}{2},y\right) = \bar{u}\left(\frac{L}{2},y\right); \quad \tau\left(-\frac{L}{2},y\right) = \tau\left(\frac{L}{2},y\right);
\]

\[
\frac{\partial p}{\partial x}\left(-\frac{L}{2},y\right) = \frac{\partial p}{\partial x}\left(\frac{L}{2},y\right) = \text{constant};
\]

\[
v\left(x,-\frac{L}{2}\right) = v\left(x,\frac{L}{2}\right) = 0; \quad \frac{\partial u}{\partial y}\left(x,-\frac{L}{2}\right) = \frac{\partial u}{\partial y}\left(x,\frac{L}{2}\right) = 0;
\]

\[
\bar{u}\left(\pm\sqrt{x^2+y^2} = \frac{D}{2}\right) = 0,
\]

where \(p\) is the local pressure, \(\bar{u}\) is the velocity vector, and \(\tau\) is the stress tensor, as in section 2.3.

### 3.4 Newtonian Simulation

Darcy’s law (Equation 1.1) applies to Newtonian fluids for flows through porous media. The law is generally valid for which the Reynolds numbers less than 1 (Marsily, 1986) and the highest Reynolds number used in this thesis is 0.0063. Low Reynolds numbers ensures that inertial effects are negligible, yielding Stokes flow. In this regime, the momentum equation is linear, the streamlines are symmetric upstream and downstream of the cylinder, and the pressure drop is directly proportional to the mass flow rate through the flow field.

Darcy’s law may be used to compare the relevant flow parameters and relate them to their respective Newtonian counterparts.

In porous media, the Reynolds number is defined as (Marsily, 1986)

\[
Re = \frac{\rho UD}{\eta},
\]

where \(D\) is the diameter of the cylinder, \(\rho\) is the density of the fluid, \(\eta\) is the fluid viscosity, and \(U\) is the bulk velocity which is,

\[
U = \frac{q}{L},
\]

where \(q\) is the flow rate per unit length.
The flow resistance can be represented the friction factor of the flow geometry can be defined as,

\[ f = -\frac{\Delta p D}{L \rho U^2}. \]  

(3.4)

Darcy’s law can also be expressed by multiplying the two groups to yield,

\[ f Re = -\frac{\Delta p D^2}{\eta L U} = \frac{D^2}{K}. \]  

(3.5)

Because Equation 3.5 is independent of the density, viscosity, and flow rate, it is a preferred way to present results.

There are various analytical solutions for predicting the resistance to flow. The one used in this paper will be that by Sangani & Acrivos (1982), who used a Fourier series method to solve the Stokes equation. According to their technique, the permeability of the porous medium, \( K \), is

\[ K = \frac{D^2}{16 \phi} \left[ \ln \left( \frac{1}{\phi} \right) - 0.738 + \phi - 0.887 \phi^2 + 2.038 \phi^3 + O(\phi^4) \right], \]  

(3.6)

where \( \phi \) is still the solid volume fraction of the array.

The flow resistance is therefore,

\[ f Re = \frac{D^2}{K} = \frac{16 \phi}{-0.5 \ln(\phi) - 0.738 + \phi - 0.887 \phi^2 + 2.038 \phi^3 + O(\phi^4)}. \]  

(3.7)

This formula was used to ensure the accuracy of the numerical results. The values for \( K \) and \( f Re \) for the three arrays are given in Table 1.

**Table 1. Permeability and flow resistance of a periodic array of cylinders based on the solution by Sangani & Acrivos (1982)**

<table>
<thead>
<tr>
<th>Solid Volume Fraction</th>
<th>Permeability, K ([m^2])</th>
<th>Flow Resistance Parameter, fRe</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5%</td>
<td>2.85e-5</td>
<td>0.355</td>
</tr>
<tr>
<td>5%</td>
<td>1.02e-5</td>
<td>0.991</td>
</tr>
<tr>
<td>10%</td>
<td>3.19e-4</td>
<td>3.17</td>
</tr>
</tbody>
</table>
The initial values of the Newtonian velocity field were set to be the bulk velocity. The simulation was then allowed to iterate until the flow rate through the domain stabilized and matched with the values in Table 1. The difference between the FLUENT flow rate and that predicted by Sangani and Acrivos was less than 0.1%, as shown in Figure 3.5.

**Figure 3.5.** Predicted simulation flow resistance vs. Reynolds number of the Newtonian flow field.

### 3.5 Simulating Elastic Stresses

After the Newtonian flow field converged, three user-defined scalars were created to represent the three UCM stresses: \( \tau_{xx}, \tau_{xy}, \) and \( \tau_{yy} \). These stresses were calculated to identify the magnitude of the polymer stress contributions to the stress field. Calculating the three UCM stresses required solving three coupled equations found in Appendix A.
To solve these three coupled equations, a user-defined function was created to define the UCM equation in FLUENT. These user-defined functions along with setup on how they were set up can be found in Appendix B.

As described earlier, the velocity field was taken to be the same as the Newtonian velocity field. The elastic stresses were initialized as the Newtonian stresses, as found previously. The simulation iterated and was considered converged when the stresses stabilized between iterations. The stresses on the cylinder and at the domain’s inlet were also monitored to ensure convergence.

It was found that at the higher Deborah numbers, it was necessary to introduce a diffusivity constant of $5\times10^{-12}$ m to prevent the polymer stresses from diverging. The constant was doubled in value to see whether it affects the results by checking the maximum UCM $\tau_{xx}$ stresses located at the top and bottom of the cylinder, which are the regions with the highest stress gradients, and thus the regions most affected by diffusivity. It was observed that doubling the diffusivity constant decreased the $\tau_{xx}$ stress at these points less than 1%. Because of this small magnitude, the introduction of diffusivity was considered negligible other than to ensure the convergence of the polymer stresses.

The stresses of a Newtonian solvent were then added to the polymer stresses to obtain the Oldroyd-B stresses:

$$\tau = \tau_p + \tau_s$$ (3.8)

where $\tau_p$ is the polymer stress tensor calculated using the UCM model and $\tau_s$ is the stress tensor created by the Newtonian solvent.

3.6 Elastic Properties

The properties of the fluid used in the simulation were the same ones measured by James et al. (2012). He measured the solvent viscosity and total viscosity and found the ratio between the two values to be 0.525. He also provided $G'$ and $N_1$ that could be used to find the relaxation time of the fluid.
James et al. (2012) used the relaxation time measured by the low oscillatory shear results calculated from $G'$ described in Equation 2.26, which requires finding a low frequency plateau when measuring $G'/\omega^2$. While he did attempt to find this plateau, there was still some variation between the lowest frequency points. Additionally, these measurements were likely unreliable due to the low shear stresses observed at the lowest oscillatory frequencies. His relaxation time from the shear test calculated from $N_1$ was, therefore, considered more appropriate due to the better reliability of the measurement and the importance of shear in this flow field. He, however, used two fluids, so the results for B2 fluid were used because it showed less variance in its $N_1$ measurements.

The relaxation time measured using a constant shear rate test in the rheometer with $N_1$ results shown in Figure 3.6. These results are plotted using the normal stress coefficient, $\psi_1$,

$$\psi_1 = \frac{N_1}{\gamma^2} = 2\eta_p \lambda .$$  \hspace{1cm} (3.9)

![Figure 3.6. First normal stress coefficient of B2 fluid adapted from Yip (2011)](image)

The Deborah Number is defined here to match that of James et al. (2012),
Later sections will use a Deborah number based on the relaxation time from the $G'$ results (3.9s) rather than the $N_1$ relaxation time of the B2 fluid (0.6s) used in the simulations in this work to maintain consistency with (James et al., 2012). The onset Deborah number will, therefore, remain at $De = 0.5$. The sole exception to this is Figure 4.15, whereby the Deborah numbers in the x-axis are scaled to use the $N_1$ relaxation time (0.6s) to make a comparison with another numerical study.

### 3.7 Summary of Assumptions

In simulations, some assumptions were necessary, and they were:

- The flow field is two-dimensional and steady, i.e. not varying with time.
- The velocity field for the Oldroyd-B fluid is exactly the same as that of the Newtonian fluid.
- The Oldroyd-B equation is a good model for the Boger fluids used in the experiment by James et al. (2012).
- Inertial effects are negligible so that a simplified momentum equation may be used.

\[
De = \frac{\lambda U}{(1 - \phi)L}.
\]  
(3.10)
4.1 Newtonian Flow Plots

As the foundation for the elastic stress calculations, the velocity field, pressure field, streamlines, and stresses are shown for the flow of a Newtonian fluid, which are displayed using two-dimensional contour plots, shown in Figures 4.1 and 4.2.

In Figure 4.1, the velocity magnitudes and pressure contour plots are shown. The velocity magnitude plots in Figure 4.1 are normalized using the bulk velocity defined earlier these are overlaid with streamlines, while the pressure field was normalized using the pressure drop across the unit cell. Considering that the Newtonian flow field scales linearly in the Stokes flow regime, only one normalized plot for each array was necessary to represent each flow field.
Figure 4.1. Normalized velocity magnitudes and pressure contours of the Newtonian flow field. Black lines on the normalized velocity contours represent streamlines, while the right-hand side of the pressure contours are uniformly 0.
As shown in the set of contour plots in Figure 4.1, the flow field is completely symmetric due to the insignificance of the non-linear inertial term. As for the pressure, the maximum and minimum pressures within the unit cell are found at the front and rear stagnation points, respectively. Additionally, the left and right hand sides are lines of constant pressure along with the vertical centreline.

Shown below are the accompanying Newtonian stresses for all three arrays as a background for the forthcoming non-Newtonian stresses. These stresses were normalized using the maximum shear stress found at the poles of the cylinder. Because these stresses are proportional to their strain rate, these plots also show the regions of high shear and extension.

![Newtonian stresses](image)

**Figure 4.2.** Newtonian stresses in a flow through a square array of cylinders. These stresses were normalized using the maximum stress in the flow field, which is the shear stress at the top and bottom poles of the cylinder.
In Figure 4.2, the Newtonian stress field is dominated by the shear stresses around the cylinder, the highest stresses being the shear stresses at the top and bottom poles of the cylinder.

Small amounts of stress due to extension fore and aft of the cylinder are slightly offset from the stagnation points, represented by \( \tau_{xx} \) and \( \tau_{yy} \). Noting where  \( \tau_{xx} \) and \( \tau_{yy} \) are positive and negative, the region fore of the cylinder experiences extension in the \( y \)-direction, while aft of the cylinder experiences extension in the \( x \)-direction. Additionally, extension has less influence for the 10% array than the other two arrays, shown by the lack of contours upstream and downstream of the cylinder in the \( \tau_{xx} \) and \( \tau_{yy} \) plots.

### 4.2 Elastic Stress Plots

After obtaining the Newtonian stresses, elastic stresses were calculated from the Oldroyd-B model using the Newtonian flow field. The Oldroyd-B model, however, includes both elastic and viscous stresses; therefore, the stresses expected from a Newtonian flow field using a fluid of the same viscosity, \( \tau_{\text{Newt}} \), were then subtracted out to obtain the additional elastic stresses, \( \Delta \tau \), which are the stresses of interest,

\[
\Delta \tau = \tau - \tau_{\text{Newt}}. \tag{4.1}
\]

These stresses were normalized by the highest Newtonian stress created by a Newtonian fluid of the same viscosity, i.e., by the Newtonian shear stress at the poles of the cylinder. They were normalized in this way to simplify comparisons between plots. The highest Newtonian stress was chosen to facilitate the normalization to allow for comparisons with the Newtonian stress field.

\( \Delta \tau \) has three components, \( \Delta \tau_{xx}, \Delta \tau_{xy}, \) and \( \Delta \tau_{yy} \), which are shown using sets of nine contour plots for Deborah numbers of 0.3, 1.0, and 2.1 and for three solid volume fractions of 2.5%, 5%, and 10% (Figures 4.3 - 4.6). The lowest Deborah number is lower than the critical Deborah number of 0.5, so elastic effects are expected to be low. The two higher Deborah numbers are two and four times higher than the critical Deborah number, so that elastic effects are expected to be substantial.
The $\Delta \tau_{xx}$ stresses are presented in Figures 4.3 and 4.4. Plotted in both figures are the exact same stresses, but the scales of the contour levels are different, as shown by their legends. In Figure 4.3, values higher than its maximum contour level of 1.0 are truncated to better reveal regions dominated by elasticity. Because each plot is shown with the same contour levels, Figure 4.3 also allows for comparisons between each plot. On the other hand, Figure 4.4 shows the $\Delta \tau_{xx}$ with a different set of contour levels to show the magnitude of the elastic stresses. It, however, fails to show the extent that $\tau_{xx}$ affects the flow field.

![Figure 4.3. $\Delta \tau_{xx}$ stresses normalized by the maximum Newtonian shear stress. Magnitudes higher than 1.0 are truncated.](image)

$\phi=2.5\%$

$\phi=5\%$

$\phi=10\%$

$D_e=0.3$ $D_e=1.0$ $D_e=2.1$
The contours in Figures 4.3 and 4.4 show that there are two main elastic regions in these contours: near the poles of the cylinder and downstream of the cylinder. The regions around the poles are dominated by shear, i.e. by $N_1$, while the downstream region is dominated by extension. As the Deborah number increases, the stresses increase and these two regions increase in size, allowing elastic stresses greater influence over the flow field. These two regions contain the highest elastic stresses, making the $\Delta \tau_{xx}$ stress component of greatest interest.

There are significant elastic stresses found near the cylinder poles even at the lowest Deborah number. The elastic stresses in this region are $N_1$ stresses caused by shear. At $De = 0.3$, however, these stresses are extremely local and would only affect a small part of the flow field around the top and bottom, and they act symmetrically. As for the larger Deborah numbers, $N_1$
stresses affect large portions of the flow field around the poles cylinder; these stresses are convected downstream slightly, creating asymmetries in the flow field.

The extensional stresses downstream of the cylinder decrease with increasing solid volume fraction, as shown in Figures 4.3 and 4.4. In fact, there are high extensional stresses only for the 2.5% and 5% arrays at De = 2.1, i.e., the extensional stresses in the 10% solid volume fraction geometry are small relative to the Newtonian stresses, despite being substantially above the critical Deborah number. This result is caused by the lower extensional rates in the region and the smaller distances that allow the polymers to stretch before the next cylinder. This fact helps to confirm the observations made by Yip (2011) that the extensional strains for the 10% solid volume fraction geometry are insufficient to cause large stresses, as well as supporting the argument that $N_1$ stresses are the main cause of the enhanced drag effect.

Shown next are the other two stress components ($\tau_{xy}$ and $\tau_{yy}$), which are provided for completion.
The $\Delta \tau_{xy}$ stress in Figure 4.5 are located near the surface of the cylinder and are significant only 60° from the x-axis. Though they increase with increasing Deborah number, these stresses affect only a small portion of the flow field.
Figure 4.6. $\Delta \tau_{yy}$ stresses normalized by the maximum shear stress.

Figure 4.6 shows the stress contours for $\Delta \tau_{yy}$. This set of contours show that high elastic stresses occur near the front stagnation point. These extensional stresses, however, are substantially smaller in magnitude than the ones downstream of the cylinder.

In Figures 4.3, 4.4, and 4.6, the three main elastic regions are shown in red: around the top and bottom surfaces of the cylinder, around the aft stagnation point, and around the forward stagnation point. The side regions are due to $N_2$ stresses created by shearing along the cylinder, while the latter two regions are caused by extension. The discussion which follows focuses on the side regions and the zone downstream of the cylinder. The stress contours found in Figure 4.3 for the 10% array at $De = 2.1$ qualitatively match with the polymer strain contours found in (Liu et al., 2017) for their 12.6% array at $Wi = 4.52$. Unfortunately, making further comparisons is difficult because they plotted contours of polymer strain rather than stresses.
4.3 Elastic Stresses Near the Cylinder

Stresses on the cylinder are directly related to the pressure drop, and so it may be informative to study the stresses acting on the cylinder. The Cartesian coordinate system, however, splits these elastic stresses into the three stress components, so it is more suitable to use stresses in cylindrical coordinates. The conversion of the stresses to a cylindrical coordinate system can be calculated as follows (Beer, Johnston, DeWolf, & Mazurek, 2009),

\[
\tau_{rr} = \frac{\tau_{xx} + \tau_{yy}}{2} + \frac{\tau_{xx} - \tau_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta,
\]

\[
\tau_{r\theta} = -\frac{\tau_{xx} - \tau_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta, \tag{4.3}
\]

\[
\tau_{\theta\theta} = \frac{\tau_{xx} + \tau_{yy}}{2} - \frac{\tau_{xx} - \tau_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta, \tag{4.4}
\]

where \( \theta \) is the angular position from the positive x-axis, \( r \) is the radial position, \( \tau_{rr} \) is the radial stress component, \( \tau_{\theta\theta} \) is the circumferential stress component, and \( \tau_{r\theta} \) is the shear stress component. The stresses at the cylinder surface are plotted in Figure 4.7 for the 5% array at a Deborah number of 2.1.

![Figure 4.7](image)

*Figure 4.7. Additional elastic stresses in cylindrical coordinates at the cylinder surface. Note that \( \Delta \tau_{r\theta} \) overlaps with \( \Delta \tau_{rr} \).*
The surface of the cylinder is a region of pure shear; therefore, the only relevant elastic stresses is from \( N_1 \). The \( N_1 \) stresses, shown in Figure 4.7, act only in the circumferential direction as \( \tau_{\theta\theta} \). Since the only elastic stress is from \( N_1 \), the circumferential stress around the cylinder should reach its steady state value of \( 2\eta_p \lambda \dot{y}^2 \) according to the Oldroyd-B model, which is shown as the red dashed line in Figure 4.7. This close match between the grey and red dashed lines in Figure 4.7 shows that the simulations are able to predict the \( N_1 \) stresses properly.

Figure 4.8. Stresses acting on a surface element in cylindrical coordinates

Figure 4.8 shows a fluid element at the surface of a cylinder with \( \tau_{\theta\theta} \), representing the \( N_1 \) stress, and \( \tau_{r\theta} \), representing the viscous stresses. The viscous stresses create a stress acting on the surface, as represented by the red arrow. The \( N_1 \) stress, however, only act tangentially to surface, creating no stress on the surface itself. This fact means that the \( N_1 \) stress around the cylinder cannot directly act on the cylinder, which will be relevant in section 4.5.

4.4 Mesh Convergence Analysis

In order to ensure that the present results are accurate and mesh independent, a mesh convergence analysis was performed. This analysis was completed by successively doubling the number of mesh elements in the mesh and monitoring \( \tau_{xx} \) at two points: one at a pole of the cylinder and one downstream at coordinates \((0.3L,0)\). These points were chosen to monitor the high \( N_1 \) stress at a pole and the high extensional stress downstream of the cylinder.

The mesh convergence analysis was performed for each geometry at \( De = 2.1 \), which was the highest Deborah number or flow rate used in this work, generating the highest elastic stresses, and thus the highest stress gradients. These high stress gradients mean that the simulations
would require the finest mesh, and the same mesh would be sufficient for simulations at lower Deborah numbers. Below are the mesh convergence analyses at a point downstream of the cylinder.

![Stress vs. Number of Mesh Elements](image)

Figure 4.9. Mesh convergence analysis of 2.5\% solid volume fraction at point (0.3L,0). (*) indicates that the downstream region was locally refined rather than the entire flow domain.
Figures 4.9 - 4.11 show that the differences in stress in the last two points are less than 1%. To reiterate, successfully converging the extensional stress downstream of the cylinder required a
highly refined mesh, so the mesh in that region was refined locally, as detailed in section 3.2. The mesh convergence analysis monitoring the stress at the cylinder pole can be found in Appendix C.

4.5 Force Acting on the Cylinder and Pressure Drop

One of the objectives of these simulations was to predict the pressure drop. Using the stresses and pressure acting on the cylinder, it is possible to calculate the force acting on the cylinder and then the pressure drop. The force acting on the cylinder, $F$, has two components: a viscous component caused by fluid stresses acting directing on the cylinder, $F_v$, and a pressure component caused by the pressure distribution around the cylinder, $F_p$.

$$ F = F_v + F_p $$ (4.5)

The viscous component was calculated by integrating the stresses around the cylinder shown in Figure 4.12.

$$ F_v = -\int_{s_1}^{s_2} \tau_{xx} dy + \int_{s_1}^{s_2} \tau_{xy} dx $$ (4.6)

where $s_1$ and $s_2$ are the start and end points of the integral. As previously explained in section 4.3, the elastic stresses around the cylinder act only tangentially to the surface; therefore, the viscous force component is identical to that of a Newtonian fluid, making the pressure component the only contributor to the enhanced pressure drop.

In order to calculate the pressure, the momentum equation was used. The inertial terms were neglected because of the Stokes flow assumption, i.e.,
\[
\frac{\partial p}{\partial x} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{4.7}
\]
\[
\frac{\partial p}{\partial y} = \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{yy}}{\partial x} \tag{4.8}
\]

The gradients of the stresses, \( \frac{\partial \tau_{xx}}{\partial x} \), \( \frac{\partial \tau_{xy}}{\partial y} \), and \( \frac{\partial \tau_{yx}}{\partial y} \), were extracted directly from the simulations.

By using these gradients, the pressure around the cylinder can be calculated,

\[
p(x, y) = \int_{x_0}^{x} \frac{\partial p}{\partial x} \, dx + \int_{y_0}^{y} \frac{\partial p}{\partial y} \, dy + p_0. \tag{4.9}
\]

This pressure was integrated around the cylinder to obtain the pressure component of the drag force acting on the cylinder. The top-bottom symmetry of the flow field means that only the x-component of the force needs to be considered,

\[
F_p = \int_{s_1}^{s_2} p(x, y) \, dy, \tag{4.10}
\]

where \( s_1 \) and \( s_2 \) are again the start and end points of the integral.

The pressure drop was then calculated by dividing the force, \( F \), acting on the cylinder, with the length \( L \) of the unit cell,

\[
\Delta p = \frac{F}{L}. \tag{4.11}
\]

To ensure the accuracy of these pressure calculations, the pressure drop using this method was first calculated to predict the Newtonian flow resistance. The integrals were calculated using a simple midpoint Riemann sum and the pressure drop was found to match with the analytical solution predicted by Sangani and Acrivos within 0.1% error.

The elastic pressure drop was then calculated using this method and normalized with the Newtonian pressure drop for each array (Figure 4.13).
Figure 4.13. Normalized predicted flow resistance vs. Deborah number

It may seem peculiar that the large elastic stresses (up to 6 times the highest viscous stress) around the cylinder (Figures 4.3 - 4.6) cause such a small change in pressure drop; however, this difference is due to the fact that the elastic stresses only act tangentially to the surface of the cylinder, as explained in section 4.3. In other words, these elastic stresses do not directly cause a force on the cylinder, only the pressure produced by them does.

4.6 Comparisons with Other Studies

Now that the flow resistance has been predicted, it is important to ensure that these values have a basis in reality by comparing them to the experimental results obtained by James et al. (2012).
Figure 4.14. Comparison of flow resistance results with experimental results by James et al. (2012).

The predicted pressure drops are considerably smaller than those found in the experiments by James et al. (2012) (Figure 4.14). For example, the predicted pressure drop for the 10% array at a Deborah number of 2.1 is only 10% compared to the 125% found experimentally. This large discrepancy means that these simulations were unsuccessful in predicting the increase in pressure drop, though the trend of a monotonic pressure increase matches what was found experimentally. Additionally, the trends found between the three geometries are similar to that found by James et al. (2012) as well (Figure 4.14): The three geometries have similar slopes until a Deborah number of roughly 1.7, where the 10% array separates from the other arrays.
Figure 4.15. Comparison of flow resistance results to numerical results from (Hemingway et al., 2018). (*) indicates that the Deborah number is calculated using $\lambda=0.6s$ for the comparison.

In Figure 4.15 are normalized pressure drop results completed in this work are compared with the numerical results by Hemingway et al. (2018) with the x-axis scaled for the comparison. The results by Hemingway et al. predict a small decrease in pressure drop before increasing again, compared to the monotonic increase found in this study. This flow resistance decrease, however, is too small to be measured experimentally. Additionally, they predicted pressure drop increases at much higher Deborah numbers. Their predicted pressure drops, however, are still considerably smaller than the experimental results by James et al. (2012).

4.7 Discussion

Considering that one of the main assumptions of these simulations was that the flow field is identical to the Newtonian flow field, the discrepancy in increased pressure drop may be caused by flow field changes. The particle image velocimetry results completed by Yip (2011)
show some decreased flow velocities around and downstream of the cylinders at a Deborah number around 2.1 for each solid volume fraction (Appendix D); hence the presented results are expected to be less accurate at Deborah numbers much higher than the critical Deborah number.

Other numerical studies which modified the flow field, however, were unable to predict these increases in the pressure drop; therefore, the above reason may not be adequate. Furthermore, other numerical studies have already used other constitutive models such as the FENE models, the Phan-Thien-Tanner model, and the Giesekus model, but they were similarly unsuccessful in predicting an increase in pressure drop (Hemingway et al., 2018; Hua & Schieber, 1998; Talwar & Khomami, 1995).

An alternate reason as to why these numerical results do not match experimental results may be related to the Oldroyd-B fluid in modelling the polymer solution. For example, the Oldroyd-B model predicts that the second normal stress difference ($N_2$), the normal stress perpendicular to $N_1$, is zero, which may not be true. $N_2$ is, however, unlikely to be a source of this discrepancy because it is generally an order of magnitude lower than $N_1$ (Barnes et al., 1989).

Also, the model predicts infinite extensibility, meaning the elastic stresses can increase without limit. Finite extensibility, however, is unlikely to be the problem because $\Delta \tau$ would need to be of order of magnitude of hundreds or thousands (Tirtaatmadja, 1993), which is implausible given that the simulations were limited to flows near the critical Deborah number. And so, the cause of the enhanced pressure drop effect due to elasticity still remains a mystery, despite all of the effort made to numerically simulate elastic flow fields.
Chapter 5 Conclusion

The stresses within a square array of cylinders in three different solid volume fractions (2.5%, 5%, and 10%) have been found using the Oldroyd-B model, with the assumption that the flow field is identical to that of a Newtonian flow field. The model predicts large non-Newtonian elastic stresses caused by shear and extension, but these do not produce pressure differences comparable to the ones found experimentally. However, the predictions are qualitatively correct with the pressure difference increasing with solid volume fraction. The elastic stresses due to extension downstream of the cylinder decrease with increasing solid volume fraction, in contrast to the increase in pressure drop which is similar for all three arrays. This finding supports the argument made in (James, 2016; James et al., 2012; Yip, 2011) that the elastic stresses due to shear cause the increase in pressure drop.

The reason why there is such a large discrepancy between experimental and numerical results still remains a mystery, which illustrates the difficulties in numerically simulating viscoelastic fluids.
References


Appendix A: Expanded Upper-Convected Maxwell and Oldroyd-B Equations

The equations for the 2-dimensional upper-convected Maxwell model in Cartesian coordinates are,

\[
\begin{align*}
\tau_{xx} + \lambda \left( \frac{\partial \tau_{xx}}{\partial t} + u \frac{\partial \tau_{xx}}{\partial x} + v \frac{\partial \tau_{xx}}{\partial y} - 2 \tau_{xx} \frac{\partial u}{\partial x} - 2 \tau_{xy} \frac{\partial u}{\partial y} \right) &= \eta \left( 2 \dot{\varepsilon}_{xx} \right) \\
\tau_{xy} + \lambda \left( \frac{\partial \tau_{xy}}{\partial t} + u \frac{\partial \tau_{xy}}{\partial x} + v \frac{\partial \tau_{xy}}{\partial y} - \tau_{xx} \frac{\partial v}{\partial x} - \tau_{yy} \frac{\partial v}{\partial y} \right) &= \eta \left( 2 \dot{\varepsilon}_{xy} \right) \\
\tau_{yy} + \lambda \left( \frac{\partial \tau_{yy}}{\partial t} + u \frac{\partial \tau_{yy}}{\partial x} + v \frac{\partial \tau_{yy}}{\partial y} - 2 \tau_{yy} \frac{\partial v}{\partial y} - 2 \tau_{xy} \frac{\partial v}{\partial x} \right) &= \eta \left( 2 \dot{\varepsilon}_{yy} \right)
\end{align*}
\]

where,

\[
\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}, \quad \dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}.
\]

The equations for the 2-dimensional Oldroyd-B equations in Cartesian coordinates are,

\[
\begin{align*}
\tau_{xx} + \lambda \left( \frac{\partial \tau_{xx}}{\partial t} + u \frac{\partial \tau_{xx}}{\partial x} + v \frac{\partial \tau_{xx}}{\partial y} - 2 \tau_{xx} \frac{\partial u}{\partial x} - 2 \tau_{xy} \frac{\partial u}{\partial y} \right) &= \eta \left( 2 \dot{\varepsilon}_{xx} + \lambda_2 \left( 2 \frac{\partial \dot{\varepsilon}_{xx}}{\partial t} + 2u \frac{\partial \dot{\varepsilon}_{xx}}{\partial x} + 2v \frac{\partial \dot{\varepsilon}_{xx}}{\partial y} - 4 \varepsilon_{xx}^2 - 2 \dot{\gamma}_{xy} \frac{\partial u}{\partial y} \right) \right) \\
\tau_{xy} + \lambda \left( \frac{\partial \tau_{xy}}{\partial t} + u \frac{\partial \tau_{xy}}{\partial x} + v \frac{\partial \tau_{xy}}{\partial y} - \tau_{xx} \frac{\partial v}{\partial x} - \tau_{yy} \frac{\partial v}{\partial y} \right) &= \eta \left( \dot{\gamma}_{xy} + \lambda_2 \left( \frac{\partial \dot{\gamma}_{xy}}{\partial t} + u \frac{\partial \dot{\gamma}_{xy}}{\partial x} + v \frac{\partial \dot{\gamma}_{xy}}{\partial y} - 2 \varepsilon_{xx} \frac{\partial \varepsilon_{xy}}{\partial x} - 2 \dot{\gamma}_{yy} \frac{\partial \varepsilon_{xy}}{\partial y} \right) \right) \\
\tau_{yy} + \lambda \left( \frac{\partial \tau_{yy}}{\partial t} + u \frac{\partial \tau_{yy}}{\partial x} + v \frac{\partial \tau_{yy}}{\partial y} - 2 \tau_{yy} \frac{\partial v}{\partial y} - 2 \tau_{xy} \frac{\partial v}{\partial x} \right) &= \eta \left( 2 \dot{\varepsilon}_{yy} + \lambda_2 \left( 2 \frac{\partial \dot{\varepsilon}_{yy}}{\partial t} + 2u \frac{\partial \dot{\varepsilon}_{yy}}{\partial x} + 2v \frac{\partial \dot{\varepsilon}_{yy}}{\partial y} - 4 \varepsilon_{yy}^2 - 2 \dot{\gamma}_{xy} \frac{\partial v}{\partial x} \right) \right)
\end{align*}
\]
Appendix B: FLUENT User Defined Function

In ANSYS FLUENT™, the upper-convected Maxwell model stresses were found in the program were defined as user-defined scalars. User defined scalars have an equation template defined as follows (ANSYS Inc., 2015),

$$\frac{\partial \alpha_k}{\partial t} + \frac{\partial}{\partial x_i} \left( F_i \alpha_k - I_k \frac{\partial \alpha_k}{\partial x_i} \right) = S_{\phi_k}, \quad \text{where } k = 1, \ldots, N_{\text{scalars}}$$

Where $\alpha_k$ are the scalars or variables, $F_i$ are the convective coefficients, $I_k$ are the diffusivity constants, and $S_{\phi_k}$ are the source terms.

The coefficient for the convective term, $F_i$, was set as $u\lambda$ to represent the $u \cdot \nabla \tau$ term, the unsteady term, $\frac{\partial \alpha_k}{\partial t}$, was set to zero, and the source terms were programmed to match with the rest of the upper-convected Maxwell equations found in Appendix A. Following is the user-defined function code used to obtain the stress results.

```c
#include "udf.h"
#include "math.h"
#define domain_ID 1
#define M_Lambda 0.6 /* Relaxation Time [s] */
#define M_Eta_P 1.9 /* Polymer Viscosity [Pa s] */
#define M_Eta_S 2.1 /* Solvent Viscosity [Pa s] */

enum
{
    Txx,
    Txy,
    Tyy,
    Txx_OB,
    Txy_OB,
    Tyy_OB,
    Txx_Newt,
    Txy_Newt,
    Tyy_Newt,
    Txx_OB_Dif,
    Txy_OB_Dif,
    Tyy_OB_Dif,
    N_REQUIRED_UDS
};
```
DEFINE_ON_DEMAND(calculate_UCM_values)
{
    Domain *d;
    cell_t c;
    cell_t c0;
    Thread *t;
    Thread *t0;
    face_t f;

    d=Get_Domain(1);

    thread_loop_c(t,d)
    {
        begin_c_loop_all(c,t)
        {
            C_UDMI(c,t,0) = C_UDSI_G(c,t,Txx_OB)[0];
            C_UDMI(c,t,1) = C_UDSI_G(c,t,Txy_OB)[0];
            C_UDMI(c,t,2) = C_UDSI_G(c,t,Txy_OB)[1];
            C_UDMI(c,t,3) = C_UDSI_G(c,t,Tyy_OB)[1];

            C_UDMI(c,t,4) = C_UDSI_G(c,t,Txx_Newt)[0];
            C_UDMI(c,t,5) = C_UDSI_G(c,t,Txy_Newt)[0];
            C_UDMI(c,t,6) = C_UDSI_G(c,t,Txy_Newt)[1];
            C_UDMI(c,t,7) = C_UDSI_G(c,t,Tyy_Newt)[1];

            C_UDMI(c,t,8) = C_UDSI_G(c,t,Txx_OB_Dif)[0];
            C_UDMI(c,t,9) = C_UDSI_G(c,t,Txy_OB_Dif)[0];
            C_UDMI(c,t,10) = C_UDSI_G(c,t,Txy_OB_Dif)[1];
            C_UDMI(c,t,11) = C_UDSI_G(c,t,Tyy_OB_Dif)[1];
        }
        end_c_loop_all(c,t)
    }
}

DEFINE_ON_DEMAND(initialize_UCM)
{
    Domain *d;
    cell_t c;
    Thread *t;

    d=Get_Domain(1);
thread_loop_c(t,d)
{
  begin_c_loop_all(c,t)
  {
    C_UDSI(c,t,Txx) = 2*M_Eta_S*C_DUDX(c,t);
    C_UDSI(c,t,Txy) = M_Eta_S*(C_DVDX(c,t)+C_DUDY(c,t));
    C_UDSI(c,t,Tyy) = 2*M_Eta_S*C_DVDY(c,t);
  }
  end_c_loop_all(c,t)
}

DEFINE_EXECUTE_ON_LOADING(define_variables,libname)
{
  if (n_uds < N_REQUIRED_UDS)
    Internal_Error("not enough user defined scalars allocated");
    offset = Reserve_User_Scalar_vars(3);
    Set_User_Scalar_Name(Txx, "txx");
    Set_User_Scalar_Name(Txy, "txy");
    Set_User_Scalar_Name(Tyy, "tyy");
    Set_User_Scalar_Name(Txx_OB, "txx OB");
    Set_User_Scalar_Name(Txy_OB, "txy OB");
    Set_User_Scalar_Name(Tyy_OB, "tyy OB");
    Set_User_Scalar_Name(Txx_Newt, "txx Newt");
    Set_User_Scalar_Name(Txy_Newt, "txy Newt");
    Set_User_Scalar_Name(Tyy_Newt, "tyy Newt");
    Set_User_Scalar_Name(Txx_OB_Dif, "txx OB_Dif");
    Set_User_Scalar_Name(Txy_OB_Dif, "txy OB_Dif");
    Set_User_Scalar_Name(Tyy_OB_Dif, "tyy OB_Dif");
}

DEFINE_SOURCE(txx_source,c,t,dS,eqn)
{
  real source;
  real x[ND_ND];

  source = 2*M_Eta_P*C_DUDX(c,t)+2*M_Lambda*C_UDSI(c,t,Txy)*C_DUDY(c,t)-
           C_UDSI(c,t,Txx)+2*M_Lambda*C_UDSI(c,t,Txx)*C_DUDX(c,t);
  dS[eqn] = 2*M_Lambda*C_DUDX(c,t)-1;
DEFINE_SOURCE(txy_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source =
        M_Eta_P*(C_DVDX(c,t)+C_DUDY(c,t))+M_Lambda*C_UDSI(c,t,Txx)*C_DVDX(c,t)+M_Lambda*C_UDSI(c,t,Tyy)*C_DUDY(c,t)-C_UDSI(c,t,Txy);

    dS[eqn] = -1;

    return source;
}

DEFINE_SOURCE(tyy_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source =
        2*M_Eta_P*C_DVDY(c,t)+2*M_Lambda*C_UDSI(c,t,Txy)*C_DVDX(c,t)-C_UDSI(c,t,Tyy)+2*M_Lambda*C_UDSI(c,t,Tyy)*C_DVDY(c,t);

    dS[eqn] = 2*M_Lambda*C_DVDY(c,t)-1;

    return source;
}

DEFINE_SOURCE(test_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source =
        C_UDSI(c,t,1)-C_UDSI(c,t,0);

    dS[eqn] = -1;

    return source;
}

DEFINE_SOURCE(Txx_OB_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];
source = 2*M_Eta_S*C_DUDX(c,t)+C_UDSI(c,t,Txx)-C_UDSI(c,t,Txx_OB);
dS[eqn] = -1;

return source;
}

DEFINE_SOURCE(Txy_OB_source,c,t,dS,eqn)
{
  real source;
  real x[ND_ND];

  source = M_Eta_S*(C_DVDX(c,t)+C_DUDY(c,t))+C_UDSI(c,t,Txy)-C_UDSI(c,t,Txy_OB);
dS[eqn] = -1;

  return source;
}

DEFINE_SOURCE(Tyy_OB_source,c,t,dS,eqn)
{
  real source;
  real x[ND_ND];

  source = 2*M_Eta_S*C_DVDY(c,t)+C_UDSI(c,t,Tyy)-C_UDSI(c,t,Tyy_OB);
dS[eqn] = -1;

  return source;
}

DEFINE_SOURCE(Txx_Newt_source,c,t,dS,eqn)
{
  real source;
  real x[ND_ND];

  source = 2*(M_Eta_S+M_Eta_P)*C_DUDX(c,t)-C_UDSI(c,t,Txx_Newt);
dS[eqn] = -1;

  return source;
}

DEFINE_SOURCE(Txy_Newt_source,c,t,dS,eqn)
{
  real source;
  real x[ND_ND];
source = (M_Eta_S+M_Eta_P)*(C_DVDX(c,t)+C_DUDY(c,t))-C_UDSI(c,t,Txy_Newt);
dS[eqn] = -1;

return source;

DEFINE_SOURCE(Tyy_Newt_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source = 2*(M_Eta_S+M_Eta_P)*C_DVDY(c,t) - C_UDSI(c,t,Tyy_Newt);
dS[eqn] = -1;

    return source;
}

DEFINE_SOURCE(Txx_OB_Dif_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source = C_UDSI(c,t,Txx)-2*M_Eta_P*C_DUDX(c,t) - C_UDSI(c,t,Txx_OB_Dif);
dS[eqn] = -1;

    return source;
}

DEFINE_SOURCE(Txy_OB_Dif_source,c,t,dS,eqn)
{
    real source;
    real x[ND_ND];

    source = C_UDSI(c,t,Txy)-(M_Eta_P)*(C_DVDX(c,t)+C_DUDY(c,t))-C_UDSI(c,t,Txy_OB_Dif);
dS[eqn] = -1;

    return source;
}

DEFINE_SOURCE(Tyy_OB_Dif_source,c,t,dS,eqn)
{
    real source;
real x[ND_ND];

source = C_UDSI(c,t,Tyy)-2*(M_Eta_P)*C_DVDY(c,t)-C_UDSI(c,t,Tyy_OB_Dif);
dS[eqn] = -1;

return source;
}

DEFINE_UDS_FLUX(uds_flux,f,t,i)
{
    real flux = 0.0;
    real rho = C_R(F_C0(f,t),THREAD_T0(t));
    flux = F_FLUX(f,t)/rho*M_Lambda;
    return flux;
}
Appendix C: Mesh Convergence Analysis at Cylinder Poles

Figure C.1: Mesh convergence analysis of 2.5% solid volume fraction at the cylinder pole.

Figure C.2: Mesh convergence analysis of 5% solid volume fraction at the cylinder pole.
Figure C.3: Mesh convergence analysis of 10% solid volume fraction at the cylinder pole.
Appendix D: Comparison of Newtonian Velocity Profiles to Particle Image Velocimetry Results from Yip (2011)

The following graphs compare Newtonian computational results from this work and particle image velocimetry results from Yip (2011) for the B2 Fluid.

![Graph comparing Newtonian and PIV results](image)

**Figure D.1.** 2.5% array velocity profiles between parallel cylinders (x/L=0). Particle image velocimetry results are adapted from Yip (2011).
Figure D.2. 2.5% array velocity profiles at the inlet boundary ($x/L = -0.5$). Particle image velocimetry results are adapted from Yip (2011).

Figure D.3. 5% array velocity profiles between parallel cylinders ($x/L = 0$). Particle image velocimetry results are adapted from Yip (2011).
Figure D.4. 5% array velocity profiles at the inlet boundary (x/L=0.5). Particle image velocimetry results are adapted from Yip (2011).

Figure D.5. 10% array velocity profiles between parallel cylinders (x/L=0). Particle image velocimetry results are adapted from Yip (2011).
Figure D.6. 10% array velocity profiles at the inlet boundary (x/L=-0.5). Particle image velocimetry results are adapted from Yip (2011).