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On the asymptotic analysis of the JWKB method via change of dependent variable in the first order Bessel's Equation

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Abstract

The first order JWKB method ($=: (JWKB)_1$) is a conventional semi-classical approximation method used in quantum mechanical systems for accurate solutions. It is known to be giving accurate energy and wave-function in the Classically Accessible Region (CAR) of the related quantum mechanical system defined by the Schroedinger's equation whereas the solutions in the Classically Inaccessible Region (CIR) requires a special treatment, conventionally known as the asymptotic matching rules. In this work, $(JWKB)_1$ solution of the Bessel Differential Equation (BDE) of the first order ($=: (BDE)_1$), chosen as a mathematical model, is studied by being transformed into the normal form via the change of dependent variable. General JWKB solution of the initial value problem where appropriately chosen initial values are applied is studied in both normal and standard form representations to be analyzed by the generalized JWKB asymptotic matching rules regarding to the S_{ij} matrix elements defined in the literature. Consequently, regions requiring first order and zeroth order JWKB approximations are determined successfully.

Keywords: JWKB, WKB, Semiclassical Approximation, Asymptotic Matching, Linear Differential Equations, Bessel's Equation.

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1. Introduction

Conventional $(JWKB)_1$ method is a well-known semiclassical approximation method used to find accurate analytical solutions in quantum mechanical systems, described by the Time Independent Schroedinger's Equation (TISE), given by:

$$u''(x) + f(x)u(x) = 0; f(x) = k^2(x) = \frac{2m}{\hbar^2} [E - V(x)] \quad (1.1)$$

whose terms are in the usual meanings, i.e., [1–11]. N th order JWKB expansion ($= (JWKB)_N$) of an analytic function, such as $u(x)$ in (1.1), involves sum of first $N + 1$ functions with indices: $i = 0, 1, \dots, N$ and conventional $(JWKB)_1$ is alternatively called as the “first order JWKB approximation” since it involves sum of only two functions with a maximum index number of 1 [1–11]. Although higher order JWKB is expected to be more accurate, the first order JWKB solution with only its two terms are accurate enough (or, even exact for some potentials such as harmonic oscillator [1–10]) to describe the analytic function under study, such as $u(x)$ in (1.1). Consequently, it is a strong and effective method to approximate the complementary solutions of the TISE in (1.1) conventionally [1–11].

In the JWKB theory, it is conventionally known that either of the complementary $(JWKB)_1$ solutions of (1.1) typically diverges to infinity (causing also the general solution to diverge) in a small region around the classical turning point where $f(x) = 0 \Rightarrow E = V(x)$, i.e., [1–4]. However, the general $(JWKB)_1$ solution is always accurate in the Classically Accessible Region (CAR) where $0 < f(x) \Rightarrow E > V(x)$, provided that the $(JWKB)_1$ applicability criterion holds. The inaccurate Classically Inaccessible Region (CIR) where $f(x) < 0 \Rightarrow E < V(x)$ needs asymptotic matching to give accurate $(JWKB)_1$ solutions if the $(JWKB)_1$ applicability criterion holds [1–4]. It is obvious that regions where the $(JWKB)_1$ applicability criterion does not hold, the $(JWKB)_1$ method is useless and other order

JWKB approximations are required. To obtain the accurate $(JWKB)_1$ solution, the system under study should pre-requisitely meet the conventional $(JWKB)_1$ applicability criterion which requires a slowly varying potential in (1.1) [1–4]. As the potential in the TISE in (1.1) gets sharper, $(JWKB)_1$ becomes useless and higher order JWKB approximation, $(JWKB)_{N>1}$, can be required for accurate-enough solutions. The conventional $(JWKB)_1$ asymptotic matching rules require either of the complementary solutions to be cancelled in the CIR, namely [1–3],

$$\tilde{u}^m(x) = \begin{cases} \tilde{u}(x), & \text{for CAR: } f(x) > 0 \\ \text{either } \tilde{k}_1 \tilde{u}_1(x) \text{ or } \tilde{k}_2 \tilde{u}_2(x) & \text{for CIR: } f(x) < 0 \end{cases} \quad (1.2)$$

In other words, asymptotically matched solutions should obey the following rules:

$$\tilde{u}^m(x) = \begin{cases} \lim_{x \rightarrow -\infty} \tilde{u}(x) = 0, & \text{if CIR lies on the LHS} \\ \lim_{x \rightarrow \infty} \tilde{u}(x) = 0, & \text{if CIR lies on the RHS} \end{cases} \quad (1.3)$$

where superscript “m.” represents the (asymptotically) matched general $(JWKB)_1$ solution. $(JWKB)_1$ solution in the CAR is always accurate and does not require asymptotic matching in the regions where the $(JWKB)_1$ applicability criterion holds [1–4]. Equation (1.3) is known as the conventional asymptotic matching rules and it implies that asymptotically diverging term for $x \rightarrow \pm\infty$ in the CIR should be cancelled in the general solution so that (1.3) can hold in the final (asymptotically matched) $(JWKB)_1$ solution [1–4].

Asymptotic matching rules given in (1.3) can be thought as a physical requirement regarding the normalizability of the physically acceptable wavefunction in the bound state problems of the quantum mechanical systems. An alternative asymptotic matching rule regarding the $(JWKB)_N$ expansion terms was studied for the Simple Linear Potential (SLP) in the TISE in (1.1) in [2]. Since the $(JWKB)_1$ applicability criterion is satisfied in the entire domain in the SLP [2], it was then applied to one of the normal forms of the $(BDE)_1$ in [3] where the criterion is partially satisfied in the domain. Normal form of the intentionally chosen $(BDE)_1$ was obtained by “change of independent variable to the standard form” and, consequently, an alternative JWKB asymptotic matching rule with superior advantages was obtained in [3]. We aim here to study this alternative asymptotic matching rule for another normal form obtained by “change of dependent variable to the standard form”. In effect, we test the alternative asymptotic matching rules suggested in [3] for this completely different TISE obtained by the change of dependent variable here.

TISE in (1.1) for the exponential potential is known to have a connection with the $(BDE)_n$ since its complementary solutions are the Bessel functions of order n and either of them cancels in the general solution for the quantum mechanical reasons (normalizability of the bound state wave functions) in the CIR [11–13]. These physical requirements obviously meet the common asymptotic matching rules given in (1.3). We aim here to search the asymptotic modifications of the $(JWKB)_1$ for the normal form $(BDE)_1$ under the transformation by change of dependent variable. mathematically without considering the physical nature of the corresponding quantum mechanical system.

Our test for the success of the asymptotically matched and unmatched $(JWKB)_1$ general solution is based on a comparison with the exact general solution where the same initial values are applied as in [2,3] as follows: For an Initial Value Problem (IVP), given in the following form:

$$u''(x) + f(x)u(x) = 0; \text{ In. val. s: } \{u[d(c)] = \alpha(c), u'[d(c)] = \beta(c)\}, \quad (1.4)$$

we have the following general solutions:

$$u_{EX}(x) = : u(x) = k_1 u_1(x) + k_2 u_2(x) \quad (1.5a)$$

$$u_{JWKB}(x) = : \tilde{u}(x) = \tilde{k}_1 \tilde{u}_1(x) + \tilde{k}_2 \tilde{u}_2(x) \quad (1.5b)$$

where primes denote derivatives with respect to x , $\{k_1, k_2\}$ & $\{\tilde{k}_1, \tilde{k}_2\}$ are the arbitrary exact and $(JWKB)_1$ constants (which will soon be determined from the given initial values in (1.4)), and $\{u_1, u_2\}$ & $\{\tilde{u}_1, \tilde{u}_2\}$ are the exact and $(JWKB)_1$ complementary solutions, respectively. If the JWKB solution of the associated linear differential equation in (1.4) (with the unknown coefficients) is really an approximate solution with respect to its exact solution, then so is the solution of the associated IVP (with the determined coefficients) where we have the same initial values as given in (1.4). We use this IVP aided comparison to test the accuracy of the unmatched and matched general $(JWKB)_1$ solution.

In this work, we intentionally start our study by the standard form Bessel Differential Equation of first order, $(BDE)_1$:

$$y'' + \frac{y'}{x} + \frac{(x^2-1)y}{x^2} = 0 \quad (1.6)$$

whose general exact solution is written by the linear combination of the two kinds of 1st order Bessel functions:

$$y_{EX}(x) = k_1 y_1(x) + k_2 y_2(x) \Rightarrow y_1(x) = J_1(x) \text{ and } y_2(x) = Y_1(x). \quad (1.7a)$$

The conventional expressions of the first order Bessel functions are given in any fundamental textbooks such as in [14,15] in terms of infinite series, namely:

$$J_1(x) = \sum_{i=0}^{\infty} \frac{(-1)^i (\frac{x}{2})^{2i+1}}{i!(i+1)!}; \quad Y_1(x) = \lim_{p \rightarrow 1} \frac{J_p(x) \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}. \quad (1.7b)$$

As to the general $(JWKB)_1$ solution of the $(BDE)_1$ in (1.6), just like (1.5b), we have:

$$y_{JWKB}(x) =: \tilde{y}(x) = \tilde{k}_1 \tilde{y}_1(x) + \tilde{k}_2 \tilde{y}_2(x) \quad (1.8)$$

However, $(JWKB)_1$ solution requires a normal form differential equation as given in (1.1) by using the $(JWKB)_1$ connection formulas [3]. Here we suggest a *change of dependent variable* to transform it to the normal form to study it like in [3] (where *change of independent variable* was studied). Normal form of (1.6) is not unique and we will go onto a completely different normal form TISE (with a completely different f function in (1.1)) to study the alternative asymptotic matching rules given in [3].

2. JWKB method

2.1. n th order JWKB approximation $((JWKB)_n)$

2.1.1. n th order JWKB solution $((JWKB)_n)$

Formal expression of the $(JWKB)_{N \rightarrow \infty}$ approximation is common for both complementary functions in (1.5b) and given by:

$$\tilde{u}_{1 \text{ and } 2}(x) = \exp \left[\frac{1}{\delta} \sum_{n=0}^N \delta^n S_n(x) \right], \quad (\delta \rightarrow 0) \quad (2.1)$$

where small parameter δ corresponds to $\hbar/i \rightarrow 0$ in the quantum mechanical system described by the the TISE and S_n terms here are the conventional JWKB expansion terms written by [1-3]:

$$S_0(x) = \pm \int \sqrt{\kappa^2(x)} dx =: \pm A_0(x) = \begin{cases} S_{01}(x) = -A_0 \\ S_{02}(x) = A_0 \end{cases} \quad (2.2a)$$

$$S_1(x) = -\frac{1}{4} \ln \kappa^2(x) =: A_1(x) = \begin{cases} S_{11}(x) = A_1(x) \\ S_{12}(x) = A_1(x) \end{cases} \quad (2.2b)$$

$$S_2(x) = \pm \int \left\{ \frac{\partial_{xx}[\kappa^2(x)]}{8\kappa^3(x)} - \frac{5\partial_x[\kappa^2(x)]}{32\kappa^5(x)} \right\} dx =: A_2(x) = \begin{cases} S_{21}(x) = -A_2(x) \\ S_{22}(x) = A_2(x) \end{cases} \quad (2.2c)$$

where $\kappa^2(x) = -k^2(x)$ as in the usual meaning. The first index ($i = 0, 1, 2, \dots, n$) in S_{ij} , represents the first n expansion terms, and the second index ($j = 1, 2$) in S_{ij} is used to label the two different sets corresponding to two complementary functions which are mixed in the formal expansion terms as a result of the two-valuedness of them [2,3]. Here, the first set with S_{i1} contributes to the first complementary $(JWKB)_1$ solution ($\tilde{u}_1$) and the second set with S_{i2} contributes to the second complementary $(JWKB)_1$ solution ($\tilde{u}_2$) in (2.1) via (1.5b).

2.1.2. Accuracy of n th order JWKB approximation $((JWKB)_n)$

It is known that, $(JWKB)_n$ solution of (1.1) given in (2.1) is accurate enough if the higher ordered $(JWKB)_n$ expansion terms are decreasing (as n increases) and if they are bounded by the $(n+1)$ th term [1-3]. So, by considering $\delta \rightarrow 1$ in (2.1), we can write these two conditions by the following inequalities as given in [2,3]:

$$1 \gg \tilde{S}_{(n+1)1}(x) < \tilde{S}_{n1}(x) < \dots < \tilde{S}_{01}(x) \quad (2.3a)$$

$$1 \gg \tilde{S}_{(n+1)2}(x) < \tilde{S}_{n2}(x) < \dots < \tilde{S}_{02}(x) \quad (2.3b)$$

where the definition of $\tilde{S}_{ij}$ are as follows:

$$\tilde{S}_{ij}(x) = \begin{cases} |S_{ij}(x)|, & \text{if } S_{ij}(x) \in \mathbb{C} \\ S_{ij}(x), & \text{if } S_{ij}(x) \in \mathbb{R} \end{cases} \quad (2.4)$$

2.1.3. Asymptotic Matching of $(JWKB)_n$

Non-obeying expansion terms (whether $\tilde{S}_{i1} \Rightarrow S_{i1}$ and/or $\tilde{S}_{i2} \Rightarrow S_{i2}$) need cancellation so that all the used expansion terms should obey (2.3a)–(2.3b) [2,3].

2.2. 1st order JWKB approximation $(JWKB)_{n=1}$

2.2.1. $n = 1$ st order JWKB solution $(JWKB)_1$:

$(JWKB)_1$ approximation involves only the first two terms with indices $n = 0, 1$ since $N = 1$ in (2.1) and it gives the conventional $(JWKB)_1$ formula:

$$\tilde{u}(x) = \exp \left[\frac{1}{\delta} S_0(x) + S_1(x) \right], \delta \rightarrow 1 \Rightarrow \begin{cases} \tilde{u}_1(x) = \exp \left[\frac{S_{01}(x)}{\delta} + S_{11}(x) \right] \\ \tilde{u}_2(x) = \exp \left[\frac{S_{02}(x)}{\delta} + S_{12}(x) \right] \end{cases}, \delta \rightarrow 1, \quad (2.5)$$

which is equivalent to (1.5b). Normally, δ is chosen as $\delta = (\hbar/i) \rightarrow 0$ for the TISE, however, our preference for the definitions of the expansion terms in (2.2a)–(2.2c) are for $\delta \rightarrow 1$. The terms $\tilde{u}_1$ and $\tilde{u}_2$ in (2.5) are the complementary solutions as defined in (1.5b). The conventional first order JWKB solution of (1.4) (which is quantum mechanically associated with the TISE in (1.1)) given in (2.5) can be written as:

$$\tilde{u}(x) = \frac{C(c)}{\sqrt{k(x)}} \exp \left[-i \int_{x_t}^x k(x) dx \right] + \frac{D(c)}{\sqrt{k(x)}} \exp \left[i \int_{x_t}^x k(x) dx \right] \quad (2.6)$$

where x_t is the classical turning point of $f(x)$ and $C(c)$ & $D(c)$ are the c dependent coefficients to be determined from the initial values of our IVP in (1.4) [1,2,4]. Note that initial values in (1.4) are defined c dependent, consequently, so are the coefficients in (2.6). $(JWKB)_1$ solution in (2.6) is used to find the accurate solution in either region (CAR or CIR) and once it has been obtained, the other adjacent region(s) can directly be determined via the conventional $(JWKB)_1$ connection formulae as in [1,2,4]:

$$CAR: \frac{2}{\sqrt{k(x)}} \sin \left[\int_{x_t}^x k(x) dx + \frac{\pi}{4} \right] \longleftrightarrow CIR: \frac{1}{\sqrt{\kappa(x)}} \exp \left[- \int_{x_t}^x \kappa(x) dx \right] \quad (2.7a)$$

$$CAR: \frac{1}{\sqrt{k(x)}} \cos \left[\int_{x_t}^x k(x) dx + \frac{\pi}{4} \right] \longleftrightarrow CIR: \frac{1}{\sqrt{\kappa(x)}} \exp \left[+ \int_{x_t}^x \kappa(x) dx \right] \quad (2.7b)$$

The definite integrals here involve turning point x_t and variable x should be connected in the correct sequence. $(JWKB)_1$ connections between CAR-CIR are made in the double arrow direction which means that (2.7a) connects CIR to CAR and (2.7b) connects CIR to CAR. However, locations of the CAR-CIR in a specific quantum mechanical problem under study can be in the opposite orientations requiring reverse connections, which are normally not allowed. To illustrate, (2.7a) enables connecting a CIR located on the right-hand-side (RHS) of the classical turning point (x_t) to the adjacent CAR located on the left-hand-side (LHS) of the classical turning point (x_t) but it could not be used to connect if the locations of the CAR and CIR were reversed (if CAR were on the RHS and CIR were on the LHS) since the correct order of integral limits would not permit. In such a case where reverse connection is required, a small phase term should be added as follows [2,4]:

$$\begin{aligned} \frac{2}{\sqrt{k(x)}} \sin \left[\int_{x_t}^x k(x) dx + \frac{\pi}{4} + \varepsilon \right] &= \frac{2}{\sqrt{k(x)}} \sin[\mu + \varepsilon] = \frac{2}{\sqrt{k(x)}} [\sin\mu \cos\varepsilon + \cos\mu \sin\varepsilon] \\ &\rightarrow \frac{\cos\varepsilon}{\sqrt{k(x)}} \exp \left[- \int_{x_t}^x \kappa(x) dx \right] + \frac{2 \sin\varepsilon}{\sqrt{k(x)}} \exp \left[+ \int_{x_t}^x \kappa(x) dx \right] \end{aligned} \quad (2.8)$$

Similarly (2.7b) can be used for reverse connection by adding a small phase term, too.

2.2.2. $(JWKB)_1$ applicability criterion:

In order that one can obtain accurate enough solution by the first order JWKB method, the physical requirement regarding to the slowly changing potential in the TISE given in (1.1) can be described by the following $(JWKB)_1$ applicability criterion [1–4]:

$$0 \leq g(x) = \left| \frac{\partial_{xx} k(x)}{2k^3(x)} - \frac{3[\partial_x k(x)]^2}{4k^4(x)} \right| \ll 1 \quad (2.9)$$

Our interest here is the region(s) obeying (2.9). For such obedient regions, according to the JWKB theories, we expect to obtain an accurate $(JWKB)_1$ solution in the CAR and an inaccurate $(JWKB)_1$ solution requiring an

asymptotic matching in the CIR. So, the alternative asymptotic matching rules we suggest should give the same result without consulting (2.9).

2.2.3. Asymptotic Matching of $(JWKB)_1$

From (2.3a) and (2.3b) with $n = 1$ for the first order JWKB, we have:

$$1 \gg \tilde{S}_{21}(x) < \tilde{S}_{11}(x) < \tilde{S}_{01} \quad (2.10a)$$

$$1 \gg \tilde{S}_{22} < \tilde{S}_{12} < \tilde{S}_{02}. \quad (2.10b)$$

For the region where this criterion holds, the $(JWKB)_1$ solution should already give accurate solution and we expect it to fall in the CAR as the JWKB theories say. But, for the regions where both do not hold, it should sign the necessity of an asymptotic matching as we similarly expect it to fall in the CIR as the JWKB theories say. Moreover, since the asymptotically matched solutions should also obey (2.10a) or (2.10b), non-obedient one should require a cancellation in the corresponding term. To illustrate, if (2.10a) fails, then non-obedient term in the first set in (2.5) cancels and if (2.10b) fails, then non-obedient term in the second set in (2.5) cancels. Note that, these two sets are due to the two-valuedness of the expansion terms and second indices in (2.10a) and (2.10b) are set numbers. So, we here also determine which set should be cancelled by the asymptotic matching process. For the regions where (2.10a) and (2.10b) fail to be bounded by 1, it should sign that the $(JWKB)_1$ general solution in both CAR and CIR fails since higher order JWKB approximation, $(JWKB)_{n>1}$, can be required for accurate JWKB solutions.

3. Standard and Normal Form Analyses

3.1. Associated IVP aided comparison method

To test the accuracy of the $(JWKB)_1$ solution, we follow the associated IVP aided comparison method given in [3]. The process being followed here involves comparison of the graphs of $(JWKB)_1$ and exact solutions where the coefficients are determined by applying the same initial values to both. Common initial values in (1.4) give for the general exact solution in (1.5a) and for the $(JWKB)_1$ solution in (1.5b) the followings:

$$u(c, x) = k_1(c)u_1(x) + k_2(c)u_2(x) \quad (3.1)$$

(where $k_1(c)$ & $k_2(c)$ are the c dependent coefficients satisfying the initial values in (1.4)), and

$$\tilde{u}(c, x) = \tilde{k}_1(c)\tilde{u}_1(x) + \tilde{k}_2(c)\tilde{u}_2(x) \quad (3.2)$$

(where similarly $\tilde{k}_1$ & $\tilde{k}_2$ are the c dependent coefficients satisfying the same initial values in (1.4)). But we should first turn it to the normal form to study in the normal form as discussed above.

3.2. Change of Dependent Variable and Re-statement of the Problem in the Normal Form

Change of dependent variable:

$$u : (-\infty, \infty) \rightarrow (0, \infty), y(x) = x^{-1/2}u(x) \quad (3.3a)$$

transforms the Bessel's equation of first order given in (1.6) into the normal form [15]:

$$u''(x) + f(x)u(x) = 0 \quad (3.3b)$$

where

$$f(x) = k^2(x) = \frac{4x^2 - 3}{4x^2}. \quad (3.3c)$$

In other words:

$$u_{EX}(x) = : u(x) = \sqrt{x}[k_1 y_1(x) + k_2 y_2(x)] \quad (3.3d)$$

where

$$y_1(x) = J_1(x) = x^{-1/2}u_1(x); y_2(x) = J_1(x) = x^{-1/2}u_2(x). \quad (3.3e)$$

Normal form obtained here by the change of dependent variable or in [3] by change of independent variable are not unique as being studied in the literature, i.e., [3,12–17]. In deed, exponential potential decorated quantum mechanical systems (TISE) are some of the examples for the normal form representations of the $(BDE)_n$, which is conventionally given in the standard form as studied in [12,13].

Some special care in choosing initial values should be taken as discussed in [3], such as, they should not be chosen at the classical turning points (where the general $(JWKB)_1$ solution typically diverges) or in the CIR (where

the general $(JWKB)_1$ solution typically fails). In other words, initial values should be chosen at the points where the $(JWKB)_1$ is accurate and finite. Accordingly, the initial values in (1.4) can safely be chosen to be:

$$d(c) = c; \alpha(c) = 0; \beta(c) = 1 \quad (3.4)$$

Here, the classical turning point where $f(x_t) = 0$ corresponds to $x_t = \pm 3/2$. Also note that, theoretically c -dependent terms: $d(c)$, $\alpha(c)$ and $\beta(c)$ in (1.4) are chosen here as constant functions and the general solutions in (3.2) and (3.3) are not violated in (3.4).

4. Exact Solution of the $(BDE)_1$ in the Suggested Normal and Standard Form

Standard form exact solution of the $(BDE)_1$ is the linear combination of the first order Bessel functions of both types given in (3.3e) and considering the form in (3.1), it can be written in the transformed normal form by

$$u(c, x) = \sqrt{x}[k_1(c)J_1(x) + k_2(c)Y_1(x)], -\infty < x < \infty \quad (3.5)$$

and by applying the initial values in (1.4), we have:

$$\begin{cases} u(c, d) = \sqrt{d}[k_1(c)y_1(d) + k_2(c)y_2(d)] = \alpha(c) \\ [\partial_x u(c, x)]_{x=d} = \{k_1(c)\partial_x[\sqrt{x}y_1(x) + k_2(c)\partial_x[\sqrt{x}y_2(x)]\}_{x=d} = \beta(c) \end{cases} \quad (3.6)$$

from which we find:

$$\begin{aligned} k_1(c) &= \left\{ \frac{\alpha(c)\partial_x[\sqrt{x}y_2(x)]_{x=d} - \beta(c)\sqrt{d}y_2(d)}{\Delta(d)} \right\}_{[d=c, \alpha(c)=0, \beta(c)=1]} \\ &= -\frac{2Y_1(c)}{\sqrt{c}[J_1(c)Y_0(c) - J_0(c)Y_1(c) + J_2(c)Y_1(c) - J_1(c)Y_2(c)]} \end{aligned} \quad (3.7a)$$

and

$$\begin{aligned} k_2(c) &= \left\{ \frac{-\alpha(c)\partial_x[\sqrt{x}y_1(x)]_{x=d} - \beta(c)\sqrt{d}y_1(d)}{\Delta(d)} \right\}_{[d=c, \alpha(c)=0, \beta(c)=1]} \\ &= -\frac{2J_1(c)}{\sqrt{c}[J_1(c)Y_0(c) - J_0(c)Y_1(c) + J_2(c)Y_1(c) - J_1(c)Y_2(c)]} \end{aligned} \quad (3.7b)$$

where the discriminant $\Delta(d)$ is the Wronskian determinant defined by

$$\begin{aligned} \Delta(d) &= \begin{vmatrix} u_1(d) & u_2(d) \\ \partial_x[u_1(x)]_{x=d} & \partial_x[u_2(x)]_{x=d} \end{vmatrix} = \begin{vmatrix} \sqrt{d}J_1(d) & \sqrt{d}Y_1(d) \\ \partial_x[\sqrt{x}J_1(x)]_{x=d} & \partial_x[\sqrt{x}Y_1(x)]_{x=d} \end{vmatrix} \\ &= \frac{d}{2}\{J_1(d)[Y_2(d) - Y_0(d)] + Y_1(d)[J_0(d) - J_2(d)]\} \end{aligned} \quad (3.8)$$

5. $(JWKB)_1$ Solution of the $(BDE)_1$ in the Suggested Normal and Standard Form

The $(JWKB)_1$ applicability criterion in (2.9) gives:

$$0 \leq g(x) = \left| \frac{9-72x^2}{(4x^2-3)^3} \right| \ll 1 \quad (3.9)$$

whose graph plotted with the graph of $f(c, x)$ for some c values are given in Fig. 1. We see that classical turning point where $f(x) = 0$ is at $x_t = 3/2$. Since the CIR where $f(x) < 0$ lies on the right-hand-side of the turning point (similarly, the CAR where $f(x) > 0$ lies on the left-hand-side of it), conventional asymptotic matching rules given in (1.2)–(1.3) shows that the correct asymptotic matching (modification) in the CIR should be as follows [1–3]:

$$u^m(x) \rightarrow \lim_{x \rightarrow \infty} u(x) = 0 \quad (3.10a)$$

$$\tilde{u}^m(x) \rightarrow \lim_{x \rightarrow \infty} \tilde{u}(x) = 0 \quad (3.10b)$$

However, we will soon see that the general $(JWKB)_1$ solution $\tilde{u}(x)$ does not need matching in this region since these equations are already hold $\tilde{u}^m(x) = \tilde{u}(x)$.

Moreover, since the two narrow sub-regions around the turning points in Fig. 1 do not obey (3.9) for all real c values in the domain, the $(JWKB)_1$ method cannot be expected to give accurate results since as typical to the $(JWKB)_1$ method. We will also see it by our suggested asymptotic matching rules soon here.

Fig. 1. Graphs of functions: $f(x)$ (on the left) and $g(x)$ (on the right).

The width of these two narrow sub-domains can be found from the solution of $g(x) > 1$ for x in (3.9) for real x values as follows:

$$D_1: [D_{1a}: x \in (-a = -1.42898, -b = -0.491652)] \cup [D_{1b}: x \in (b = 0.491652, a = 1.42898)] \quad (3.11a)$$

and the remaining one narrow and one wide domains can be found to be as follows:

$$D_2: -b = -0.491652 << x << b = 0.491652 \Leftrightarrow$$

$$D_3: [D_{3a}: -\infty < x << (-a = -1.42898)] \cup [D_{3b}: (a = 1.42898) << x < \infty]. \quad (3.11b)$$

Here, the general (JWKB)₁ solution in D_3 should directly be accurate-enough when compared to the general exact solution without requiring any matching since it is in the CAR and $g(x) < 1$. For D_1 , (JWKB)₁ solution seems useless since $g(x) \geq 1$ and D_2 seems to be in need of special analyses in order to be matched since it is in the CIR and $g(x) < 1$.

Since the regions in the ascending order (as LHS-RHS) are located here as CAR-CIR, we can start by calculating the (JWKB)₁ solution in the CAR and connect it to the CIR by using the (JWKB)₁ connection formulas given in (2.7) in the reverse direction as in (2.8). Since the orientations of the CAR and CIR are the same as in [2, see example 1], it gives the same formulas for $\tilde{u}_L(c, x)$ and $\tilde{u}_R(c, x)$ (But in $\tilde{u}(c, x)$ here):

$$\tilde{u}(c, x) = \begin{cases} \tilde{u}_L(c, x), & \text{for } 0 < x \leq \frac{\sqrt{3}}{2} \\ \tilde{u}_R(c, x), & \text{for } \frac{\sqrt{3}}{2} \leq x < \infty \end{cases} \quad (3.12)$$

where

$$\tilde{u}_L(c, x) = \frac{A(c)}{\sqrt{k(x)}} \sin[\eta(x) + \gamma(c)] \quad (3.13a)$$

and

$$\tilde{u}_R(c, x) = \frac{A(c)}{2\sqrt{\kappa(x)}} \cos[\gamma(c) - \frac{\pi}{4}] \exp[-\zeta(x)] + \frac{A(c)}{\sqrt{\kappa(x)}} \sin[\gamma(c) - \frac{\pi}{4}] \exp[\zeta(x)] \quad (3.13b)$$

Now, the constituents (functions: η and ζ) in (3.13a)–(3.13b) here rather reads:

$$\eta(x) = \int_{\frac{\sqrt{3}}{2}}^x k(x) dx = -\frac{\sqrt{3}\pi}{4} + \frac{\sqrt{4x^2 - 3}}{2} + \frac{\sqrt{3}}{2} \operatorname{arccot} \left(\sqrt{\frac{4x^2 - 3}{3}} \right)$$

$$\zeta(x) = \int_x^{\frac{\sqrt{3}}{2}} \kappa(x) dx = \frac{i}{4} \left[\sqrt{3}\pi - 2\sqrt{4x^2 - 3} - 2\sqrt{3} \operatorname{arccot} \left(\sqrt{\frac{4x^2 - 3}{3}} \right) \right] \quad (3.14a)$$

(where k and κ are in usual meanings: $\kappa^2(x) = -k^2(x)$) and applications of the initial values in (3.4) gives:

$$\left[\begin{array}{l} \tilde{u}(c, d) = \tilde{u}_L(c, d) = \alpha(c) \\ [\partial_x \tilde{u}(c, x)]_{x=d} = [\partial_x \tilde{u}_L(c, x)]_{x=d} = \beta(c) \end{array} \right]_{[d=c, \alpha(c)=0, \beta(c)=1]} \quad (3.14b)$$

$$A(c) = \frac{\sqrt{2}c(4-\frac{3}{c^2})^{1/4}}{\sqrt{4c^2-3}}, \text{ for } 0 < c \neq x_t = \frac{\sqrt{3}}{2} < \infty \quad (3.14c)$$

$$\gamma(c) = \frac{1}{2} \left[-\sqrt{4c^2-3} + \frac{\sqrt{3}\pi}{2} - \sqrt{3} \operatorname{arccot} \left(\sqrt{\frac{4c^2-3}{3}} \right) \right] \quad (3.14d)$$

Note that, since we have chosen the initial values in the LHS in the CAR with CAR: $-\infty < d = c < x_t = 3/2$, according to (3.12), the general $(JWKB)_1$ solution $\tilde{u}(c, d)$ in (3.14b) corresponds to $\tilde{u}_L(c, d)$. General $(JWKB)_1$ solution in the other form given in (1.5b), can then be written as follows [1,2]:

$$\tilde{u}_1(x) = \begin{cases} \frac{2}{\sqrt{\kappa(x)}} \sin \left[\eta(x) + \frac{\pi}{4} \right], \sqrt{3}/2 < |x| < \infty \\ \frac{1}{\sqrt{\kappa(x)}} \exp[-\zeta(x)], 0 \leq |x| < \sqrt{3}/2 \end{cases} \quad (3.15a)$$

$$\tilde{u}_2(x) = \begin{cases} \frac{1}{\sqrt{\kappa(x)}} \sin[-\eta(x) + \pi/4], \sqrt{3}/2 < |x| < \infty \\ \frac{1}{\sqrt{\kappa(x)}} \exp[\zeta(x)], 0 \leq |x| < \sqrt{3}/2 \end{cases} \quad (3.15b)$$

where $\eta(c, x)$ and $\zeta(c, x)$ are as given in (3.14a), and by applying the initial values in (1.4) and (3.4) we find the followings:

$$\begin{cases} \tilde{u}(c, d) = \tilde{k}_1(c)\tilde{u}_1(d) + \tilde{k}_2(c)\tilde{u}_2(d) = \alpha(c) \\ [\partial_x \tilde{u}(c, x)]_{x=d} = [\tilde{k}_1(c)\partial_x \tilde{u}_1(x) + \tilde{k}_2(c)\partial_x \tilde{u}_2(x)]_{x=d} = \beta(c) \end{cases} \quad (3.16)$$

$$\begin{aligned} \Rightarrow \tilde{k}_1(c) &= \left[\frac{\alpha(c)\partial_x [\tilde{u}_2(x)]_{x=d} - \beta(c)\tilde{u}_2(d)}{\tilde{\Delta}(c, d)} \right]_{[d=c, \alpha(c)=0, \beta(c)=1]} = \frac{A(c)}{2} \cos[\alpha(c) - \pi/4] \\ &= \frac{(4c^4-3c^2)^{1/4}}{\sqrt{8c^2-6}} \sin \left\{ \frac{1}{4} \left[-\sqrt{4c^2-3} + \pi + \sqrt{3}\pi - 2\sqrt{3} \operatorname{arccot} \left(\sqrt{\frac{4c^2-3}{3}} \right) \right] \right\} \end{aligned} \quad (3.17a)$$

and

$$\begin{aligned} \Rightarrow \tilde{k}_2(c) &= \left[\frac{-\alpha(c)\partial_x [\tilde{u}_1(x)]_{x=d} - \beta(c)\tilde{u}_1(d)}{\tilde{\Delta}(c, d)} \right]_{[d=c, \alpha(c)=0, \beta(c)=1]} = A(c) \sin[\alpha(c) - \pi/4] \\ &= \frac{\sqrt{2}c}{(4c^2-3)^{1/4}} \sin \left\{ \frac{1}{4} \left[2\sqrt{4c^2-3} + \pi - \sqrt{3}\pi + 2\sqrt{3} \operatorname{arccot} \left(\sqrt{\frac{4c^2-3}{3}} \right) \right] \right\} \end{aligned} \quad (3.17b)$$

where the discriminant $\tilde{\Delta}(c, d)$ is the Wronskian determinant defined by:

$$\Delta(d) = \begin{vmatrix} \tilde{u}_1(d) & \tilde{u}_2(d) \\ \partial_x [\tilde{u}_1(x)]_{x=d} & \partial_x [\tilde{u}_2(x)]_{x=d} \end{vmatrix} = -2, \text{ for } d \neq \sqrt{3}/2 \quad (3.18)$$

6. $(JWKB)_1$ Asymptotic Matching of the Standard and Suggested Normal Form $(BDE)_1$

Both exact and $(JWKB)_1$ general solutions of our IVP in the normal and standard form is given in Fig. 2, from which we see that $(JWKB)_1$ solution is sure to be successful for D_3 as expected from Fig. 1 according to the result of the criterion in (3.9). Note that, although the other domains are not much clear in Fig. 2, one can consult Fig. 4 where a detailed graph focused on these relatively narrow domains are given along with the asymptotically matched solutions. We can see that $(JWKB)_1$ solutions in the CAR for D_3 are accurate when compared with the exact solutions without requiring any matching where as $D_1 \cup D_2$ are not accurate as typical consequences of the JWKB theories as discussed above. Now, the questions are whether we can see this without either interfering the exact results or applying the conventional asymptotic matching rules in (1.2)–(1.3) (and hence, (3.10b) here), and how they can be asymptotically matched in terms of expansion terms which we are studying here.

Let us follow testing our alternative suggestion in (2.10a)–(2.10b). Inequalities in (2.10a)–(2.10b) and the definition in (2.4) gives:

$$\tilde{S}_{ij}(x) = \begin{cases} |S_{ij}(x)|, \forall x - \left\{ \pm \frac{\sqrt{3}}{2} \right\}, \because S_{ij}(x) \in \mathbb{C} \text{ in the CAR} \cup \text{CIR} \Leftrightarrow n = 0, 2 \\ \left\{ \begin{array}{l} |S_{ij}(x)|, |x| < \sqrt{3}/2 \because S_{ij}(x) \in \mathbb{C} \text{ in the CIR} \\ S_{ij}(x), \frac{\sqrt{3}}{2} < |x| \because S_{ij}(x) \in \mathbb{R} \text{ in the CAR} \end{array} \right\} \Leftrightarrow n = 1 \end{cases} \quad (3.19)$$

since from (2.2a)–(2.2c) we have:

$$S_{01}(x) = -S_{02}(x) = \pm \frac{i}{2} \left[\sqrt{4x^2 - 3} + \sqrt{3} \operatorname{arccot} \left(\sqrt{\frac{4x^2 - 3}{3}} \right) \right] \quad (3.20a)$$

$$S_{11}(x) = S_{12}(x) = -\frac{1}{4} \ln \left(\frac{3 - 4x^2}{4x^2} \right) \quad (3.20b)$$

$$S_{21}(x) = -S_{22}(x) = \frac{\pm i \left[6(2x^2 + 1) \sqrt{4x^2 - 3} - \sqrt{3} (4x^2 - 3)^2 \operatorname{arccot} \left(\sqrt{\frac{4x^2 - 3}{3}} \right) \right]}{12(4x^2 - 3)^2} \quad (3.20c)$$

from which we have: $S_{ij}(x) \in \mathbb{C}$ in the CAR (where $x < 3/2$) and $S_{ij}(x) \in \mathbb{R}$ in the CIR (where $3/2 < x$).

Fig. 2. Graphs of exact and JWKB solutions in **a)** normal form: $u(x)$, **b)** standard form: $y(x)$.

Before achieving the comparison given in (2.3a)–(2.3b) (and (2.10a)–(2.10b) for $n = 1$), we should first check whether the elements are real or complex as given in due to (2.4) via [2,3]. If both (2.3a)–(2.3b) (and (2.10a)–(2.10b) for $n = 1$) hold (we obviously see that this happens in the CAR), then the general $(JWKB)_1$ solution involves both $(JWKB)_1$ complementary functions ($\tilde{y}_1$ and $\tilde{y}_2$). Reviewing equation (2.1) along with (2.2a)–(2.2c) we see that S_{i1} (and hence $\tilde{S}_{i1}$) contributes to $\tilde{y}_1$ and similarly, S_{i2} (and hence $\tilde{S}_{i2}$) contributes to $\tilde{y}_2$. So, any non-obedient S_{i1} terms should require a cancellation of $\tilde{y}_1$ and similarly, any non-obedient S_{i2} terms should require a cancellation of $\tilde{y}_2$ in the related subdomains for a successful asymptotic matching in terms of expansion terms as stated in [3].

Graphs of $\tilde{S}_{i1}(c, x)$ and $\tilde{S}_{i2}(c, x)$ for some c values are given in Fig. 3. We can see that: i) $D_1: D_{1a} \cup D_{1b}$ is useless since (2.10a)–(2.10b) does not hold unless all the expansion terms are cancelled, ii) $D_2: D_{2a} \cup D_{2b}$ can be matched by cancelling S_{21} and S_{22} , which means: $(JWKB)_0$, so that (2.10a)–(2.10b) can hold. iii) $D_3: D_{3a} \cup D_{3b}$ does not need matching since (2.10a)–(2.10b) already hold. These results are in consistence with our analyses via the graph of $g(c, x)$ given in Fig. 3. Indeed, ours here are more informative since also showing us which expansion term to be cancelled in the CIR. This remarkable result is obtained by our pure $(JWKB)_1$ analysis regarding the expansion terms via Fig. 3 without consulting either the exact solutions or reasoning the physical nature of the corresponding quantum mechanical system. From Fig. 3, we see that need of asymptotic matching in the normal form corresponds to D_2 and by applying (2.10a)–(2.10b) to these graphs, it can be calculated as follows:

$$\tilde{y}^m(c, x) = x^{-\frac{1}{2}} \tilde{u}^m(c, x) (\text{everywhere}) \quad (3.21a)$$

$$\tilde{u}^m(c, x) = \tilde{k}_1(c) \exp[-\zeta(x)] + \tilde{k}_2(c) \exp[\zeta(x)] \text{ for } D_2: -b \leq x \leq b$$

where non-obeying terms (S_{11} & S_{12}) in (2.5) are not included, in effect, zeroth-order JWKB ($= (JWKB)_0$) is necessarily used in D_2 . Superscript "m." shows successfully asymptotically matched solutions here. Constant coefficients $\tilde{k}_1$ & $\tilde{k}_2$ can be found from initial values selected on D_2 . To show its correctness for a specific c value, we choose the initial values by using the exact solution as follows:

$$u(2, 0.491652) = -1.30114 \approx: \tilde{u}(2, 0.491652) \quad (3.21b)$$

$$\left[\frac{\partial u(c, x)}{\partial x} \right]_{\{c=2, x=0.491652\}} = 1.06959 \approx: \left[\frac{\partial \tilde{u}(c, x)}{\partial x} \right]_{\{c=2, x=0.491652\}}$$

$$\Rightarrow \tilde{k}_1 = -0.37939(503040768785), \tilde{k}_2 = -0.757057(729122614). \quad (3.21c)$$

From our analyses we see that, asymptotic modification is required only in D_2 and the other regions do not need it since (2.10a)–(2.10b) already holds. This result is also in agreement with the results obtained by the conventional asymptotic matching rules discussed above. Graphs of our modified (asymptotically matched) solutions for a specific

c value ($c = 2$) are given along with the graphs of un-modified (asymptotically unmatched) solutions for comparison in Fig. 4, from which we see that D_2 has been successfully matched whereas the conventional asymptotic matching rules have no way to match (or even predict the accuracy) this region.

Fig. 3. Graphs of $\tilde{S}_{i1}(c, x)$ (left column) and $\tilde{S}_{i2}(c, x)$ (right column) (where $i = 0, 1, 2$) (solid red curves: for $i = 0$, dashed green curves: for $i = 1$, and dotted blue curves: for $i = 2$).

Fig. 4. a) normal form solutions for $c = 2$, **b)** standard form solutions for $c = 2$. (Black-dashed curve: Exact, Blue-dotted curve: $(JWKB)_1$, Red&orange-solid curve: $(JWKB)^{(m)}$ ($\therefore$ order modified)

(a) $u_{EX}, u_{JWKB}, u_{JWKB}^m$

(b) $y_{EX}, y_{JWKB}, y_{JWKB}^m$

7. Conclusion

We can see that application of our alternative asymptotic matching rules given in (2.3a)–(2.3b)–(2.4) gives consistent results with the present conventional asymptotic matching rules given in (1.2)–(1.3). For the

intentionally chosen $(BDE)_1$ (since $(JWKB)_1$ applicability criterion both holds and fails in some subdomains of both CAR&CIR as in [3]) we studied one of the normal forms obtained by the change of dependent variable. For the normal form we have studied here, we see that D_3 does not need any asymptotic matching since it satisfies (2.3a)–(2.3b) as the conventional asymptotic matching rules in (1.2)–(1.3) concludes. We also see that present conventional asymptotic matching rules in (1.2)–(1.3) works only in D_3 , but ours in (2.3a)–(2.3b) work in D_2 , too. In deed, the subdomain in D_2 is matched correctly and it implies the zeroth order JWKB, $(JWKB)_0$ for the necessary complementary solution. So it can also be thought as a "JWKB order matching" as a consequence of our intended asymptotic matching. We can see that conventional asymptotic matching rules in (1.2)–(1.3) are useless for D_2 , but ours in (2.3a)–(2.3b), as shown in red&orange-solid curve in Fig. 4, have successfully matched it in D_2 , too. Consequently, our alternative modification seems enabling a correct determination of which complementary function (solution) and where in the semiclassically solvable (sub)domain to cancel. Our pure semiclassical alternative asymptotic matching rules in (2.3a)–(2.3b) enable a modification not only asymptotically (see (1.2)–(1.3)) but in the whole domain where the JWKB application criterion holds. It also shows such applicable domains where (2.3a)–(2.3b) can be maintained by cancelling either of the complementary solutions. It is tested in our analyses here via the $(BDE)_1$ for the $(JWKB)_N$ where $N = 0, 1$ via (2.10a)–(2.10b) and higher order JWKB analyses (with $N > 1$) seem promising for the upcoming studies.

References

- [1] C. M. Bender and S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers Asymptotic Methods and Perturbation Theory*, Springer-Verlag, NY. 1999.
- [2] C. Deniz. *Ann. Phys.*, 326, 1816(2011). (for *erratum: Ann. Phys.*, 371, 478 (2016)).
- [3] C. Deniz. *Math. Comp. Appl.*, 21, 1 (2016).
- [4] A. K. Ghatak, R.L. Gallawa, and I. C. Goyal. *Modified Airy Functions and WKB Solutions to the Wave Equation*, NIST, Washington. 1991.
- [5] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics, Non-Relativistic Theory*, second ed., *Course of Theoretical Physics*, vol. 3, Pergamon, NY. 1965.
- [6] R. E. Langer. *Phys. Rev.*, 51, 669 (1937).
- [7] A. K. Ghatak and S. Lokhanatan. *Quantum Mechanics: Theory and Applications*, Springer, Netherland, 2004. pp: 423-477.
- [8] A. K. Ghatak, E. G. Sauter and I. C. Goyal. *Eur. J. Phys.*, 18, 199 (1997). doi: 10.1088/01430807/18/3/013.
- [9] M. Hruska, W.-Y. Keung, and U. Sukhatme. *Phys. Rev. A*, 55, 3345 (1997). doi: 10.1103/PhysRevA.55.3345.
- [10] H. Friedrich and J. Trost. *Phys. Rev. A*, 54, 1136 (1996). doi: 10.1103/PhysRevA.54.1136.
- [11] C. M. Fabre and D. Guéry-Odelin. *Am. J. Phys.*, 79, 755 (2011). doi: 10.1119/1.3567002.
- [12] D. W. L. Sprung, S. Akbar, and N. Sator. *Am. J. Phys.*, 79, 755 (2011). (for *erratum: Am. J. Phys.*, 80, 734 (2012). doi: 10.1119/1.3699066.
- [13] A. Safari; Some Recent Developments in WKB Approximation, *Open Access Dissertations and Theses*, Paper 8172. (2013).
- [14] M. L. Boas. *Mathematical Methods in the Physical Sciences*, 3rd ed., Wiley, NY. 2006.

- [15] H. J. Arfken and G. B. Weber. Mathematical Methods for Physicists, 6th ed., Elsevier Academic Press, International Edition. 2005.
- [16] R. L. Langer. Trans. Am. Math. Soc., 33, 23 (1931).
- [17] R. E. Langer. Trans. Am. Math. Soc., 36, 90 (1934).

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