Canadian Journal of Civil Engineering
 Revue canadienne de génie civil

**Shear deformable hybrid finite-element formulation for
 buckling analysis of composite columns**

Journal:	<i>Canadian Journal of Civil Engineering</i>
Manuscript ID	cjce-2017-0159.R2
Manuscript Type:	Article
Date Submitted by the Author:	19-Nov-2017
Complete List of Authors:	Niki, Vida; University of Technology, Sydney, Civil and Environmental Engineering Erkmen, Emre; University of Technology, Sydney, Civil And Environmental Engineering
Is the invited manuscript for consideration in a Special Issue? :	N/A
Keyword:	Hellinger-Reissner functional, Hybrid element, Shear deformation, composite columns

SCHOLARONE™
 Manuscripts

Shear deformable hybrid finite-element formulation for buckling analysis of composite columns

By Vida Niki¹ and R. Emre Erkmen²

Abstract

The objective of this study is to develop a shear deformable hybrid finite element formulation for the flexural buckling analysis of fiber-reinforced laminate composite columns with doubly symmetric cross-sections. The hybrid finite element formulation is developed by using the Hellinger-Reissner functional which is obtained by introducing the conditions of compatibility as auxiliary conditions to the potential energy functional. The shear deformation effects due to bending are included by equilibrating shear stress. In comparison to the displacement-based formulations the current hybrid formulation has the advantage of incorporating the shear deformation effects easily by using the strain energy of the shear stress field without modifying the basic kinematic assumptions of the beam theory. The agreement with Engesser formulation for flexural buckling analysis of columns with shear-weak cross-sections shows the applicability and accuracy of the current hybrid finite element method for composite structural elements. The applicability of the developed method herein to sandwich and built-up columns are also illustrated.

KEY WORDS: Hellinger-Reissner functional; Hybrid element; Shear deformation composite beam-columns;

¹ Graduate Research Assistant, School of Civil and Environmental Engineering, University of Technology, Sydney, NSW 2007, Australia. E-mail: vida.niki@student.uts.edu.au

² Corresponding author, Lecturer, School of Civil and Environmental Engineering, University of Technology, Sydney, NSW 2007, Australia. E-mail: emre.erkmen@uts.edu.au

Introduction

Fiber-reinforced composite materials are being increasingly used in many engineering fields due to various desirable characteristics such as high ratio of stiffness and strength to weight and high corrosion resistance. Thin-walled beams are also used in different structures because of the large axial and bending stiffness to weight ratios. In thin-walled elements made of fiber reinforced laminates materials buckling is the governing failure mode happening before yielding or cracking. For axially loaded columns that are not laterally braced, the buckling resistance is usually governed by their flexural behavior. (**Fig. 1**)

Fig. 1. An Axially Loaded Column and the Free-body Diagram of a Small Segment

34 The shear deformation effects can play an important role in the buckling analysis of members
35 with built-up or composite sections (Kant et al. 1991; Huang et al. 2002) or when materials with
36 relatively low shear modulus are used such as FRP (Bank 1989; Silvestre et al. 2003). In
37 particular, shear deformation can lead to reduction in the buckling load prediction and thus,
38 neglecting its effects may cause overestimation of the buckling load. In displacement-based
39 finite element formulations by modifying the kinematic assumptions of the basic beam theory the
40 effects of shear deformation can be included in the buckling analysis, (Moon-Young et al. 2003;
41 Saadé et al. 2004; Erkmén et al. 2011). Alternatively, Erkmén and Mohareb (2008) included the
42 shear deformation effects without modifying the basic kinematic assumptions by using a
43 complementary energy-based formulation and correspondingly developed a finite element
44 formulation. However, in that method the inter-element force equilibrium conditions have to be
45 a-priori satisfied which complicates the assemblage procedure. In this study, following Erkmén
46 (2014), a hybrid finite element formulation is developed based on the potential energy
47 functional. By relaxing the strain-displacement equations as auxiliary conditions in the potential
48 energy functional, the hybrid functional is developed. In this approach the shear deformation can
49 be included via equilibrating shear stress without modifying the kinematic assumptions. In
50 addition, there is no need for satisfying the inter-element force equilibrium conditions. This is a
51 desirable advantage of this method over force-based method especially when analyzing the
52 structures with non-collinear elements. It is shown that the developed hybrid finite element
53 method is in a very good agreement with Engesser's formula, rather than Haringx's in capturing
54 the shear deformation effect in the flexural buckling of homogenous columns. In order to extend
55 the applicability of developed hybrid finite element formulation to fiber reinforced laminates, the
56 constitutive relations of these types of materials are developed and incorporated in the hybrid
57 functional.

This paper is organized as follows: kinematic assumptions are given in section 2. Constitutive relations for fiber reinforced laminates and stresses and stress resultants relations are presented in sections 3 and 4, respectively. Section 5 introduces the variational formulations for flexural buckling analysis of composite thin-walled members including shear deformation effects and in section 6 the hybrid finite element formulation is developed. In section 7, in order to validate current method, highlight the effects of shear deformation and show the applicability and efficiency of the method in capturing the behavior of special elements such as built-up columns or columns with sandwich cross-sections, some numerical examples are presented and the results are compared with analytical solutions or other numerical solutions available in the literature.

Kinematics

The coordinate system that is adopted in this paper is orthogonal Cartesian coordinate system (x,y,z) in which the x axis is parallel to the longitudinal axis of the column, the bending of the column due to buckling deformations occur around the y axis and z axis lie on the plane of the cross-section (**Fig. 2**). It is assumed that the cross-section will remain plane after deformation. In agreement with the inextensional buckling assumption, the shortening of the column is assumed to be negligible during buckling. Therefore, the longitudinal normal strain induced by the flexural buckling deformations can be given as

$$\varepsilon_x = -zv'' \quad (1)$$

where z is measured from the weighted centroid (or neutral axis) of the cross-section. It is assumed that buckling induced normal strain is small thus only the first order term is retained while the higher order terms are neglected in Equation (1). On the other hand, the shear strain of

the cross-section is assumed to be equal to zero. Thus, Euler-Bernoulli beam kinematics is adopted for the analysis.

$$\gamma_{xz} = 0 \quad (2)$$

Constitutive relation

Consider a laminate composed of n orthotropic layers in which the fibre orientation of each layer with respect to the global coordination is determined by angle ϕ about the z -axis. (**Fig. 2**) Assuming that perfect inter-laminar bond exists between the layers; the stress-strain relationship for the k^{th} layer is given as:

$$\boldsymbol{\sigma}^{(k)} = \bar{\mathbf{Q}}^{(k)} \boldsymbol{\varepsilon} \Rightarrow \begin{Bmatrix} \sigma_x^{(k)} \\ \sigma_y^{(k)} \\ \tau_{xz}^{(k)} \\ \tau_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & 0 & \bar{Q}_{16}^{(k)} \\ \bar{Q}_{12}^{(k)} & \bar{Q}_{22}^{(k)} & 0 & \bar{Q}_{26}^{(k)} \\ 0 & 0 & Q_{44}^{(k)} & 0 \\ \bar{Q}_{16}^{(k)} & \bar{Q}_{26}^{(k)} & 0 & \bar{Q}_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad (3)$$

In here, subscripts 1 to 6 are consistent with the convention adopted for laminated composites (Wang 1997). In Equation (3)

$$\bar{Q}_{11}^{(k)} = Q_{11}^{(k)} \cos^4 \phi_k + 2(Q_{12}^{(k)} + 2Q_{66}^{(k)}) \sin^2 \phi_k \cos^2 \phi_k + Q_{22}^{(k)} \sin^4 \phi_k \quad (4)$$

$$\bar{Q}_{12}^{(k)} = (Q_{11}^{(k)} + Q_{22}^{(k)} - 4Q_{66}^{(k)}) \sin^2 \phi_k \cos^2 \phi_k + Q_{12}^{(k)} (\sin^4 \phi_k + \cos^4 \phi_k) \quad (5)$$

$$\bar{Q}_{22}^{(k)} = Q_{11}^{(k)} \sin^4 \phi_k + 2(Q_{12}^{(k)} + 2Q_{66}^{(k)}) \sin^2 \phi_k \cos^2 \phi_k + Q_{22}^{(k)} \cos^4 \phi_k \quad (6)$$

$$\bar{Q}_{26}^{(k)} = (Q_{11}^{(k)} - Q_{22}^{(k)} - 2Q_{66}^{(k)}) \sin^3 \phi_k \cos \phi_k + (Q_{12}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)}) \sin \phi_k \cos^3 \phi_k \quad (7)$$

$$\bar{Q}_{16}^{(k)} = (Q_{11}^{(k)} - Q_{12}^{(k)} - 2Q_{66}^{(k)}) \sin \phi_k \cos^3 \phi_k + (Q_{12}^{(k)} - Q_{22}^{(k)} + 2Q_{66}^{(k)}) \sin^3 \phi_k \cos \phi_k \quad (8)$$

$$\bar{Q}_{66}^{(k)} = (Q_{11}^{(k)} + Q_{22}^{(k)} - 2Q_{12}^{(k)} - 2Q_{66}^{(k)}) \sin^2 \phi_k \cos^2 \phi_k + Q_{66}^{(k)} (\sin^4 \phi_k + \cos^4 \phi_k) \quad (9)$$

$$Q_{44}^{(k)} = G_{13}^{(k)} \quad (10)$$

where

$$Q_{11}^{(k)} = \frac{E_1^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, \quad Q_{12}^{(k)} = \frac{\nu_{12}^{(k)} E_1^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, \quad Q_{22}^{(k)} = \frac{E_2^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, \quad Q_{66}^{(k)} = G_{12}^{(k)} \quad (11)$$

where $E_1^{(k)}$ and $E_2^{(k)}$ are Young's moduli of the k^{th} layer in the local x_k and y_k directions, respectively, $G_{12}^{(k)}$ is the shear moduli in $x_k y_k$ plane of the k^{th} layer, $G_{13}^{(k)}$ is the shear moduli in $x_k z_k$ plane, $\nu_{12}^{(k)}$ is the Poisson's ratio defined as the ratio of the transverse strain in the y_k direction to the axial strain in x_k direction due to the normal stress in x_k direction, and $\nu_{21}^{(k)}$ is the Poisson's ratio defined as the ratio of the transverse strain in the x_k direction to the axial strain in y_k direction due to the normal stress in y_k direction. In this paper laminate configuration is limited to symmetric angle-ply stacking sequence with respect to mid-plane. (Omidvar et al. 1996; Reddy 2003)

(a) Thin-walled column

(b) Fibre orientations

(c) Laminates across the thickness

Fig. 2. Thin-walled Column Composed of Fibre-reinforced Laminates

99 As one can see in **Fig. 2**, the y direction is assumed to be perpendicular to the element thickness
 100 so in the web the coordinate system rotates 90 degrees around the x axis. Adopting the
 101 assumption of free stress in y direction $\sigma_y = 0$, Equation (3) will be reduced to

$$\begin{Bmatrix} \sigma_x^{(k)} \\ \tau_{xz}^{(k)} \\ \tau_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{(k)*} & 0 & \bar{Q}_{16}^{(k)*} \\ 0 & Q_{44}^{(k)} & 0 \\ \bar{Q}_{16}^{(k)*} & 0 & \bar{Q}_{66}^{(k)*} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad (12)$$

102 in which

$$\bar{Q}_{11}^{(k)*} = \bar{Q}_{11}^{(k)} - \frac{\bar{Q}_{12}^{(k)2}}{\bar{Q}_{22}^{(k)}}, \quad \bar{Q}_{16}^{(k)*} = \bar{Q}_{16}^{(k)} - \frac{\bar{Q}_{12}^{(k)}\bar{Q}_{26}^{(k)}}{\bar{Q}_{22}^{(k)}}, \quad \bar{Q}_{66}^{(k)*} = \bar{Q}_{66}^{(k)} - \frac{\bar{Q}_{26}^{(k)2}}{\bar{Q}_{22}^{(k)}} \quad (13)$$

103 In addition, it is assumed that $\tau_{xy}^{(k)} = 0$ and thus the constitutive relations can be obtained as

$$\begin{Bmatrix} \sigma_x^{(k)} \\ \tau_{xz}^{(k)} \end{Bmatrix} = \begin{bmatrix} \hat{\bar{Q}}_{11}^{(k)*} & 0 \\ 0 & Q_{44}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{xz} \end{bmatrix} \quad (14)$$

104 in which

$$\hat{\bar{Q}}_{11}^{(k)*} = \bar{Q}_{11}^{(k)*} - \frac{\bar{Q}_{16}^{(k)*2}}{\bar{Q}_{66}^{(k)*}} \quad (15)$$

105 Stresses and stress resultants relations

106 Substituting the longitudinal strains in the constitutive relations (Equation (14)), the
 107 longitudinal stress is obtained as

$$\sigma_x^{(k)} = -\hat{\bar{Q}}_{11}^{(k)*} z v'' \quad (16)$$

Therefore, the stress resultant functions can be obtained in terms of the displacement derivatives

$$M = -\int_A \sigma_x z dA = J_{yy} v'' \quad (17)$$

In which, $J_{yy} = \int_A \hat{Q}_{11}^{(k)*} z^2 dA$ is defined. Using Eqs. (17) and (16), the expression for the normal stress in terms of stress resultant function is obtained i.e.

$$\sigma_x^{(k)} = -\hat{Q}_{11}^{(k)*} M \frac{z}{J_{yy}} \quad (18)$$

The shear stress $\tau_{xz}^{(k)}$ is due to the shear force and can be obtained as follows: The shear force F_s can be obtained from the longitudinal equilibrium of the free-body **Fig. 3**.

$$dF_s + \int_{A'} (\sigma_x + d\sigma_x) dA - \int_{A'} \sigma_x dA = 0 \quad (19)$$

in which A' is the cross sectional area of the piece cut from the segment dx . Substituting Equation (19) in relation $\tau_{xz} = dF_s/dx$, the shear stress is calculated

$$\tau_{xz} = \int_{A'} \frac{-d\sigma_x}{dx} dA = -\int_{A'} \sigma'_x dA \quad (20)$$

The longitudinal stress obtained in Equation (16) is substituted into the above expressions and the shear stress expression in terms of displacements is obtained.

$$\tau_{xz} = \int_{A'} \hat{Q}_{11}^{(k)*} z v''' dA \quad (21)$$

Fig. 3. Free-body Diagram of a Piece Cut from the Column

On the other hand, the shear stress resultant is given as,

$$V = M' = J_{yy} v''' \quad (22)$$

From Equation (22), substituting in Equation (21), the shear stress is expressed in terms of shear stress resultant.

$$\tau_{xz}^{(k)} = \frac{\bar{S}_y}{J_{yy}} V \quad (23)$$

in which $\bar{S}_y = \int_{A'} \hat{Q}_{11}^{(k)*} z dA$ has been used. The column is subjected to axial compressive load N^p which do not cause any pre-buckling shear forces V^p and associated bending moments M^p (a superscript p will relate to the pre-buckling state). Throughout buckling, increment in axial force is zero, i.e. $\Delta N = 0$, in accordance with the inextensional buckling assumption. The linearized buckling analysis is performed in two steps: firstly, linear static analysis is performed under applied load. Secondly; the internal forces obtained from the previous step are substituted into the stability condition and an eigenvalue problem is obtained.

130 Variational formulation

131 The hybrid formulation can be developed from the potential energy functional by relaxing the
 132 strain-displacement equations as auxiliary conditions through Lagrange multipliers method.
 133 Total potential energy of the buckled configuration of a system is expressed as the sum of strain
 134 energy and work done by external forces,

$$\Pi = U + W \quad (24)$$

135 in which the strain energy is given by

$$U = \frac{1}{2} \int_{V_0} (\sigma_x \varepsilon_x + \tau_{xz} \gamma_{xz}) dV \quad (25)$$

136 where V_0 = Volume of the column. The external potential due to flexural buckling can be written
 137 as

$$W = -\frac{1}{2} \int_L N^p v'^2 dx \quad (26)$$

138 where L = Length of the column. The conditions of compatibility are introduced to the total
 139 potential energy expression through the Lagrange multipliers method.

$$\Pi_{II} = U + W - \int_{V_0} \beta_1 (\varepsilon_x + zv'') dV - \int_{V_0} \beta_2 (\gamma_{xz}) dV \quad (27)$$

140 In which β_1 and β_2 are the Lagrange multipliers. From the variations of the functional Π_{II} with
 141 respect to the strains one obtains

$$\frac{\partial U_d}{\partial \varepsilon_x} - \beta_1 = 0, \quad \frac{\partial U_d}{\partial \gamma_{xz}} - \beta_2 = 0 \quad (28)$$

142 where U_d is the internal strain energy density and is defined as $U = \int_{V_0} U_d dV$. Equations (28)
 143 show that β_1 and β_2 are the energy conjugate of the strains ε_x and γ_{xz} , respectively. Therefore,
 144 they can be replaced with stresses σ_x and τ_{xz} in Equation (27), i.e.

$$\Pi_{II} = U + W - \int_{V_0} \sigma_x (\varepsilon_x + zv'') dV - \int_{V_0} \tau_{xz} (\gamma_{xz}) dV \quad (29)$$

145 By substituting Eqs. (25) and (26) and the inverse of constitutive relations for the laminate
 146 composite and using Equation (17) in Equation (29), the hybrid functional Π_{III} can be expressed
 147 as

$$\Pi_{III} = -\frac{1}{2} \int_{V_0} \begin{bmatrix} \sigma_x^{(k)} & \tau_{xz}^{(k)} \end{bmatrix} \begin{bmatrix} \hat{Q}_{11}^{(k)*} & 0 \\ 0 & Q_{44}^{(k)} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_x^{(k)} \\ \tau_{xz}^{(k)} \end{bmatrix} dV + \int_L M v'' dx - \frac{1}{2} \int_L \lambda N^p v'^2 dx \quad (30)$$

148 By substituting Eqs. (18) and (23) into Equation (30), the final form of hybrid formulation, Π_{III} ,
 149 in the buckled state is obtained

$$\Pi_{III} = \int_L \left[-\frac{1}{2} \left(\frac{M^2}{J_{yy}} + \frac{V^2}{A_{yy}} \right) + M v'' - \frac{1}{2} \lambda N^p v'^2 \right] dx \quad (31)$$

150 where $A_{yy}^{-1} = \int_A \frac{1}{Q_{44}^{(k)}} \frac{\bar{S}_y^2}{J_{yy}^2} dA$.

151 Finite element formulation

152 Interpolation of the stress resultant and displacement fields for buckling analysis

153 For an element i with a span of L the buckling internal bending moment is assumed as linear, i.e.

$$M = \mathbf{L}^T \mathbf{M}_i \quad (32)$$

$$V = \mathbf{L}^T \mathbf{M}_i \quad (33)$$

154 in which, $\mathbf{L}^T = \langle (1-x/L) \quad x/L \rangle$ and $\mathbf{M}_i^T = \langle M(0) \quad M(L) \rangle$. The finite element formulation is
 155 obtained using two different interpolations for the lateral displacement, i.e.

$$v_i = \mathbf{N}^T \mathbf{v}_i \quad (34)$$

156 First set of interpolation functions are based on cubic interpolation where the vector $\mathbf{N}$ can be
 157 written as $\mathbf{N}^T = \langle (1-3x^2/L^2+2x^3/L^3) \quad (x-2x^2/L+x^3/L^2) \quad (3x^2/L^2-2x^3/L^3) \quad (-x^2/L+x^3/L^2) \rangle$,
 158 $\mathbf{v}_i^T = \langle v_i(0) \quad v'_i(0) \quad v_i(L) \quad v'_i(L) \rangle$ are defined. Second set of interpolation functions are based
 159 on trigonometric functions and in this case vector $\mathbf{N}$ can be written as $\mathbf{N}^T = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle$
 160 where $N_1 = 0.5[\cos(\pi x/L) + 1]$, $N_2 = 0.25[L \cos(\pi x/L) + (2L/\pi) \sin(\pi x/L) + 2x - L]$,
 161 $N_3 = 0.5[1 - \cos(\pi x/L)]$ and $N_4 = 0.25[L \cos(\pi x/L) - (2L/\pi) \sin(\pi x/L) + 2x - L]$.

162 *Discretized form of the hybrid functional for buckling analysis*

163 By substituting Eqs. (32) to (34) into Equation (31), the discretized form of the hybrid functional
 164 for buckling analysis can be written as

$$\Pi_{III} = \sum_{i=1}^N \int_L \left(\mathbf{M}_i^T \mathbf{L} \mathbf{N}^T \mathbf{v}_i - \frac{1}{2} \lambda N^p \mathbf{v}_i^T \mathbf{N}' \mathbf{N}'^T \mathbf{v}_i - \frac{1}{2J_{yy}} \mathbf{M}_i^T \mathbf{L} \mathbf{L}^T \mathbf{M}_i - \frac{1}{2A_{yy}} \mathbf{M}_i^T \mathbf{L}' \mathbf{L}'^T \mathbf{M}_i \right) \quad (35)$$

165 By applying $\partial \delta^2 \Pi_{III} / \partial \mathbf{M}_i = 0$, one obtains

$$\mathbf{M}_i = \mathbf{H}_b \mathbf{G}_b \mathbf{v}_i \quad (36)$$

166 Where

$$\mathbf{H}_b = \left\{ \int_L \frac{1}{J_{yy}} \mathbf{L} \mathbf{L}^T + \frac{1}{A_{yy}} \mathbf{L}' \mathbf{L}'^T dx \right\}^{-1} \quad (37)$$

167 and

$$\mathbf{G}_b = \int_L \mathbf{L} \mathbf{N}''^T dx \quad (38)$$

168 It should be noted that the stationary condition with respect to the stress resultant parameters can
 169 be written at the element level given that to the relaxation of inter-element equilibrium the nodal
 170 stress resultant parameters are not coupled between the elements. Using Equation (36), the
 171 functional in Equation (35) becomes

$$\Pi_{III} = \frac{1}{2} \sum_{i=1}^N \int_L (\mathbf{v}_i^T \mathbf{G}_b^T \mathbf{H}_b \mathbf{G}_b \mathbf{v}_i) dx - \frac{1}{2} \lambda \sum_{i=1}^N \int_L \mathbf{v}_i^T \mathbf{N}^p \mathbf{N}' \mathbf{N}'^T \mathbf{v}_i dx \quad (39)$$

172 From the stationary condition with respect to $\mathbf{v}_i$, i.e. $\partial \delta^2 \Pi_{III} / \partial \mathbf{v}_i = 0$, the discretised equilibrium
 173 equations of the system can be obtained as

$$\sum_{i=1}^N [\mathbf{K}_{bi} - \lambda \mathbf{K}_{gi}] \mathbf{v}_i = 0 \quad (40)$$

174 In which $\mathbf{K}_{bi}$ is the element stiffness matrix for buckling analysis and $\mathbf{K}_{gi}$ is the element
 175 geometric stiffness matrix.

176 Numerical examples

177 *Simply supported column with isotropic cross-section*

178 To validate the current hybrid finite element model a simply supported column with an isotropic
 179 cross-section subjected to a uniform axial load is analyzed and the obtained flexural buckling

load are compared with the results from closed form solutions ($P_e = \pi^2 EI/L^2$) and ABAQUS FEA shell element model in which the four noded shell element (S4R) is used. The finite element mesh is illustrated in **Fig. 4**, in which the values indicate the number of finite elements used in each direction.

Fig. 4. The finite element mesh used in ABAQUS

Additionally, in this example the buckling load obtained by the current model is compared with the buckling load calculated by the closed form solution suggested by Kollár (2001) for the flexural-torsional buckling load of composite columns. The kinematic assumptions of the Kollar formulation is based on the fact that the axis of the deformed beam does not rotate with the same amount of that of the cross-section as in the Timoshenko beam kinematics.

The column span is 2000mm and the cross-section which is a symmetric I-section is demonstrated in **Fig. 5**. The material is steel and the material properties are as follows: $E = 2 \times 10^5 \text{ MPa}$, $G = 77 \times 10^3 \text{ MPa}$, $\nu = 0.3$. An axial load P is applied at the centroid C of the cross-section. The flexural buckling happens around the weak axis of the column which is y axis. In addition to the shear flexible model, the column is modeled without considering the shear deformation effects by setting $A_{yy}^{-1} = 0$ in the developed finite element formulation. Results with

both cubic interpolation and trigonometric interpolation for the lateral displacement are presented.

Fig. 5. Simply Supported column with Symmetric Cross-section

The values of flexural buckling load obtained by the current methods are showed in **Table 1** along with the results from other mentioned methods. From **Table 1**, it is verified that the predicted values of the buckling load by the current shear deformable hybrid model with 0.15% difference is in a very good agreement with the results of the ABAQUS model. In addition, the results from the analysis without shear deformation effects are equal to the buckling load obtained by the closed-form solution (0% difference). The small number of elements required for relatively accurate results shows the efficiency of the current model.

Table 1. Buckling Loads of columns (in Newtons)

Element type	Number of Elements	P_{cr} (cubic)	P_{cr} (trigonometric)
Current study with shear	2	12968.6	11971.7
	4	12857.3	12562.7
	8	12844.4	12766.9
	16	12842.3	12822.5
	32	12841.8	12836.7
	64	12841.8	12840.3
	128	12841.7	12841.2
Current study without shear	2	13271.1	12208.3
	4	13178.9	12863.3

	8	13172.7	13089.6
	16	13172.3	13151.2
	32	13172.3	13167.0
	64	13172.3	13171.0
	128	13172.3	13172.0
ABAQUS	6000		12822
$P_e = \pi^2 EI / L^2$	-		13172.3

209 It can be observed that when the shear deformation effects are considered, the magnitude of the
 210 buckling load decreases due to additional flexibility of the column. It is interesting to note that
 211 while cubic interpolation leads to upper bound convergence, trigonometric functions lead to
 212 lower bound convergence property. For more detailed analysis on the convergence property in
 213 buckling formulations where both force and displacement field is interpolated one may refer to
 214 Erkmén et al. (2009). For the rest of the examples only cubic interpolation of the displacement is
 215 used for the analysis.

216 The closed form solution developed by Kollár (2001) for the flexural-torsional buckling load is
 217 given as

$$\frac{1}{\hat{N}_{\text{crz}}} = \frac{1}{\hat{N}_{\text{crz}}^{\text{B}}} + \frac{1}{\hat{S}_{\text{yy}}} \quad (41)$$

218 in which $\hat{N}_{\text{crz}}$ is the buckling load of the column, $\hat{N}_{\text{crz}}^{\text{B}}$ is the buckling load when the shear
 219 stiffness is infinite (the Euler buckling load) and $\hat{S}_{\text{yy}}$ is the buckling load when the bending
 220 stiffness is infinite. In the current model, the value of $\hat{N}_{\text{crz}}^{\text{B}}$ is the buckling load obtained in the
 221 analysis without the shear deformation effect and the value of $\hat{S}_{\text{yy}}$ is equivalent to A_{yy} . These
 222 values are obtained by the current method as: $\hat{N}_{\text{crz}}^{\text{B}} = 13172.3 \text{ N}$ and $A_{\text{yy}} = 514000 \text{ N}$. Therefore,
 223 using Equation (41) the buckling load is calculated as

$$\hat{N}_{crz} = 12843.2N$$

which is in a very good agreement with the critical buckling load obtained by the current method.

Agreement with Engesser's buckling load

There are two well-known approaches in the literature that include the shear deformation effects in the column buckling analysis. The first one is developed by Engesser (1891) and the second one by Haringx (1948-1949). The difference between two methods is due to the different assumptions for the normal load orientation at the deformed state of the beam. Engesser assumes that the normal load is parallel to the beam axis in the loading state whereas in Haringx theory, the normal load is assumed to be perpendicular to the cross section of the beam. The buckling load expressions developed by Engesser and Haringx are as follows:

$$\text{Engesser: } P_{cr} = \frac{P_s P_b}{P_s + P_b} \quad (42)$$

$$\text{Haringx: } P_{cr} = \frac{1}{2} \left(-P_s + \sqrt{P_s^2 + 4P_s P_b} \right) \quad (43)$$

in which $P_b = \pi^2 EI / l^2$ is the Euler buckling load where EI is the flexural rigidity and $P_s = GA$ is the shear buckling load in which GA is the shear rigidity. For very large shear rigidity cases, the critical load will approach P_b in both methods. In the case of very large flexural rigidity (shear-weak), Engesser solution yields $P_{cr} = P_s$. However, for this extreme case Haringx theory obtains a significantly different result of $P_{cr} = \sqrt{P_b P_s}$.

240 In order to show the agreement of developed hybrid finite element model with the Engesser
 241 solution, the hybrid functional is obtained for two special cases of pure bending and pure shear.
 242 From the hybrid functional (Equation (31)) being equal to zero, the Rayleigh factor can be
 243 obtained as

$$\lambda N^p = \frac{\int_L \left[2Mv'' - \left(\frac{M^2}{J_{yy}} + \frac{V^2}{A_{yy}} \right) \right] dx}{\int_L v'^2 dx} \quad (44)$$

244 Considering the fact that the first variation of the hybrid formulation vanishes:

$$\delta \Pi_{III} = \int_L \left[\delta M \left(v'' - \frac{M}{J_{yy}} + \frac{V'}{A_{yy}} \right) - \delta v' (M' + \lambda N^p v') \right] dx - \delta M \frac{V}{A_{yy}} \Big|_L - \delta v' M \Big|_L = 0 \quad (45)$$

245 One can obtain

$$\left(v'' - \frac{M}{J_{yy}} + \frac{V'}{A_{yy}} \right) = 0 \quad (46)$$

246 In the presence of both bending and shear, the differential equation is (Timoshenko et al. 1961)

$$\left(1 + \frac{P}{GA} \right) v'' - \frac{P}{EI} v = 0 \quad (47)$$

247 Substitution of the solution $v = A \sin(\beta x)$ in Equation (47) yields $\beta^2 - P/GA\beta^2 - P/EI = 0$. For the
 248 simply supported column $\beta L = \pi$. In the case of pure bending, i.e. $A_{yy}^{-1} = 0$, from Eqs. (44) and
 249 (46) one can obtain the buckling load as

$$\lambda N^p = \frac{\int_L J_{yy} v''^2 dx}{\int_L v'^2 dx} = \frac{\pi^2 J_{yy}}{L^2} \quad (48)$$

250 which is equal to the Euler buckling load. In the case of pure shear, i.e. $J_{yy}^{-1} = 0$ the differential
 251 equation is $(1 - nP/GA)v'' = 0$ where n is the shape factor. Using Eqs. (44) and (46) one can
 252 obtain the buckling load as

$$\lambda N^p = \frac{\int_L \left(2Mv'' - \frac{V^2}{A_{yy}} \right) dx}{\int_L v'^2 dx} = A_{yy} \quad (49)$$

253 where A_{yy} can be shown to be equal to GA/n . It can be observed that, in the case of small shear
 254 rigidity, critical buckling load has the upper limit of GA/n as in the Engesser's theory.
 255 Therefore, current hybrid finite element solution is able to capture the buckling behavior of
 256 shear-weak columns. The applicability and accuracy of the model is verified in the following
 257 section through numerical example for a full range of shear flexibilities.

258 In this section, a buckling analysis of an axially loaded column is studied and the results obtained
 259 by the current hybrid finite element method are presented and compared with the results
 260 calculated by Engesser and Haringx solutions. The column is simply supported and the cross
 261 section is shown in **Fig. 6**. The analysis is done for various lengths. The material properties of
 262 the element are $E = 144 \text{ MPa}$, $G = 4.14 \text{ MPa}$. The flexural buckling happens around the weak
 263 axis of the column which is y axis.

Fig. 6. Simply Supported column and its Cross-section

In **Fig. 7** the critical buckling load is presented for the full range of P_b/P_s in terms of dimensionless ratios of P_{cr}/P_s and P_{cr}/P_b . In the right-hand part of figure (shear-rigid cases) the values of P_{cr}/P_b and in the left-hand side (shear-weak cases) P_{cr}/P_s will be discussed. It can be observed, when $P_b = P_s$ the results obtained by the current hybrid finite element method is in a very good agreement with the results of both Engesser and Haringx theories. As shear rigidity decreases, the difference between Engesser and Haringx figures increases significantly. One can see that the results of the current finite element analysis are very close to the results of Engesser theory.

Fig. 7. Comparison with Engesser and Haringx formulations

Column with laminate composite cross-section with various boundary conditions

In this example, a thin-walled composite column with various stacking sequences subjected to an axially compressive force at the centroid is modeled with simply supported (S-S) and clamped-free (C-F) boundary conditions. FRP members are generally made of high-strength glass or carbon fibres and polyester or vinylester resins as the binding matrix. In this example, the column is assumed to be made of glass-epoxy with the following material properties:

$$\begin{aligned} E_1 &= 53.78MPa, E_2 = 17.93MPa \\ G_{12} &= G_{13} = 8.96MPa \\ \nu_{12} &= 0.25 \end{aligned}$$

The subscript 1 corresponds to direction parallel to fiber orientation and subscripts 2 and 3 are related to the directions perpendicular to it. The cross-section dimensions are demonstrated in **Fig. 8**. The web and flanges are laminated symmetrically with respect to the mid-plane and consist of 16 layers with equal thickness. The length of the column is assumed to be equal to 4m and 1m for S-S case and C-F case, respectively. The flexural buckling happens around the weak axis of the column which is y axis.

Fig. 8. Cross-sectional Dimensions

A schematic configuration of symmetric angle-ply stacking sequence of $[\phi^\circ / -\phi^\circ]_{2s}$ with respect to mid-plane in a segment of the cross-section is illustrated in **Fig. 9**.

Fig. 9. Schematic configuration of symmetric angle-ply stacking sequence $[\phi^\circ / -\phi^\circ]_{2s}$

The column is modeled efficiently by using four elements for S-S boundary conditions and eight elements for C-F boundary conditions. The values of the buckling loads obtained by the current model for various lay-ups are presented and compared with the results obtained by analytical solutions from the exact stiffness matrix method in paper of Nguyen et al. (2015) in **Table 2**.

Table 2. Buckling Loads of columns (in Newtons)

Lay-up	This study		Nguyen et al. (2015)		Differences (%)	
	SS	CF	SS	CF	SS	CF
$[0^\circ]_{16}$	1502.6	6008.4	1438.8	5755.2	4.25	4.21
$[15^\circ / -15^\circ]_{4s}$	1207.7	4829.6	1300	5199.7	7.10	7.12
$[30^\circ / -30^\circ]_{4s}$	820.4	3280.8	965.2	3861	15.0	15.03
$[45^\circ / -45^\circ]_{4s}$	615	2459.8	668.2	2672.7	7.96	7.97
$[60^\circ / -60^\circ]_{4s}$	530.2	2120.6	528.7	2114.8	0.28	0.27
$[75^\circ / -75^\circ]_{4s}$	504.2	2016.6	487.1	1948.3	3.39	3.39
$[0^\circ / 90^\circ]_{4s}$	1001.8	4006	959.3	3837.3	4.24	4.21
$[0^\circ / -45^\circ / 90^\circ / 45^\circ]_{2s}$	808.5	3233.3	813.8	3255.3	0.65	0.68

In addition, the buckling loads obtained by the current method are compared with the loads calculated by the closed-form solution proposed by Kollár (2001) for the flexural-torsional buckling load of composite columns (Equation (41)) in **Table 3**. As mentioned in the first example, the value of $\hat{N}_{crz}^B$ is the buckling load when the shear stiffness is infinite (the Euler buckling load= $\pi^2 EI / (kL)^2$) and the value of $\hat{S}_{yy}$ is equivalent to A_{yy} in the current model.

Table 3. Buckling Loads of columns (in Newtons)

Lay-up	This study		Kollár (2001)		Differences (%)	
	SS	CF	SS	CF	SS	CF
$[0^\circ]_{16}$	1502.6	6008.4	1499.8	5981.8	0.19	0.44
$[15^\circ / -15^\circ]_{4s}$	1207.7	4829.6	1205.6	4811.3	0.17	0.38
$[30^\circ / -30^\circ]_{4s}$	820.4	3280.8	819.2	3271.4	0.14	0.29
$[45^\circ / -45^\circ]_{4s}$	615	2459.8	614.5	2455.0	0.09	0.20
$[60^\circ / -60^\circ]_{4s}$	530.2	2120.6	530.0	2117.9	0.04	0.13
$[75^\circ / -75^\circ]_{4s}$	504.2	2016.6	504.3	2015.1	-0.02	0.07
$[0^\circ / 90^\circ]_{4s}$	1001.8	4006	1000.0	3990.5	0.18	0.39
$[0^\circ / -45^\circ / 90^\circ / 45^\circ]_{2s}$	808.5	3233.3	807.3	3222.8	0.15	0.32

As one can observe, the results obtained from current model is in a very good agreement with the solution suggested by Kollár (2001).

Cantilever column with laminate composite cross-section

In this example, an axially loaded cantilever column is studied. The dimensions of the column are showed in **Fig. 10**. The web and flanges are assumed to be laminated symmetrically with respect to the mid-plane with four layers of equal thickness and the stacking sequences are $[0^\circ]_4$, $[0^\circ / 90^\circ]_s$, $[45^\circ / -45^\circ]_s$. The material is graphite-epoxy (AS4/3501) with the following properties:

$$E_1 = 144 \text{ MPa}, E_2 = 9.65 \text{ MPa}, G_{12} = 4.14 \text{ MPa}, \nu_{12} = 0.3$$

Fig. 10. Cantilever column with Symmetric Cross-section

The column is modeled by only 2 elements in the current study. The flexural buckling happens around the weak axis of the column which is y axis. The predicted values of flexural buckling load along with the eigenvalue results calculated previously by an alternative finite element model developed by Cortínez et al. (2006) are presented in **Table 4**. It can be seen that the results calculated by current method are in an acceptable agreement with the results from other finite element model (about 8% differences). It should be noted that, the considerable difference between the values of buckling load for the first case (stacking sequence of $[0^\circ]_4$) is due to the different assumptions for values of cross section stiffness. The value of modulus of elasticity is set to be $\hat{Q}_{11}^{(k)*}$ in the current study which is calculated for composite laminated cross-sections based on Eqs. (3) to (15), whereas in Cortínez's paper for this special stacking sequence the modulus of elasticity is assumed to be equal to E_1 . If in the current hybrid model, $\hat{Q}_{11}^{(k)*}$ is replaced by E_1 the same results as in Cortínez's paper will be obtained.

Table 4. Buckling Loads of columns (in Newtons)

stacking sequence	Current study	Cortínez and Piovan (2006)
-------------------	---------------	----------------------------

$[0^\circ]_4$	162.3	148.7
$[0^\circ / 90^\circ]_s$	86.9	80
$[45^\circ / -45^\circ]_s$	12.2	15.5

The effects of shear deformation in short column

In this example, the importance of shear deformation effects and the capability of the current hybrid method for catching these effects are discussed. The cross-section dimensions and the boundary conditions of the column are illustrated in **Fig. 11**. The flexural buckling happens around the weak axis of the column which is y axis.

Fig. 11. Fixed-fixed Short column with Symmetric Cross-section

The web and flanges are laminated with layers of equal thickness $t=2.5mm$ and the fiber orientation is along the column axis. The material properties of the column are as follow $E_1 = 144MPa, E_2 = 9.65MPa, G_{12} = 4.14MPa, \nu_{12} = 0.3$. To see the effects of shear deformation, the analyses are done with and without the shear deformation for two different lengths of column: $L=1200mm$ and $L=700mm$. In **Table 5** the critical buckling load obtained from the shear flexible analysis and the one that does not include the shear deformation ($A_{yy}^{-1} = 0$) are presented for each column. It can be observed that by decreasing the column length the difference between two analyses increases from 5.1% to 16.3%. Therefore, ignoring the effects of shear deformation will underestimate the actual value of critical buckling load.

346

Table 5. Buckling Loads of Columns (in Newtons)

Length(mm)	with shear	No shear	Difference (%)
1200	13802.3	14540.3	5.1
700	36732.1	42700.3	16.3

347 It should be noted that in short columns the local buckling may occur before the global buckling.

348 In order to avoid that, the cross sectional dimensions and length of the column should be chosen

349 in a way that the local buckling load of web and flanges be more than the global buckling load of

350 the column. This has been done here using the explicit expressions developed by Kollár (2003)

351 for calculating the local buckling load of web and flanges. Based on Kollar's expressions, in this

352 example the flanges will buckle locally before the web. The value of the local buckling load of

353 flanges is calculated as 43724.2 N which is greater than the global buckling loads shown in

354 **Table 5.**

355 *Flexural buckling in sandwich column*

356 In this section, an axially loaded column with a sandwich cross-section is analyzed. The cross

357 section and boundary conditions are depicted in **Fig. 12**. The material properties of the core and

358 faces and the cross sectional dimensions are as follows

359 $E_f = 30000MPa, G_f = 10000MPa, E_c = 75MPa, G_c = 30MPa, t = 1mm, c = 20mm, b = 10mm$

360

Fig. 12. Simply Supported Column with Sandwich Cross-section

The flexural buckling happens around the weak axis of the column which is y axis. In here, the critical buckling load P_{cr} is calculated by adding up the load obtained from shear deformable analysis of the core P_c and the Euler buckling load ($A_{yy}^{-1} = 0$) of faces $P_{E,f}$. The analyses are done for various lengths varying from 500mm to 40mm. In **Table 6** the predicted buckling loads obtained by current hybrid finite element method are presented and compared with the results from the analytical solution of Allen (1969), the formula developed by Blaauwendraad (2010) and the hyperelastic formulation of Attard et al. (2008).

Table 6. Buckling Loads of Columns (in Newtons)

$L \text{ (mm)}$	500	250	100	50	40
Allen (1969)	2.63E+03	1.05E+04	6.58E+04	2.63E+05	4.10E+05
Blaauwendraad (2010)	4.49E+03	1.45E+04	7.13E+04	2.68E+05	4.15E+05
Attard and Hunt (2008)	2.63E+03	1.05E+04	6.54E+04	2.61E+05	4.08E+05
Current model	2.63E+03	1.05E+04	6.58E+04	2.63E+05	4.15E+05

As one can observe, the results obtained by the current shear deformable hybrid finite element method is in a very good agreement with the results from other methods. It is of interest to note that for a very short column the critical buckling load predicted by other methods can be obtained in the current model by simply adding the Euler buckling load of the faces $P_{E,f}$ and shear buckling load of the core $P_{S,c} = G_c A_c$. It is due to the fact that, compared to the long columns, in short columns the Euler buckling load of the core $P_{E,c}$ is very large compared to the shear buckling load $P_{S,c}$. Therefore, it is not the dominant buckling load for the core. As an example, two columns with different lengths are compared here. As we can see in last row of

378 **Table 7**, the difference between P_{cr} and $P_{cr,Short}$ for the long column is very considerable and
 379 cannot be ignored whereas for the shorter column it is negligible.

380 **Table 7.** Comparisons of Buckling Predictions

L (mm)	500	40
$P_{S,c}$	5.00E+03	5.00E+03
$P_{E,c}$	1.97E+01	3.08E+03
$P_{E,f}$	2.61E+03	4.09E+05
$P_{cr,Short} = P_{S,c} + P_{E,f}$	7.61E+03	4.14E+05
$P_{cr} = P_c + P_{E,f}$	2.63E+03	4.10E+05
difference	189.2%	0.76%

381 *Flexural buckling in built-up column*

382 In this section, laced built-up column is studied. In the current hybrid finite element method the
 383 non-collinear elements can be connected easily. Unlike in the complementary energy based
 384 methods, in this method there is no need to satisfy the inter-element equilibrium. Therefore, the
 385 assemblage procedure is as simple as in the displacement based methods. The built-up column
 386 consists of two I-shaped elements spaced by and a lacing system connecting them. It has been
 387 modeled by using 48 elements. The material properties of the elements are
 388 $E = 205MPa$, $G = 82MPa$, $\nu = 0.3$. The configuration of the laced column, dimensions and the
 389 boundary conditions are illustrated in **Fig. 13**.

Fig. 13. Laced Built-up Column and its Cross-section

The first mode of buckling is the overall bending of the column around the y axis and therefore the critical flexural buckling load excluding shear deformation effects is comparable with the equivalent Euler buckling load of the column, $P_E = \pi^2 EI_{col} / (kL)^2$. In which $EI_{col} = 0.5h^2 EA_{ch}$ is the effective bending rigidity of the built-up column and $k=2$ is the effective length factor (Kalochairitis et al. 2011). As one can observe in **Table 8**, the predicted buckling load by the current hybrid model is in a very good agreement with the Euler buckling load.

Table 8. Buckling Loads of Columns (in Newtons)

Current study with shear	Current study without shear	P_E
10974.5	11036.8	10991.5

It can also be noted that the buckling modes involving buckling of segments of the built-up column can be captured by the formulation as well. For instance, the fifth buckling mode of the

column can be observed in **Fig. 14**, which has a buckling load of 801.0 *kN*. It should be noted that because of the local deformations such a mode cannot be captured with a single element.

Fig. 14. Fifth buckling mode of the built-up column

Conclusions

The finite element solution developed in this study is able to accurately capture shear deformation effects, a feature that is neglected in classical buckling solutions. The current hybrid formulation has the advantage of incorporating the shear deformation effects by using the strain energy of the equilibrating shear stress field without modifying the basic kinematic assumptions of the beam theory. It was shown that conventional solutions can overestimate the predicted buckling load in some cases, especially when the E/G ratio is relatively high which may be the case in practice for composite columns. It was shown that the developed hybrid finite element method is in a very good agreement with Engesser's formula, rather than Haringx's in capturing the shear deformation effect in the case of homogenous columns. Non-collinear elements can be easily connected using the developed finite element

416 formulation as only inter-element displacement continuity is required for the assemblage
417 procedure.

418 References

419 Allen, H. G. (1969). Analysis and design of structural sandwich panels. Pergamon, Oxford.

420

421 Attard, M. M. and Hunt, G. W. (2008). "Sandwich column buckling – A hyperelastic
422 formulation." International Journal of Solids and Structures **45**(21): 5540-5555.

423

424 Bank, L. C. (1989). "Flexural and Shear Moduli of Full-section Fiber Reinforced Plastic (FRP)
425 Pultruded Beams." Journal of Testing and Evaluation **17**(1): 40-45.

426

427 Blaauwendraad, J. (2010). "Shear in Structural Stability: On the Engesser–Haringx Discord."
428 Journal of Applied Mechanics **77**(3): 031005-031005.

429

430 Cortínez, V. H. and Piovan, M. T. (2006). "Stability of composite thin-walled beams with shear
431 deformability." Computers & Structures **84**(15–16): 978-990.

432

433 Engesser, F. (1891). "Die Knickfestigkeit gerader Stäbe " Zentralblatt der Bauverwaltung **11**:
434 483-486.

435

436 Erkmen, R. E. (2014). "Shear deformable hybrid finite element formulation for buckling analysis
437 of thin-walled members." Finite Elements in Analysis and Design **82**: 32-45.

438

439 Erkmen, R. E. and Attard, M. M. (2011). "Lateral–torsional buckling analysis of thin-walled
440 beams including shear and pre-buckling deformation effects." International Journal of
441 Mechanical Sciences **53**(10): 918-925.

442

- 443 Erkmen, R. E. and Mohareb, M. (2008). "Buckling analysis of thin-walled open members—A
444 complementary energy variational principle." *Thin-Walled Structures* **46**(6): 602-617.
- 445
- 446 Erkmen, R. E., Mohareb, M. and Bradford, M. (2009). "Complementary energy based on
447 formulation for torsional buckling of columns." *Journal of Engineering Mechanics* **135**(12):
448 1420-1426.
- 449
- 450 Haringx, J. A. (1948-1949). "On highly compressible helical springs and rubber rods, and their
451 application for vibration-free mountings, I - III ." *Phillips Research Reports* **3**, **Eindhoven,**
452 **Holland** 1-97.
- 453
- 454 Huang, H. and Kardomateas, G. A. (2002). "Buckling and initial postbuckling behavior of
455 sandwich beams including transverse shear." *AIAA Journal* **40**(11): 2331-2335.
- 456
- 457 Kalochairetis, K. E. and Gantes, C. J. (2011). "Numerical and analytical investigation of collapse
458 loads of laced built-up columns." *Computers & Structures* **89**(11): 1166-1176.
- 459
- 460 Kant, T. and Patil, H. S. (1991). "Buckling loads of sandwich columns with a higher-order
461 theory." *Journal of Reinforced Plastics and Composites* **10**(1): 102-109.
- 462
- 463 Kollár, L. P. (2001). "Flexural–torsional buckling of open section composite columns with shear
464 deformation." *International Journal of Solids and Structures* **38**(42): 7525-7541.
- 465
- 466 Kollár, L. P. (2003). "Local buckling of fiber reinforced plastic composite structural members
467 with open and closed cross sections." *Journal of Structural Engineering* **129**(11): 1503-1513.
- 468
- 469 Moon-Young, K. and Nam Yi, K. (2003). "Exact dynamic and static stiffness matrices of shear
470 deformable thin-walled beam-columns." *Journal of Sound and Vibration* **267**(1): 29-55.
- 471
- 472 Nguyen, X.-H. and Kim, N.-I. (2015). "Optimum design of thin-walled composite beams for
473 flexural–torsional buckling problem." *Composite Structures* **132**: 1065-1074.

474

475 Omidvar, B. and Ghorbanpoor, A. (1996). "Nonlinear FE solution for thin-walled open-section
476 composite beams." *Journal of Structural Engineering* **122**(11): 1369-1377.

477

478 Reddy, J. N. (2003). *Mechanics of laminated composite plates and shells: Theory and analysis*,
479 2nd ed., CRC Press.

480

481 Saadé, K. and Espion, B. (2004). "Non-uniform torsional behavior and stability of thin-walled
482 elastic beams with arbitrary cross sections." *Thin-Walled Structures* **42**(6): 857-881.

483

484 Silvestre, N. and Camotim, D. (2003). "GBT buckling analysis of pultruded FRP lipped channel
485 members." *Computers and Structures* **81**(18-19): 1889-1904.

486

487 Timoshenko, S. P. and Gere, J. M. (1961). *Theory of elastic stability*. McGrawHill-Kogakusha
488 Ltd, Tokyo.

489

490 Wang, S. (1997). "Free vibration analysis of skew fibre-reinforced composite laminates based on
491 first-order shear deformation plate theory." *Computers & Structures* **63**(3): 525-538.

492

493

494