Controllability of Networks of Linear Systems: 
a Graph Theoretic Approach

by

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Abstract

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In this thesis, we study the controllability of networked single-input single-output linear time-invariant systems. We show that necessary conditions for the controllability of networks of single-integrators such as dilation, input-symmetry and equitable partitioning do not apply, in general, to more complex networks. We introduce new necessary and sufficient conditions for the controllability of networks in terms of the interconnection topology and sub-systems’ properties. We also provide a number of graph-theoretic necessary conditions for controllability of networks. As a particular case of our results, we will provide necessary conditions for the controllability of diffusively coupled networks. Finally, we illustrate our results by studying the controllability of a simple neuronal network model.
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To my amazing family who always believed in me
With ideas it is like with dizzy heights you climb: At first they cause you discomfort and you are anxious to get down, distrustful of your own powers; but soon the remoteness of the turmoil of life and the inspiring influence of the altitude calm your blood; your step gets firm and sure and you begin to look - for dizzier heights.

Nikola Tesla
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6.1 Diffusive digraph of networks of 9 neurons. Neurons with different colors have different parameters.

6.2 Interconnection digraph of networks of 9 neurons. Neurons with different colors have different parameters.
List of Symbols

\( \mathbb{R} \) Real numbers
\( \mathbb{C} \) Complex numbers
\( \mathbb{N} \) Natural numbers

\( 1_n \) All-one column vector of length \( n \)
\( 0_n \) All-zero column vector of length \( n \)

\( A_d \) \( A_d = \text{diag}(A_1, A_2, \ldots, A_N) \)
\( B_d \) \( B_d = \text{diag}(b_1, b_2, \ldots, b_N) \)
\( C_d \) \( C_d = \text{diag}(c_1, c_2, \ldots, c_N) \)

\( G \) Interconnection digraph
\( G_c \) Diffusive digraph
\( G_e \) Extended digraph

\( L \) Laplacian matrix of diffusive digraph \( G_c \)

\( \Sigma \) Interconnection matrix

\( \Gamma \) Input-connection matrix

NEP Nontrivial equitable partitioning
NINP Nontrivial identical neighbor partitioning
HNL Heterogeneous network of linear systems
Chapter 1

Introduction

Some of the most exciting scientific and technological challenges of the modern age involve systems that are composed of a large number of interconnected units. A cooperating network of robots, a network of neurons in our brain, and a genetic network in a cell are examples of such systems.

Current research efforts in the control and dynamical systems communities have created a distinct area of research at the intersection of systems theory and graph theory \[1, 2, 3, 4\]. Our goal in this thesis is to study when the network dynamics can (or cannot) be influenced by external inputs (e.g., a decision, stimulus, or control action, according to the application). To quantify this property, we use the notion of controllability: a dynamical system is controllable if, with a suitable choice of inputs, it can be driven from any initial state to any desired final state within finite time.

Networks controllability (as well as other dynamical properties), depend on two independent factors: i) the network architecture, represented by a graph and encapsulating which sub-systems interact and which do not; and ii) the dynamical properties of the components.

In recent years, new technologies led to a considerable progress in identifying the
topological properties of engineering and biological networks, thus providing good qualitative information on factor i) [5], [6]. This leads us to the main question of this thesis, that is, ”how the interconnection topology and the dynamics affects network controllability”. Despite some progress has been made (we will review some of the main results in the next subsection and in Chapter 4), the question for weighted, directed networks, and sub-systems with general linear dynamics is open. It is worth noting that all the results in the literature are focused on single-integrator networks while there is evidence that the controllability properties of networks depend both on the interconnection topology and dynamics [7]. One of the goals of this thesis is to shed some light on this question.

The study of network’s controllability can be of interest in a variety of applications, ranging from Biological to engineering domains.

- Control of neuronal circuits: Abnormal regulation in certain cerebral areas can
lead to pathological disorders such as Parkinson’s disease, epilepsy and akinesia. Deep brain stimulation (DBS) is a symptomatic treatment of several neurological diseases and it consists in an electrical stimulation of deep brain structures through implanted electrodes. The current DBS treatment still suffers from considerable limitations. It is often based on heuristics and experimental deduc- tions. Moreover, little is known about the exact functioning of DBS and the relationship with neuronal synchronization is still not completely understood [8]. This problem inspires a plethora of interesting questions in control theory.

One fundamental question is to identify which neuronal subpopulations should be stimulated in deep brain stimulation treatments. Recasting this problem as a controllability one could, in the future, help to improve current treatments and shed some light on the functioning of DBS. In Chapter 6 we will present an application of the results of this thesis to a simple neuronal network model.

• Control of gene regulatory networks: The major goal of functional genomics is screening genes that determine specific cellular phenotypes (diseases) and to develop therapies based on the disruption or mitigation of aberrant gene function contributing to a certain disease. Mitigation can be accomplished by the use of drugs that act on receptors that enhance/deactivate the gene transcription. Recasting this problem as controllability one could, in the future, help to select drugs that effect the dynamics of the undesired genetic function.

• Control of robotic swarms: Swarm robotics will have an impact in several application areas, including rescue missions, agriculture, localization and monitoring [9]. All these applications involve a set of robots which are communicating through a network to accomplish a task. Suppose that, in order to perform a
specific task, some of the robots (leaders) can be influenced by external commands (e.g., by humans). One problem is to identify how many (and which) leaders must be selected in order to steer the whole network to perform the desired task. This problem can be formulated as a network controllability problem.

- **Control of social networks:** With the rise of the internet and social networks, their neutrality has become a concern. Essentially, in these networks, people are connected to each other. They share information and can affect each other’s behavior. If one would be able to control these networks by influencing a certain number of people, it would be against Net Neutrality. Thus, the goal is to reshape the network’s topology such that it would be uncontrollable, unless a large number of people are influenced.

In this thesis, we will consider networks of single-input single-output (SISO) systems with linear time-invariant dynamics. The network topology is assumed to be
fixed. The problem of identifying the network structures that prevent controllability can, as we will see in the next chapters, be addressed with the tools presented in this thesis. The dynamics, unless stated otherwise, are not assumed to be identical across the network.

1.1 Literature review

Networks and their properties have been studied in many different fields such as biology, mathematics, computer science and electrical engineering. Extensive applications of networks have been reviewed in [1]. Various properties of networks with an especial focus on multi-layer networks have been overviewed in [10]. Recently, in the systems and control community, some attention has been devoted to the study of networks controllability. The relationship between graph properties and controllability dates back to [11], when the concept of structural controllability has been introduced and related to the topological properties of the so called extended graph. In [12] and [13], structural controllability of composite systems has been studied. While structural controllability is a necessary condition (not sufficient) for the controllability, it provides a graph theoretic characterization of controllability using properties of extended digraph. In [2], structural controllability of interconnected systems with interconnection topologies experimentally determined from real systems has been studied. Similarly, in [3] the structural controllability of brain networks was investigated. The same approach was also applied to transcriptional regulatory networks in [5]. In all these approaches, each system is modeled with a single integrator dynamics. In [6], it was suggested that considering the dynamics in a biological network can lead to less restrictive conditions on the controllability.

Recently, controllability of consensus networks of single-integrators [14] has been
a popular topic [15], [16], [17]. The main focus was in providing graph theoretic conditions such that the network is uncontrollable. While the literature on controllability of consensus networks is mostly focused on undirected graphs, [18] extended some of the results to the directed case.

Results in [19] are a first effort to characterize network controllability where the standard assumption of single-integrator dynamics is dropped but the sub-systems are assumed to have identical dynamics. Results in [3], [5] suggest that modeling biological networks as multi-layer networks can improve their understanding and reduces the computational complexity of the calculations involved in determining their structural properties.

All the results mentioned above either assume single integrator dynamics for the subunits or homogeneity of the network (i.e., each subunit has identical dynamics). The goal of this thesis is to investigate the more general case where the network is heterogeneous and the dynamics of each unit is a (general) SISO LTI system.

As we will see further in the thesis, controllability properties of networks of dynamical systems is related to the spectral properties of different notions of interconnection graphs. Characterization of spectral properties of graph matrices have been studied extensively in the past. Next, we will review some key results.

In [20], determinant of the adjacency matrix of a graph is characterized in terms of its matchings. Later, in [21], the characteristic polynomial of graphs was formulated in terms of tree count matchings. As F. Harary noted in [20], there was a conjecture in the beginning that spectra of graphs can be used to discriminate non-isomorphic graphs. Later, this was proved to be false. In [22] and [23], authors introduced methods for constructing non-isomorphic cospectral graphs. But, still there is no method able to find all cospectral, non-isomorphic graphs for a given number of nodes. Expressing the nullity of the adjacency matrix of a graph in terms of its
structural properties has many applications in chemical graph theory and quantum theory \cite{24, 25}. Nullity of graphs is still an open problem. Check \cite{26} for a survey of famous results regarding this problem. In \cite{27} and \cite{28}, authors investigated spectral properties of the Laplacian matrices of graphs in terms of their structural properties. In \cite{29}, algebraic relations between elements of the adjacency matrix and eigenvalue-eigenvector pairs of adjacency matrix have been studied. Sign-patterns and their relation with spectrum of adjacency matrix have been studied in \cite{30}, \cite{31}. In \cite{32}, using matrix pencil theory, authors were able to determine the size of Jordan blocks of a union graph based on its base graphs’ structural properties. There are also many other interesting topics in spectral graph theory which we encourage readers to find more about them in \cite{33}, \cite{34}, \cite{35} and \cite{36}.

1.2 Statement of contribution

The following is a list of original contributions of this thesis.

1. Chapter \texttt{5}

- **Homogenous networks**: Lemma \texttt{2} characterizes the spectrum of homogenous networks. Theorem \texttt{16} characterizes the controllability of homogenous networks based on interconnection topology and systems dynamics. Corollary \texttt{3} provides necessary conditions for controllability of homogenous networks.

- **Heterogeneous networks**: Lemma \texttt{3} provides a characterization of the network’s spectrum. Corollary \texttt{4} provides an alternative way for computing eigenvalues of $A$. Theorem \texttt{17} provides a sufficient and necessary
condition for the controllability of heterogeneous networks. Corollary 5 provides an alternative condition when the sub-systems have the same transfer function. Corollary 6 characterizes controllability for cartesian products of networks. Corollary 7 relates input-symmetry of the extended graph to controllability. Theorem 18 relates the input-connectedness of the extended graph to controllability. Proposition 9 provides a sufficient condition for uncontrollability. Theorem 20 relates uncontrollability and nontrivial identical neighbor partitioning (NINP). Theorem 19 provides a sufficient condition for uncontrollability based on dilation sets of extended digraph. Theorem 21 relates nontrivial equitable partitioning (NEP) of interconnection digraph and uncontrollability. Theorem 23 provides a set of necessary conditions for controllability of heterogeneous diffusively-coupled networks.

1.3 Thesis Outline

This thesis contains seven chapters.

- **Chapter 2** Preliminaries
  
  This chapter contains basic mathematical background, graph theory and linear algebra.

- **Chapter 3** Problem Formulation
  
  We define a heterogeneous network of SISO LTI systems. We define extended digraph properties used for controllability analysis in the next chapters.

- **Chapter 4** The Single Integrator Case and Consensus Networks
  
  In this chapter, we review some famous results regarding the controllability of
networks of single-integrators. We discuss structural controllability of networks and their extended digraph characterization. We define consensus networks. Sufficient conditions for uncontrollability of consensus networks based on diffusive digraph is discussed in this chapter.

• Chapter 5: Network Controllability: Main Results

This chapter contains the main result of this thesis. First, we study homogenous networks. Spectrum and controllability of homogenous networks is characterized in terms of interconnection topology and systems dynamics. Then controllability of heterogeneous networks is characterized. We discuss set of sufficient conditions for uncontrollability of networks in terms of interconnection topology. Finally, a set of necessary graph-theoretic conditions for controllability of consensus networks is discussed.

• Chapter 6: Example

In this chapter, we show the application of results in chapter 5 by studying controllability of model of a neuronal network.

1.4 Notation

Throughout this paper, we will denote matrices by capital letters. Matrix $A$ is denoted by $A = [a_{ij}]$, where $a_{ij}$ denotes the $(i, j)$ element of $A$. We have omitted the dimension, as it is clear from the context. The right null space and the right image of matrix $A$ are denoted by ker($A$) and Img($A$) respectively. The conjugate transpose (transpose) of matrix $A$ is denoted by $A^*$ ($A^T$). The determinant and the adjugate of the square matrix $A$ are denoted by det($A$) and adj($A$), respectively. Let $A$ be a full column (row) rank matrix. Then, $A^+ = (A^*A)^{-1}A^*$ ($A^+ = A^*(A^*A)^{-1}$) denotes the left
(right) pseudo-inverse of $A$. All-one (all-zero) column vector of length $n$ is denoted by $1_n(0_n)$. Column vectors $e_1, e_2, \ldots, e_n$ denote the standard basis vectors for $\mathbb{R}^n$. Let $\text{col}(x_1, x_2, \ldots, x_n)$ denote the column vector comprised of these vectors. The identity matrix of size $n$ is denoted by $I_n$. We denote the Kronecker product of two matrices $A$, $B$ by $A \otimes B$. Let $\{A_1, A_2, \ldots, A_t\}$ be a set of same size square matrices; then $\text{diag}(A_1, A_2, \ldots, A_t)$ denotes the block diagonal matrix comprised of $A_i$. Let $\mathcal{S}$ be a set; We denote its cardinality by $|\mathcal{S}|$. The cartesian product of sets $\mathcal{A}$, $\mathcal{B}$ is denoted by $\mathcal{A} \times \mathcal{B}$. Let $\mathcal{W} \subseteq \mathbb{C}^n$ be a linear subspace. Then $\dim(\mathcal{W})$ denotes the minimum number of linearly independent vectors spanning $\mathcal{W}$ or simply, its geometric dimension. Let $V$ and $W$ be two finite-dimensional vector spaces; Then we will denote the direct sum of $V$ and $W$ by $V \oplus W$. Consider a countable set of elements $\Pi$; We denote the index set of $\Pi$ by $\mathcal{I}_\Pi$. For example, for the node set $\mathcal{V} = \{n_1, n_2, \ldots, n_N\}$ we have $\mathcal{I}_\mathcal{V} = \{1, 2, \ldots, N\}$. Consider $a, b, c \in \mathbb{N}$. $a \equiv b \pmod{c}$ means that $a$, $b$ have the same modul when divided by $c$.

We define the selection matrix $E(\mathcal{I})$ as $E(\mathcal{I}) = \sum_{i \in \mathcal{I}} e_i e_i^T$, where $\mathcal{I}$ is a subset of $\{1, 2, 3, \ldots, n\}$. Consider a set of linear systems $\Pi = \{(A_i, b_i, c_i)\}, \ i = 1, 2, \ldots, N$. Let $\mathcal{N} \subseteq \{1, 2, \ldots, N\}$. Then, $\sigma_c(\mathcal{N})$ denotes the set of common eigenvalues of $A_i$ associated with the linear system $(A_i, b_i, c_i)$, for $i \in \mathcal{N}$. 
Chapter 2

Preliminaries

2.1 Linear algebra

A matrix $A$ with $m$ rows and $n$ columns is denoted by $A = [a_{ij}]$, where $a_{ij}$ is the $(i,j)$ entry of the matrix $A$. We denote the determinant of a square matrix $A$ by $\det(A)$, and its trace by $\text{trace}(A)$. A square matrix $A$ is called symmetric if $A = A^T$. A square matrix $A$ is called self-adjoint if $A = A^*$.

We will make use of the next theorem to compute the determinant of the sum of two matrices.

Theorem 1 (Generalized Sylvester’s Determinant Theorem [37]) Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times m}$ and $X \in \mathbb{R}^{m \times m}$. Suppose $X$ is an invertible matrix. Then, we have

$$\det(X + AB) = \det(X) \det(I_n + BX^{-1}A). \quad (2.1)$$

We denote the maximum eigenvalue of square matrix $A$ by $\lambda_{\text{max}}(A)$. Then for a symmetric matrix $A$ we can find its largest eigenvalue by solving the following optimization problem.
Lemma 1 Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Then,

\[
\lambda_{\text{max}}(A) = \sup_{x \in \mathbb{C}^n, \|x\| = 1} x^T A x.
\]

Square matrix \( A \) is a normal matrix if it commutes with its conjugate transpose, i.e., \( AA^* = A^* A \). Square matrix \( A \) is diagonalizable if and only if it is a normal matrix \[38\].

A nonnegative matrix is a matrix with elements that are equal to or greater than zero.

A matrix with nonnegative off-diagonal elements is called a Metzler matrix.

A square nonnegative matrix with each row (column) summing to 1 is called a right (left) stochastic matrix.

### 2.2 Graph theory

Let \( \mathcal{G} = \{ \mathcal{V}, \mathcal{E} \} \) be a weighted directed graph, or digraph, where \( \mathcal{V} = \{ n_1, n_2, \ldots, n_N \} \) denotes its node set and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) denotes its edge set. A pair \( (n_i, n_j) \in \mathcal{E} \) implies that there is an oriented arrow going from the node \( n_j \) to \( n_i \). Also, we say that the nodes \( n_i \) and \( n_j \) are incident to the edge \( (n_i, n_j) \). The weight of the edge \( (n_i, n_j) \in \mathcal{E} \) is denoted by \( \sigma_{ij} \). If \( (n_i, n_j) \notin \mathcal{E} \) then we set \( \sigma_{ij} = 0 \). We call the matrix \( \Sigma = [\sigma_{ij}]_{i,j=1,2,\ldots,N} \) the interconnection matrix of \( \mathcal{G} \).

An ordered sequence of vertices in \( \mathcal{G} \) is a path if any ordered pair of vertices appearing consecutively is an edge of \( \mathcal{G} \). Digraph \( \mathcal{G} \) is called strongly connected if there is a path between any two distinct vertices of \( \mathcal{G} \).

Digraph \( \mathcal{G}' = \{ \mathcal{V}', \mathcal{E}' \} \) is called a subgraph of \( \mathcal{G} = \{ \mathcal{V}, \mathcal{E} \} \) if \( \mathcal{V}' \subseteq \mathcal{V} \), \( \mathcal{E}' \subseteq \mathcal{E} \cap (\mathcal{V} \times \mathcal{V}) \).

Digraph \( \mathcal{G}' \) is called an induced subgraph of \( \mathcal{G} \) if \( \mathcal{V}' \subseteq \mathcal{V} \), \( \mathcal{E}' = \{(v, w) \in \mathcal{E} \mid v, w \in \mathcal{V}' \} \).
\( \mathcal{V} \) and an edge \((v, w) \in \mathcal{E}'\) have the same weight as \((v, w) \in \mathcal{E}\). We denote the interconnection matrix of \(G'\) by \(\Sigma'\).

Let \(\sigma_i^{\text{out}} = \sum_{j \in I} \sigma_{ji}\) denote the sum of the weights of the outgoing edges of the node \(n_i\), and similarly let \(\sigma_i^{\text{in}} = \sum_{j \in I} \sigma_{ij}\) denote the sum of the weights of the ingoing edges of the node \(n_i\). Digraph \(G\) is called out-regular (in-regular) if \(\sigma_i^{\text{out}} = \sigma_i^{\text{out}} (\sigma_i^{\text{in}} = \sigma_i^{\text{in}})\) for some \(\sigma_i^{\text{out}} \neq 0 (\sigma_i^{\text{in}} \neq 0)\) and all \(n_i \in \mathcal{V}\). If \(G\) is both out-regular and in-regular, it is called a regular digraph. Column vector \(1_n\) is a left (right) eigenvector of \(\Sigma\) associated with nonzero eigenvalue \(\sigma_i^{\text{out}} (\sigma_i^{\text{in}})\), if and only if \(G\) is out-regular (in-regular) \([39]\).

**Definition 1** A digraph \(G = \{\mathcal{V}, \mathcal{E}\}\), \(\mathcal{V} = \{n_1, n_2, \ldots, n_N\}\) which is a path, i.e., \(\mathcal{E} = \bigcup_{i=1}^{N-1} (n_{i+1}, n_i)\), is called a path digraph.

**Definition 2** Let \(G = \{\mathcal{V}, \mathcal{E}\}\), \(\mathcal{V} = \{n_1, n_2, \ldots, n_N\}\) be an oriented path digraph. Digraph \(G' = \{\mathcal{V}', \mathcal{E}'\}\), where \(\mathcal{V}' = \mathcal{V}\) is a cycle digraph if \(\mathcal{E}' = \mathcal{E} \cup (n_1, n_N)\).

**Definition 3** A digraph in which any two nodes are connected by exactly one path is called a tree.

**Definition 4** Consider a digraph \(G\). Tree \(T\) which contains all the nodes of \(G\) and minimum number of edges is called a spanning tree of \(G\).

**Definition 5** A matching is a subset of edges such that no two edges share a common node. A matching \(M\) is a perfect matching if each node of the digraph is incident to one edge of \(M\).

### 2.2.1 Cartesian product of digraphs

Consider digraphs \(G_1 = \{\mathcal{V}_1, \mathcal{E}_1\}\) and \(G_2 = \{\mathcal{V}_2, \mathcal{E}_2\}\). Digraph \(G = \{\mathcal{V}, \mathcal{E}\}\) is a Cartesian product of \(G_1\), \(G_2\) denoted by \(G = G_1 \square G_2\) if the following holds:
Chapter 2. Preliminaries

1. \( \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \),

2. There is an edge from the node \((u_1, u_2)\) to \((v_1, v_2)\) in \( G \) if either \( u_1 = v_1 \) and \((v_2, u_2) \in \mathcal{E}_2\) or \( u_2 = v_2 \) and \((v_1, u_1) \in \mathcal{E}_1\).

Based on the above specifications, if \( G = G_1 \square G_2 \) we have

\[
\Sigma = \Sigma_1 \otimes I_{n_2} + I_{n_1} \otimes \Sigma_2,
\]

where \( n_1 = |\mathcal{V}_1|, \ n_2 = |\mathcal{V}_2| \), and \( \Sigma_1, \Sigma_2 \) and \( \Sigma \) are the interconnection matrices of \( G_1 \), \( G_2 \) and \( G \), respectively.

A digraph which can not be written as a Cartesian product of two digraphs is called a prime digraph.

**Proposition 1 ([40])** Consider digraphs \( G_1 = \{ \mathcal{V}_1, \mathcal{E}_1 \} \), \( G_2 = \{ \mathcal{V}_2, \mathcal{E}_2 \} \) and \( G = \{ \mathcal{V}, \mathcal{E} \} \), where \( G = G_1 \square G_2 \). Let \( \Sigma_1, \Sigma_2 \) and \( \Sigma \) be the interconnection matrices of \( G_1 \), \( G_2 \) and \( G \), respectively. Let \((\omega_1, \lambda_1)\) and \((\lambda_2, \omega_2)\) be left eigenvalue-eigenvector pairs of \( \Sigma_1 \) and \( \Sigma_2 \), respectively. Then \((\lambda, \omega)\) is a left eigenvalue-eigenvector pair of \( \Sigma \), where

\[
\begin{align*}
\begin{cases}
\lambda = \lambda_1 + \lambda_2, \\
\omega = \omega_1 \otimes \omega_2,
\end{cases}
\end{align*}
\]

Proposition 1 characterizes the spectral properties of a cartesian product of two digraphs in terms of the spectral properties of the isolated digraphs.

### 2.2.2 Digraph partitioning

A *Cell* is a subset of nodes of a digraph. Consider a digraph \( G = \{ \mathcal{V}, \mathcal{E} \} \) and let \( \pi = \{C_1, C_2, \ldots, C_l\} \) be a partitioning of \( \mathcal{V} \). Each digraph comes with two trivial
partitioning:

1. The partitioning where all the nodes are contained in one cell,

2. The partitioning where each cell contains only one node.

A partitioning that is not trivial is called a nontrivial partitioning.

Definition 6 Consider a partition \( \pi = \{C_1, C_2, \ldots, C_l\} \) of nodes on a digraph \( \mathcal{G} \). Let \( T \in \mathbb{R}^{V \times |\pi|} \) be a zero-one matrix, where \( t_{ij} = 1 \) if the \( i \)th node is contained in the cell \( C_j \), and \( t_{ij} = 0 \) otherwise. Since \( \pi \) is a partitioning, \( T \) is a full column rank matrix. We call \( T \) the characteristic matrix of \( \pi \). Let \( \mathcal{G}_i \) be the induced subgraph of \( \mathcal{G} \) associated with the cell \( C_i \). The interconnection matrix of \( \mathcal{G}_i \) is denoted by \( \Sigma_i \).

Definition 7 Consider a digraph \( \mathcal{G} = \{V, E\} \) and a partitioning \( \pi = \{C_1, C_2, \ldots, C_l\} \). Let \( \Sigma \) denote the interconnection matrix of \( \mathcal{G} \). We call the partitioning \( \pi \) almost equitable (AEP) if for any distinct \( C_s, C_t \in \pi \), we have

\[
\sum_{j \in C_t} \sigma_{ij} = \sum_{j \in C_t} \sigma_{kj} = \sigma_m(C_t, C_s) \tag{2.4}
\]
for any $i,k \in C_s$. An almost equitable partitioning which is not a trivial partitioning is called a nontrivial almost equitable partitioning (NAEP).

An NAEP in which (2.4) holds for any pair of cells (including equal cells) is called a nontrivial equitable partitioning (NEP).

**Definition 8** Consider a digraph $G$ with an NEP $\pi = \{C_1, C_2, \ldots, C_l\}$. Let $\Sigma^\pi = [\sigma^\pi_{ij}]$ be a square matrix, where

\[ \sigma^\pi_{ij} = \sigma_{in}(C_i, C_j), \]

for $C_i, C_j \in \pi$. Let $V^\pi = \{n^\pi_1, n^\pi_2, \ldots, n^\pi_l\}$, and let

\[ \mathcal{E}^\pi = \{(n^\pi_i, n^\pi_j) \in V^\pi \times V^\pi | \sigma^\pi_{ij} \neq 0\}. \]

The digraph $G^\pi = \{V^\pi, \mathcal{E}^\pi\}$ with the interconnection matrix $\Sigma^\pi$ is called the quotient digraph of $G$ over $\pi$, and we call $\Sigma^\pi$ the reduced interconnection matrix of the digraph $G$ with respect to the partitioning $\pi$.

Figure 2.2a shows a digraph $G$ with the interconnection matrix

\[
\Sigma = \begin{bmatrix}
0 & 1.5 & 0 & 0 \\
1.5 & 0 & 0 & 0 \\
1.0 & 2.0 & 1.2 & 0 \\
0 & 3.0 & 0 & 1.2
\end{bmatrix},
\]

and the partitioning $\pi = \{C_1, C_2\}$, where $C_1 = \{n_1, n_2\}$ and $C_2 = \{n_3, n_4\}$. We can check that $\pi$ is an NEP on the digraph $G$. 
Figure 2.2: Figure 2.2a shows a digraph $G$ with the NEP $\pi = \{C_1, C_2\}$, where $C_1 = \{n_1, n_2\}$ and $C_2 = \{n_3, n_4\}$. The reduced interconnection matrix is $\Sigma^\pi = \begin{bmatrix} 1.5 & 0 \\ 3 & 1.2 \end{bmatrix}$. Figure 2.2b shows the quotient digraph $G^\pi$.

**Proposition 2** Consider a digraph $G$ with an NEP $\pi$. Let $T$ denote the characteristic matrix of $\pi$ and $\Sigma^\pi$ denote the reduced interconnection matrix of $G$ with respect to $\pi$. Then $\Sigma T = T \Sigma^\pi$, where $\Sigma$ denotes the interconnection matrix of $G$.

**Proof:** The proof is a straightforward extension of the proof of Theorem 2 in [18], and therefore it is omitted.

**Definition 9** An NAEP $\pi = \{C_1, C_2, \ldots, C_l\}$ is called a nontrivial identical neighbor partitioning (NINP), if for any $C_i, C_k \in \pi$, $C_i \neq C_k$, the weight of the edges connecting $n_i \in C_i$ to $n_j \in C_k$ are the same.

Set $\sigma^\pi_{ii} = 0$, for $i \in \{1, 2, \ldots, l\}$. The interconnection matrix of the digraph $G$ can be written as:

\[
\Sigma = \text{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_l) + T \Sigma^\pi (T^T)^{-1} T,
\]

where $T$ is the characteristic matrix of NINP $\pi$. 
Figure 2.3: Digraph $\mathcal{G}$ with NINP $\pi = \{C_1, C_2, C_3\}$, where $C_1 = \{n_1, n_2\}$, $C_2 = \{n_3\}$ and $C_3 = \{n_4, n_5, n_6\}$.

Figure 2.3 shows a digraph $\mathcal{G}$ with the interconnection matrix

$$
\Sigma = \begin{bmatrix}
1.3 & 0 & -1.2 & 0 & 0 & 0 \\
0 & 0 & -1.2 & 0 & 0 & 0 \\
2.1 & 2.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.2 & 0 & 2.2 & 0 \\
0 & 0 & 3.2 & 0 & 0 & 0 \\
0 & 0 & 3.2 & 0 & 1.1 & 0
\end{bmatrix},
$$

the reduced interconnection matrix

$$
\Sigma^\pi = \begin{bmatrix}
0 & -1.2 & 0 \\
2.1 & 0 & 3.2 \\
0 & 0 & 0
\end{bmatrix},
$$

and its NINP.
2.3 Linear systems

In this subsection, we recall some technical results that we will need in the following chapters of the thesis.

Consider a linear time-invariant system $\Lambda = (A, B, C)$

\[ \dot{x} = Ax + Bu, \]
\[ y = Cx, \tag{2.6} \]

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$.

Consider (2.6). Given two times $t_1 > t_0 \geq 0$, the controllable on $[t_0, t_1]$ subspace $C[t_0, t_1]$ consists of all states $x_0$ for which there exists an input $u : [t_0, t_1] \mapsto \mathbb{R}^m$ that transfers the state from $x(t_0) = x_0$ to $x(t_1) = 0$.

**Definition 10** Given two times $t_1 > t_0 \geq 0$, the system (2.6) is controllable on $[t_0, t_1]$, if $C[t_0, t_1] = \mathbb{R}^n$, i.e., if the origin can be transferred to every state.

**Theorem 2 (Popov-Belevitch-Hautus (P.B.H) test)** System (2.6) is controllable if and only if the matrix

\[ [A - \lambda I_n | B] \]

is full rank for all $\lambda \in \mathbb{C}$.

We can restate the above theorem as the following corollary.

**Corollary 1** System (2.6) is controllable if and only if for any left eigenvector $w$ of $A$, $\omega^* B \neq 0$.

From now on, every-time we refer to the P.B.H test, we are referring to the above corollary.
Consider (2.6). Given two times $t_1 > t_0 \geq 0$, the unobservable subspace on $[t_0, t_1]$ $UO[t_0, t_1]$ consists of all states $x_0 \in \mathbb{C}^n$ for which

$$C \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} A^k x_0 = 0,$$

for all $t \in [t_0, t_1]$.

**Definition 11** Given two times $t_1 > t_0 \geq 0$, the system (2.6) is observable if its unobservable subspace contains only the zero vector; i.e., $UO[t_0, t_1] = 0$.

Using the duality of controllability and observability, we have the following corollary.

**Corollary 2** System (2.6) is observable if and only if for any right eigenvector $w$ of $A$, $C\omega \neq 0$.

**Definition 12** Let $S = (A, B)$ and $S' = (A', B')$, where $A, A' \in \mathbb{R}^{n \times n}$ and $B, B' \in \mathbb{R}^{n \times m}$. We say that $S$ and $S'$ have the same sign-pattern if the following holds:

1. $a_{ij} \neq 0 \iff a'_{ij} \neq 0$, for $i, j = 1, 2, \ldots, N$,

2. $b_{ij} \neq 0 \iff b'_{ij} \neq 0$, for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, m$.

**Definition 13** Let $S = (A, B)$. The pair $S$ is structurally controllable if for any real $\epsilon > 0$ there exists a controllable pair $S' = (A', B')$ with the same sign pattern as $S$ such that $\|A - A'\| < \epsilon$ and $\|B - B'\| < \epsilon$, where $\|\cdot\|$ denotes a matrix norm.

Notice that if a system is controllable it is also structurally controllable, but the converse is not true in general.
2.4 Topics in spectral graph theory

Spectra of adjacency matrices of graphs have been studied extensively in the literature. In this section, we will review some results regarding the spectra of Laplacian matrices and adjacency matrices of graphs. Interested readers can find the proofs of these results in [26], [27]-[29].

Common assumption in the spectral graph theory is that the graph is undirected and unweighted with no self-loop. As a result, for the rest of this chapter, unless it has been mentioned otherwise, we will assume that the graph is unweighted and undirected containing no self-loops.

Generally speaking, complexity of a graph depends on its number of edges and nodes. Motivated to reduce this complexity, one interesting idea is to study the spectral properties of a graph in terms of its complement.

Complement of graph $G = \{V, E\}$ is $G^c = \{V, E^c\}$, where $(n_i, n_j) \in E$ if and only if $(n_i, n_j) \notin E$ for $n_i, n_j \in V$. Figure 2.4b shows a graph on 4 nodes and Figure 2.4b shows its complement. Let $L$ denote the Laplacian matrix of $G$ and let $L'$ be the Laplacian matrix of $G^c$. We can observe that $L + L' = nI_n - n_1 n_1^T$, where $n = |V|$. 

Figure 2.4: Figure 2.4a shows graph $G$ with 4 nodes and Figure 2.4b shows complement of graph $G$. 

(a) graph $G$ on 4 nodes. 

(b) Complement of graph $G$. 

2.4 Topics in spectral graph theory
Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L$, and $\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_n$ be the eigenvalues of $L'$. Then we have:

$$\lambda'_{n-i} = n - \lambda_i,$$

for $i = 1, 2, \ldots, n - 1$. First conclusion is that $L'$ has only one eigenvalue $\lambda'_n = n$ if and only if $G$ is strongly connected.

Moreover, $n_i$ is a nontrivial (left or right) eigenvector of $L$ affording eigenvalue $\lambda_i$ if and only if it is a nontrivial eigenvector of $L'$ affording eigenvalue $n - \lambda_i$.

Another interesting topic in spectral graph theory is the spectrum of the unions of graphs. The following theorem states one of the most basic results.

**Theorem 3** Consider two graphs $G_1 = \{V_1, E_1\}$, $G_2 = \{V_2, E_2\}$ such that $V_1 \cap V_2 = \emptyset$.

Let $L_1$ and $L_2$ be the Laplacian matrices of $G_1$ and $G_2$, respectively. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues of $L_1$, where $r = |V_1|$ and $\mu_1, \mu_2, \ldots, \mu_s$ be the eigenvalues of $L_2$, where $s = |V_2|$.

Then the eigenvalues of of the Laplacian matrix of $G_1 \cup G_2$ are $\mu_1 + r, \mu_2 + r, \ldots, \mu_s + r, 0, \lambda_1 + s, \lambda_2 + s, \ldots, \lambda_r + s$.

The following theorem states an insightful result about the eigenvectors of $L$.

**Theorem 4** Let $G = \{V, E\}$ be a graph on $n$ nodes with the Laplacian matrix $L$.

Let $(\lambda, \nu)$ be an eigenvalue-eigenvector pair of $L$ such that $0 < \lambda < n$. Then the $i$th element of $\nu$ is zero if the node $n_i$ has degree $n - 1$.

The following theorem considers the case when adding or deleting an edge does not affect some eigenvalue-eigenvector pairs of $L$. 
Theorem 5 Consider a graph $G = \{V, E\}$ with the Laplacian matrix $L$. Let $(\lambda, \nu)$ be an eigenvalue-eigenvector pair of $L$. Suppose $\nu_i = \nu_j$. Then $(\lambda, \nu)$ is an eigenvalue-eigenvector pair of $L'$, the Laplacian matrix of graph $G' = \{V, E'\}$, where

1. if $(n_i, n_j) \notin E$ then $E' = E \cup (n_i, n_j)$,
2. if $(n_i, n_j) \in E$ then $E' = E \setminus (n_i, n_j)$.

The following theorem relates the spectra of Laplacian matrix and adjacency matrix of regular graphs.

Theorem 6 Consider a graph $G$ with the adjacency matrix $A \in \mathbb{R}^{n \times n}$ and Laplacian matrix $L \in \mathbb{R}^{n \times n}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L$ and $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ be the eigenvalues of $A$. Suppose $G$ is a $r$-regular graph. Then $\lambda_i = r - \sigma_i$, for $i = 1, 2, \ldots, n$.

Proof: For a $r$-regular digraph we have $L = rI - A$. By the definition of eigenvalues and characteristic matrix, the proof can be deduced.

Next, we will characterize the spectra of circulant matrices. Later, this will help us in the controllability analysis of cycle and path graphs.

2.4.1 Circulant Matrices

Circulant matrix is matrix with the form:

$$M = \begin{bmatrix}
  c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\
  c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
  c_2 & c_1 & c_0 & \cdots & c_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0
\end{bmatrix} \quad (2.8)$$
In other words, each column (row) is the same as the previous one but the elements are shifted one position down (right) and wrapped around at the bottom of the column (beginning of the row). The following theorem provides an algebraic characterization of spectra of circulant matrices.

**Theorem 7** Consider the circulant matrix \((2.8)\). For \(j \in \{0, 1, \ldots, n - 1\}\) let

\[
\begin{align*}
\nu_j &= \frac{1}{\sqrt{n}} \text{col}(1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1}) \\
\lambda_j &= c_0 + c_{n-1} \omega_j + c_{n-2} \omega_j^2 + \cdots + c_1 \omega_j^{n-1},
\end{align*}
\]

where \(\omega_j = \exp\left(\frac{2\pi j}{n}\right)\) is the \(n\)th root of unity and \(i\) is the imaginary unit.

Then \((\lambda_j, \nu_j)\) is a right eigenvalue-eigenvector of \((2.8)\) for \(j \in \{0, 1, \ldots, n - 1\}\).

Path and cycle graphs have circulant adjacency matrices and the above theorem can be used for characterization of their spectrum.
Chapter 3

Problem Formulation

In this chapter, we will define the general class that we will investigate in this thesis. We will associate to the model a graph structure by using the notion of extended digraph. This notion will be used throughout the thesis to provide a graph theoretic connotation of the controllability properties of the network. We will also present and prove some preliminary results that will be instrumental in the forthcoming chapters.

3.1 Network Model

Consider a set of single-input single-output (SISO) linear time-invariant systems

\[
\dot{x}_i = A_i x_i + b_i u_i, \\
y_i = c_i x_i,
\]

(3.1)

where \( x_i \in \mathbb{R}^n, u_i \in \mathbb{R}, y_i \in \mathbb{R}, i = 1, 2, \ldots, N \) and

\[
u_i = \sum_{j=1}^{N} \sigma_{ij} y_j + \sum_{j=1}^{m} \gamma_{ij} v_j,
\]

(3.2)
where \( \Sigma = [\sigma_{ij}]_{i,j=1,2,\ldots,N} \in \mathbb{R}^{N \times N} \), \( \Gamma = [\gamma_{ij}]_{i=1,2,\ldots,N, j=1,2,\ldots,m} \in \mathbb{R}^{N \times m} \), and \( v = \text{col}(v_1, v_2, \ldots, v_m) \in \mathbb{R}^m \). We call \( \Sigma \) the \textit{interconnection matrix} while \( \Gamma \) is called the \textit{input-connection matrix}. Now, we assign to (3.1), (3.2) a digraph structure. We pair each linear system \( i \) in (3.1) with a node \( n_i \) and we define a digraph \( G = \{V, E\} \), where \( V = \{n_1, n_2, \ldots, n_N\} \) and \( E \subseteq V \times V \) such that \((n_i, n_j) \in E\) if and only if \( \sigma_{ij} \neq 0 \). We will call the digraph \( G \) defined above the \textit{interconnection digraph} associated with \( \Sigma \).

We associate the (scalar) exogenous input \( v_i \) for \( i = 1, 2, \ldots, m \), to a node \( o_i \), that we call an \textit{origin} \[1\]. Let \( V_e = \{n_1, n_2, \ldots, n_N, o_1, o_2, \ldots, o_m\} \) and \( E_e \subseteq V \times V_e \) such that \((n_i, n_j) \in E_e\) if and only if \((n_i, n_j) \in E\) and \((n_i, o_j) \in E_e\) if and only if \( \gamma_{ij} \neq 0 \). We call \( G_e = \{V_e, E_e\} \) the \textit{extended digraph} associated with the pair \((\Sigma, \Gamma)\). An example of extended digraphs with two exogenous inputs is shown in Fig. 3.1, where

\[
\Sigma = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}.
\]

We now rewrite (3.1), (3.2) in a compact form. By defining the column vector \( x = \text{col}(x_1, x_2, \ldots, x_N) \), we can rewrite (3.1), (3.2) as

\[
\dot{x} = Ax + Bv, \quad (3.3)
\]

where \( B = B_d \Gamma \) and

\[
A = A_d + B_d \Sigma C_d, \quad (3.4)
\]

\( A_d = \text{diag}(A_1, A_2, \ldots, A_N) \), \( B_d = \text{diag}(b_1, b_2, \ldots, b_N) \) and \( C_d = \text{diag}(c_1, c_2, \ldots, c_N) \).

We call (3.3) a \textit{heterogeneous network of linear systems} (HNL).
Definition 14 Given two times $t_1 > t_0 \geq 0$, the network (3.3) is controllable on $[t_0, t_1]$, if $\forall x_0 \in \mathbb{C}^{nN}$, $x(t_0) = x_0$, there exists an input $u : [t_0, t_1] \mapsto \mathbb{R}^m$ such that $x(t_1) = 0$.

The goal of this thesis is to study the controllability of (3.3) in terms of the dynamics of the isolated systems and the interconnection structure. We will now state some definitions and preliminary results that will be useful in the following chapters.

### 3.2 Extended digraph properties

Definition 15 Consider an extended digraph $G_e = \{V_e, E_e\}$. The extended digraph $G_e$ is called input-connected if for every node $n_i \in V$, there exists an origin $o_j \in V_e$, such that there exists a path from $o_j$ to $n_i$ following the edges of $G_e$.

Proposition 3 (Π) Consider an extended digraph $G_e = \{V_e, E_e\}$ associated with the pair $(\Sigma, \Gamma)$. $G_e$ is input-connected if and only if there is no permutation matrix $T$
such that

\[ T^T \Sigma T = \begin{bmatrix} \Sigma_1 & 0_l \\ \Sigma_2 & \Sigma_3 \end{bmatrix}, \quad T^T \Gamma = \begin{bmatrix} 0_l \\ \Gamma_1 \end{bmatrix}, \]

for some \( l \in \mathbb{N} \).

**Definition 16** Let \( S \) be a subset of \( \mathcal{V} \). Let \( T(S) \) denote a subset of \( \mathcal{V}_e \) such that:

1. An origin \( o_i \) is in \( T(S) \) if there is an edge from \( o_i \) to a node in \( S \),
2. A node \( n_i \) is in \( T(S) \) if there is an edge from \( n_i \) to a node in \( S \).

If \( |S| > |T(S)| \), then \( S \) is called the dilation set of \( \mathcal{G}_e \), and \( \mathcal{G}_e \) is said to be dilated.

**Proposition 4** [11] Consider an extended digraph \( \mathcal{G}_e = (\mathcal{V}_e, \mathcal{E}_e) \) associated with the pair \((\Sigma, \Gamma)\). If \( \mathcal{G}_e \) is dilated with dilation set \( S \), then \([\Sigma|\Gamma]^{*}\) is a singular matrix and

\[ \text{dim}(\ker([\Sigma|\Gamma]^{*})) \leq |S| - |T(S)|. \]

Figure 3.2 shows a dilated extended digraph, where

\[ \Sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]

If we choose \( S = \{n_1, n_2, n_4\} \), then \( T(S) = \{n_3, o\} \). As a result, we have \( |S| > |T(S)| \), which means the extended digraph is dilated.

**Definition 17** The extended graph \( \mathcal{G}_e = (\mathcal{V}_e, \mathcal{E}_e) \) with the interconnection digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is called a stem, if \( \mathcal{G} \) is a path digraph and \( \mathcal{E}_e = \mathcal{E} \cup (n_1, o) \).
Chapter 3. Problem Formulation

Figure 3.2: Dilated extended digraph $G_e$. While $G_e$ is input-connected, we can check that for $S = \{n_1, n_2, n_4\}$, we have $T(S) = \{n_3, o\}$. Thus, $|S| > |T(S)|$, and $G_e$ is dilated.

**Definition 18** Consider a cycle $G_c = \{V_c, E_c\}$. The extended graph $G_e = \{V_e, E_e\}$ is called a bud with the source vertex $n_i$, if $V_e = V_c \cup \{o\}$ and $E_e = E_c \cup P$, where the pair $P = (n_j, n_i) \in V_c \times V_e$.

Figures 3.3a, 3.3b show two examples of bud extended digraphs. Figure 3.3a shows a bud with the origin as its source, while in Figure 3.3b we have a bud with the source $n_4$.

**Definition 19** Consider an extended digraph $G_e$ with a single origin. $G_e$ is a cactus if it can be written as $G_e = S \cup B_1 \cup B_2 \cup \ldots \cup B_p$, where $S$ is a stem and $B_i$ denotes a bud for $i = 1, 2, \ldots, p$. We say that an extended digraph $G_e$ is spanned by cacti, if by removing none or some of its edges, it becomes a cactus.

Figure 3.4 shows a cactus.

**Definition 20** Consider an extended digraph $G_e = \{V_e, E_e\}$. We call $G_e$ input-symmetric if there exists a nontrivial permutation $T$ such that $\Sigma T = T \Sigma$ and $T \Gamma = \Gamma$. 
Figure 3.3: The blue circles denote the nodes and the green diamond denotes the origin. Figure 3.3a shows an extended digraph where $\Sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, while in Figure 3.3b we have $\Sigma = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Figure 3.4: Extended digraph $G_e$ which is a cactus.
Figure 3.5: If change the labels of $n_1$ and $n_4$ in $G_e$, we get $G'_e$. As a result, $G_e$ and $G'_e$ are input-symmetric.

Figure 3.5a shows an example of input-symmetric extended digraph.

The following result provides a sufficient condition for uncontrollability of the pair $(\Sigma, \Gamma)$ in terms of a topological property of $G_e$.

**Proposition 5** Consider the extended digraph $G_e = \{V_e, E_e\}$ associated with $(\Sigma, \Gamma)$. Suppose $G_e$ is input-symmetric, then $(\Sigma, \Gamma)$ is uncontrollable.

**Proof:** Let $(\lambda, \omega)$ be an arbitrary left eigenvalue-eigenvector pair of $\Sigma$. Then

$$\omega^* \Sigma = \lambda \omega^*. \quad (3.5)$$

Based on Definition 20 there is a nontrivial permutation matrix $T$ such that $\Sigma T = T \Sigma$ and $T \Gamma = \Gamma$. If we multiply both sides of (3.5) by $T$, we have

$$\omega^* \Sigma T = \lambda \omega^* T.$$ 

By using $\Sigma T = T \Sigma$ the above equation becomes

$$\omega^* T \Sigma = \lambda \omega^* T.$$
We conclude that $(\lambda, T^*\omega)$ is a left eigenvalue-eigenvector pair of $\Sigma$. Define $\bar{\omega}^* = \omega^* - \omega^*T$. One can easily check that $(\lambda, \bar{\omega})$ is also a left eigenvalue-eigenvector pair of $\Sigma$. Consider $\bar{\omega}^*\Gamma$. We have

$$\bar{\omega}^*\Gamma = \omega^*\Gamma - \omega^*T\Gamma.$$  

Since $TT = \Gamma$, then we have

$$\bar{\omega}^*\Gamma = \omega^*\Gamma - \omega^*\Gamma = 0.$$  

By P.B.H test (Corollary 1), we conclude that $(\Sigma, \Gamma)$ is uncontrollable. 

This proposition proves a more general result than Proposition 5.8 proved in [15], as we are not assuming that the interconnection digraph is diffusive.

**Proposition 6** Consider the extended digraphs $G_{e_1} = \{V_{e_1}, E_{e_1}\}$ and $G_{e_2} = \{V_{e_2}, E_{e_2}\}$ associated with the pairs $(\Sigma_1, \Gamma_1)$ and $(\Sigma_2, \Gamma_2)$, respectively. Let $G_1$ and $G_2$ denote the respective interconnection digraphs.

Suppose

1. $G = G_1 \square G_2,$

2. $\Gamma = \Gamma_1 \otimes \Gamma_2.$

Then the pair $(\Sigma, \Gamma)$ associated with the extended digraph $G_e$ is controllable if and only if $(\Sigma_1, \Gamma_1)$ and $(\Sigma_2, \Gamma_2)$ are controllable.

**Proof:** According to the P.B.H test (Corollary 1), $(\Sigma, \Gamma)$ is controllable if and only if $\Sigma$ has no left eigenvector $\omega \in \mathbb{R}^n$ such that $\omega^*\Gamma = 0$. From Lemma 1 we obtain
\((\omega_1 \otimes \omega_2)^*(\Gamma_1 \otimes \Gamma_2) \neq 0\), where \(\omega_1, \omega_2\) are left eigenvectors of \(\Sigma_1\) and \(\Sigma_2\) respectively.

We conclude that \(\omega_1^*\Gamma_1 \neq 0\) and \(\omega_2^*\Gamma_2 \neq 0\), which is equivalent to controllability of \((\Sigma_1, \Gamma_1)\) and \((\Sigma_2, \Gamma_2)\).
Chapter 4

Networks of Single Integrators:
Main Results from the Literature

In this chapter we will consider the special case where (3.1) are single-integrators. Since this special case has been considered in the literature, we will report here the main results pertaining our work. As a particular case, we will consider the case where the interconnection digraph is diffusive. Networks of diffusively-coupled single-integrators are commonly called consensus networks, and have been a popular subject in recent years.

4.1 Networks of Single Integrators

The special case of (3.1) where each system is a single integrator

\[
\begin{align*}
\dot{x}_i &= u_i, \\
y_i &= x_i
\end{align*}
\]

(4.1)
for $i = 1, 2, \ldots, N$, where

$$ u_i = \sum_{j=1}^{N} \sigma_{ij} y_j + \gamma_i v, \quad (4.2) $$

where $v \in \mathbb{R}$ denotes the exogenous input and $\gamma_i$ and $\sigma_{ij}$ for $i, j = 1, 2, \ldots, N$ are real constants. We can rewrite (4.1) with (4.2) in a vector form as:

$$ \dot{x} = \Sigma x + \gamma v, \quad (4.3) $$

where $x = \text{col}(x_1, x_2, \ldots, x_N)$, $\gamma = \text{col}(\gamma_1, \gamma_2, \ldots, \gamma_N)$ and $\Sigma = [\sigma_{ij}]$ for $i, j \in \{1, 2, \ldots, N\}$.

Notice that, since each isolated system is a single-integrator, the dynamics is governed exclusively by the extended digraph $G_e$ and its associated weights.

### 4.2 Structural Controllability of Networks

In this section, we will characterize the structural controllability of (4.3) in terms of its extended digraph’s properties.

**Theorem 8** ([11]) Consider (4.3) and the associated extended digraph $G_e$. Network (4.3) is structurally controllable if and only if $G_e$ is input-connected and not dilated.

The above theorem states that the structural controllability of a network is equivalent to the extended digraph being input-connected and not dilated. While the notion of input-connectedness is very intuitive, the notion of dilation is less intuitive. One may interpret dilation as a bottleneck in the information flow in the extended digraph.
Chapter 4. Networks of Single Integrators: Main Results from the Literature

For example, in Figure 3.2, the nodes in set $S = \{n_1, n_4, n_2\}$ receive information only from $\{n_3, o\}$, which can be visualized as a bottleneck.

Next theorem states another graph theoretical characterization for the structural controllability.

**Theorem 9 ([11])** Network (4.3) is structurally controllable if and only if $G_e$ is spanned by cacti.

The above theorem implies that given a number of single-integrators (4.1), the smallest interconnection topology that makes (4.3) structurally controllable is a cactus.

The generalization of the above theorem to networks with more than one exogenous inputs can be found in [2].

**Theorem 10 ([2])** Consider a structurally controllable network (4.3). Then the interconnection digraph $G$ has a perfect matching.

The generalization of the above theorem to the case where there is more than one exogenous input is based on the size of maximum matching. See [2] for more details.

4.3 Consensus Networks

In this section, we will review the main results on the controllability of consensus networks.

Suppose (4.2) can be written as

$$u_i = \sum_{j=1}^{N} a_{ij}(y_j - y_i) + b_i v, \quad (4.4)$$
where $a_{ii} = 0, \ i = 1, 2, \ldots, N$.

We can rewrite (4.1) and (4.4) in a compact form as

$$\dot{x} = -Lx + bv,$$

(4.5)

where $b = \text{col}(b_1, b_2, \ldots, b_N)$, $x = \text{col}(x_1, x_2, \ldots, x_N)$, $L = D - A$, $D = \text{diag}(A_{1_n})$. Let $G_c = \{V_c, E_c\}$, where $V_c = \{n_1, n_2, \ldots, n_N\}$ and $E_c = \{(n_i, n_j)|a_{ij} \neq 0\}$. We call $G_c$ the diffusive digraph and we call $L$ the Laplacian matrix of $G_c$.

Equation (4.5) with the state feedback (4.4) represent a consensus network. The reason for this name is that, when $v = 0$, under suitable assumptions on the coefficients $a_{ij}$, the outputs converge to each other, thereby reaching consensus.

**Definition 21** Consider (4.5) with the coupling digraph $G_c$.

(4.5) asymptotically synchronizes if for any initial state $x_0 = \text{col}(x_1(0), x_2(0), \ldots, x_N(0))$, we have

$$\lim_{t \to \infty} \|y_i(t) - y_j(t)\| = 0,$$

(4.6)

for all $i, j \in \{1, 2, \ldots, N\}$.

**Theorem 11 ([41])** Network (4.5) asymptotically synchronizes if and only if $G_c$ contains a spanning tree.

**Theorem 12 ([15])** If there exists a nontrivial permutation matrix $P$ such that $PA = AP$ and $P^Tb = b$, where $A$ denotes the adjacency matrix of $G_c$, then (4.5) is uncontrollable.

The next theorem provides a connection between the controllability of (4.5) and equitable partitionings of $G_c$. 
Theorem 13 ([18]) Consider (4.5). Suppose $G_c$ has an NEP $\pi = \{C_1, C_2, \ldots, C_l\}$ such that for any cell $C_i \in \pi$ and $n_j, n_k \in C_i$ we have $b_j = b_k$. Then (4.5) is uncontrollable and
\[
\dim(W_{uc}) \geq \sum_{C_i \in \pi} (|C_i| - 1),
\]
where $W_{uc}$ denotes the uncontrollable subspace of (4.5).

Theorems 12 and 13 state that after a nontrivial relabeling of nodes of $G_d$ if the pair $(L, b)$ remains unchanged, then (4.5) is not controllable.

Input-symmetry and equitable partitioning are only sufficient conditions for uncontrollability of (4.4), as shown in [16].

The following theorem provides a necessary and sufficient condition for the structural controllability of consensus networks.

Theorem 14 ([42]) The network (4.5) is structurally controllable if and only if $G_c$ is strongly-connected.

4.4 Results for graphs with special structure

By applying the PBH test to (4.5), we can characterize controllability from the spectral properties of the Laplacian matrix $L$ which depends on $G_c$. Next, we will discuss controllability of (4.5) when $G_c$ is a path or cycle digraph.

4.4.1 Path and Cycle Graphs

Proposition 7 ([29]) Consider a path graph on $n$ vertices with adjacency matrix $A$ and Laplacian matrix $L$. The pairs $(\lambda_i, \nu_i)$ and $(\epsilon_i, \nu_i)$ are eigenvalue-eigenvector
pairs of $A$ and $L$, respectively, where

$$
\begin{align*}
\lambda_i &= 2 \cos \left( \frac{\pi j}{n+1} \right) \\
\epsilon_i &= 2 + 2 \cos \left( \frac{\pi j}{n+1} \right) \\
\nu_i &= \text{col} \left( \sin \left( \frac{\pi j}{n+1} \right), \sin \left( \frac{2\pi j}{n+1} \right), \ldots, \sin \left( \frac{n\pi j}{n+1} \right) \right)
\end{align*}
$$

(4.7)

for $i = 1, 2, 3, \ldots, n$.

From the above proposition, we can conclude that Laplacian matrices of path graphs do not have repeated eigenvalues. Thus, we need one exogenous input to make (4.5) controllable.

**Theorem 15** ([29]) Consider a consensus network (4.5) with diffusive coupling graph $G$. Let $G$ be a path graph with $n$ vertices. Let $P = \{p_1, p_2, \ldots, p_k\}$ be a set containing all the prime odd numbers dividing $n$.

Suppose the exogenous input is only connected to the $i$th single-integrator, i.e., $b = e_i$, $1 \leq i \leq N$. Then (4.5) is controllable if and only if

$$n - i \mod p \neq i - 1,$$
for any $p \in P$.

The following result characterizes the eigenvalues and eigenvectors of cycle graphs.

**Proposition 8** ([36]) Consider a cycle graph $G$ with $n$ vertices. Let $A$ and $L$ denote the adjacency and Laplacian matrices of $G$, respectively.

Vectors $v_{2j-1}$, $v_{2j}$ are right eigenvectors of $A$ associated with eigenvalue $\lambda_j$, where

$$
\begin{align*}
\lambda_j &= 2 \cos\left(\frac{2\pi j}{n}\right) \\
v_{2j} &= \text{col}(\sin\left(\frac{2\pi j}{n}\right), \sin\left(\frac{4\pi j}{n}\right), \ldots, \sin\left(\frac{2n\pi j}{n}\right)) \\
v_{2j-1} &= \text{col}(\cos\left(\frac{2\pi j}{n}\right), \cos\left(\frac{4\pi j}{n}\right), \ldots, \cos\left(\frac{2n\pi j}{n}\right))
\end{align*}
$$

for $j = 1, 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Similarly, $(2 - \lambda_j, v_{2j-1})$, $(2 - \lambda_j, v_{2j})$ are right eigenvalue-eigenvector pairs of $L$, where $\lambda_j$, $v_{2j}$, $v_{2j-1}$ are as defined in (4.8), for $j = 1, 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

The above proposition tells us that each eigenvalue of Laplacian and adjacency matrices of cycle graphs have multiplicity 2. Thus, we need at least two exogenous inputs to control the network.
Chapter 5

Network Controllability: Main Results

This chapter studies the controllability of heterogeneous and homogenous networks. The main result is a necessary and sufficient condition for controllability. This result is used to derive necessary conditions for controllability based on the topological properties of the extended digraph. We will briefly comment on how the results of this chapter generalize previous work.

5.1 Homogenous Networks

This section assumes that the case when the systems \( (3.1) \) are identical.

Consider the network \( (3.1), (3.2) \), where \( (A_i, b_i, c_i) = (A_0, b_0, c_0) \) for \( i = 1, 2, \ldots, N \). Accordingly, the vector representation of the network \( (3.3) \) can be written in compact form as

\[
\dot{x} = Ax + Bv,
\] (5.1)
where \( B = \Gamma \otimes b_0 \), and

\[
A = I_N \otimes A_0 + \Sigma \otimes b_0 c_0. \tag{5.2}
\]

\( \Sigma \) and \( \Gamma \) are the interconnection and input-connection matrices, respectively (section 3.1). We call (5.1) a homogenous network of linear systems.

The following lemma is essential for some of the results of this chapter.

**Lemma 2** Consider (5.1). Let \((\epsilon, \omega)\) be a left eigenvalue-eigenvector pair of \( \Sigma \) and \((\lambda, \nu)\) be a left eigenvalue-eigenvector pair of \( A_0 + \epsilon b_0 c_0 \). Then \((\lambda, \bar{\omega})\) is a left eigenvalue-eigenvector pair of \( A \), where \( \bar{\omega} = \omega \otimes \nu \).

**Proof:** Let \( \bar{\omega} = \omega \otimes \nu \), where \( \omega, \nu \) and \( \lambda \) are defined in the statement of the Theorem. From the definition of \( A \)

\[
\bar{\omega}^* A = (\omega \otimes \nu)^* (I_N \otimes A_0 + \Sigma \otimes b_0 c_0). 
\]

Using the properties of the kronoccker product, we can expand the above equation as

\[
\bar{\omega}^* A = \omega^* \otimes (\nu^* A_0) + (\omega^* \Sigma) \otimes (\nu^* b_0 c_0).
\]

Since \((\epsilon, \omega)\) is a left eigenvalue-eigenvector pair of \( \Sigma \), we have

\[
\bar{\omega}^* A = \omega^* \otimes [\nu^*(A_0 + \epsilon b_0 c_0)].
\]

Again, since \((\lambda, \nu)\) is a left eigenvalue-eigenvector pair of \( A_0 + \epsilon b_0 c_0 \), we conclude

\[
\bar{\omega}^* A = \lambda \bar{\omega}^*,
\]

proving that \((\lambda, \bar{\omega})\) is a left eigenvalue-eigenvector pair of \( A \).
5.1.1 Controllability Conditions

In Lemma 2, we characterized the eigenvalue-eigenvectors of $A$. Now, using Lemma 2 and P.B.H test (Corollary 1), we will characterize the controllability of homogenous networks (5.1).

**Theorem 16** The homogenous network (5.1) is controllable if and only if $(A_0, b_0)$ and $(\Sigma, \Gamma)$ are controllable pairs.

**Proof:** Let $(\lambda, \omega)$ be a left eigenvalue-eigenvector pair of $A$. From Lemma 2, we know that $\bar{\omega} = \omega \otimes \nu$, where $(\epsilon, \omega)$ and $(\lambda, \nu)$ are left eigenvalue-eigenvector pairs of $\Sigma$ and $A_0 + \epsilon b_0 c_0$, respectively. Since (5.1) is controllable, by P.B.H test (Corollary 1) we conclude that $\bar{\omega}^* B \neq 0$,

which can be written as

$$(\omega \otimes \nu)^*(\Gamma \otimes b_0) \neq 0.$$  

The above equation can be expanded as

$$(\omega^* \Gamma) \otimes (\nu^* b_0) \neq 0. \quad (5.3)$$

Equation (5.3) implies that $\omega^* \Gamma \neq 0$ and $\nu^* b_0 \neq 0$. Since $\omega$ is a left eigenvector of $\Sigma$ and $\omega^* \Gamma \neq 0$, from the P.B.H test, we conclude that $(\Sigma, \Gamma)$ is controllable. Similarly, $\nu^* b_0 \neq 0$ implies that $S = (A_0 + \epsilon b_0 c_0, b_0)$ is controllable. We know that $S$
is controllable if and only if \((A_0, b_0)\) is controllable. Thus, we conclude that \((A_0, b_0)\) is controllable. All the previous steps are reversible and the proof is complete.

Theorem 16 is a generalization of the previous results on controllability of consensus networks presented in chapter 4. Theorem 16 says that a homogenous network (5.1) is controllable if and only if its components are controllable and \((\Sigma, \Gamma)\) is a controllable pair. As we will see in the upcoming chapters, this condition is neither necessary nor sufficient when we drop the homogeneity assumption.

The following corollary links controllability of the homogeneous network (5.1), and properties of the extended digraph \(G_e = \{V_e, E_e\}\).

**Corollary 3** Consider a controllable homogenous network (5.1). Then the extended digraph \(G_e\) is input-connected and contains no dilation set.

**Proof:** Since (5.1) is controllable, based on Theorem 16 we conclude that \((\Sigma, \Gamma)\) is a controllable pair. Based on the definition of structural controllability, if \((\Sigma, \Gamma)\) is controllable, then it is structurally controllable and from Theorem 8 we conclude the extended digraph \(G_e\) is input-connected and contains no dilation set. This ends the proof.

**Remark 1** Theorem 16 generalizes previous results on networks of single-integrators ([15], [18]) to networks of general SISO LTI systems.

In the next section, we will drop the homogeneity assumption.


5.2 The Heterogeneous case

As in the previous chapter, our first step is to study the spectral properties.

Lemma 3 Consider (3.3) and assume that \((A_i, b_i)\) and \((A_i, c_i)\) are controllable and observable, respectively, for \(i = 1, 2, \ldots, N\). Let \(f_i(s) = \frac{z_i(s)}{p_i(s)}\) denote the transfer function of the \(i\)th sub-system, and let \(\zeta_i(s) = c_i \text{adj}(sI - A_i)\), for \(i = 1, 2, \ldots, N\). Let

\[
\Psi(s) = \zeta(s)^+ \left[ P(s) - Z(s)\Sigma \right],
\]

where \(Z(s) = \text{diag}(z_1(s), z_2(s), \ldots, z_N(s))\), \(P(s) = \text{diag}(p_1(s), p_2(s), \ldots, p_N(s))\), \(\zeta(s) = \text{diag}(\zeta_1(s), \zeta_2(s), \ldots, \zeta_N(s))\) and \(\zeta(s)^+\) denotes the right pseudo-inverse of \(\zeta(s)\). The pair \((s_0, \omega)\) is a left eigenvalue-eigenvector pair of \(A\) if and only if

\[
\omega^* \Psi(s_0) = 0, \ \omega \neq 0.
\]

Proof: First, we will show that \(\zeta(s)\) is full row rank. Thus, we have to show \(\zeta_i(s) \neq 0, \ \forall s \in \mathbb{C}, \ \forall i = 1, 2, \ldots, N\).

Let \(i \in \{1, 2, \ldots, N\}\). By assumption, the pair \((A_i, c_i)\) is observable. Thus, there exists a coordinate transformation \(T \in \mathbb{C}^{n \times n}, \ \det(T) \neq 0\) such that

\[
TA_iT^{-1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & -a_n \\
1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -a_1
\end{bmatrix}, \ c_iT^{-1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 1
\end{bmatrix}, \ \ (5.5)
\]

where

\[
p_i(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n.
\]
Consider $\zeta_i(s) = c_i \text{adj}(sI - A_i)$. As $\det(T) \det(T^{-1}) = 1$, we have

$$\zeta_i(s) = c_i \text{adj}(sI - A_i) \det(T) \det(T^{-1}).$$

Since $T^{-1} \text{adj}(T^{-1}) = \det(T^{-1})I_n$ and $T \text{adj}(T) = \det(T)I_n$, then

$$\zeta_i(s) = c_i T^{-1} \text{adj}(T^{-1}) \text{adj}(sI - A_i) \det(T)T.$$

Based on the properties of adjoint matrix,

$$\text{adj}(T^{-1}) \text{adj}(sI - A_i) \det(T) = \text{adj}(sI - TA_iT^{-1}).$$

In conclusion, we have $\zeta_i(s) = c_i T^{-1} \text{adj}(sI - TA_iT^{-1})T$. By considering (5.5) and computing the last row of $\text{adj}(sI - TA_iT^{-1})$, we have

$$\zeta_i(s) = c_i T^{-1} \text{adj}(sI - TA_iT^{-1})T = \begin{bmatrix} 1 & s & s^2 & \ldots & s^{n-1} \end{bmatrix} T.$$

Thus, $\zeta_i(s) \neq 0$, for all $s \in \mathbb{C}$. Consequently, $\zeta(s)$ is full row rank for every $s \in \mathbb{C}$.

Now, suppose $(s_0, \omega)$ is a left eigenvalue-eigenvector pair of $A$. Then

$$\omega^*(s_0I - A) = 0.$$

Let $\omega^*_c = \omega^* \zeta^+(s_0)$. Thus, we have:

$$\omega^*_c \zeta(s_0)(s_0I - A_d - B_d \Sigma C_d) = 0. \quad (5.6)$$
We can easily check that

\[ \zeta(s_0)(s_0 I - A_d) = C_d(P(s_0) \otimes I_n), \]

since \( \text{adj}(s_0 I - A_i)(s_0 I - A_i) = p_i(s_0)I_n, \ i = 1, 2, \ldots, N, \) and \( \zeta(s_0)B_d = Z(s_0). \) Thus, (5.6) can be rewritten as

\[ \omega^r_*(C_d(P(s_0) \otimes I_n) - Z(s_0)\Sigma C_d) = 0. \tag{5.7} \]

Let \( C^+_d \) denote the right pseudo-inverse of \( C_d \) (which is full row rank). By multiplying (5.7) from the right side by \( C^+_d \), we obtain

\[ \omega^r_*(P(s) - Z(s)\Sigma) = 0, \]

since \( C_d(P(s_0) \otimes I_n)C^+_d = P(s_0). \) Rewrite \( \omega^r_* = \omega^r\zeta(s_0)^+. \) We conclude that \( \omega^r\Psi(s_0) = 0. \)

All the previous steps are reversible, and the proof is complete.

\[ \square \]

The above lemma characterizes the spectral properties of \( A \) in terms of the spectrum of \( \Psi(s). \)

Lemma 3 can also be used for finding the eigenvalues of \( A. \) The following corollary provides a computationally efficient method for finding the eigenvalues of \( A. \)

**Corollary 4** Consider an (3.3) with controllable, observable sub-systems; \( s_0 \in \mathbb{C} \) is an eigenvalue of \( A \) if and only if \( s_0 \) is a root of

\[ \det(P(s) - Z(s)\Sigma) = 0. \tag{5.8} \]
**Proof:** Let \((s_0, \omega)\) be an eigenvalue-eigenvector pair of \(A\). By Lemma 3, we know that \(\Psi(s_0)\) is singular. In the proof of Lemma 3, we showed that \(\zeta(s)\) is full row rank. This means that \(\zeta^+(s)\) is full column rank. Thus,

\[
\text{rank}(\Psi(s_0)) = \text{rank}([P(s_0) - Z(s_0)\Sigma])
\]

Singularity of the square matrix

\[
\left[ P(s_0) - Z(s_0)\Sigma \right]
\]

is equivalent to \(s_0\) being the root of (5.8). The proof is complete. 

\[\blacksquare\]

## 5.2.1 Main Results

In this subsection, we characterize the controllability of (3.3). Later, in the next subsection, we will provide graph theoretic conditions interpretations as corollaries of the results of this subsection.

The following theorem characterizes the controllability of (3.3).

**Theorem 17** Consider (3.3). Suppose \((A_i, c_i)\), \(i = 1, 2, \ldots, N\) is observable. Let

\[z_i(s) = c_i \text{adj}(sI_n - A_i)b_i,\]

\[p_i(s) = \det(sI - A_i),\]

\(i \in \{1, 2, \ldots, N\}\). Let \(P(s) = \text{diag}(p_1(s), p_2(s), \ldots, p_N(s))\) and \(Z(s) = \text{diag}(z_1(s), z_2(s), \ldots, z_N(s))\).

Consider the rectangular matrix

\[
\Psi_c(s) = \left[ P(s) - Z(s)\Sigma \quad Z(s)\Gamma \right].
\]

(5.9)

Then the HNL (3.3) is controllable if and only if
(i) for any \( i \in \{1, 2, \ldots, N\} \), the linear system \((A_i, b_i)\) is controllable,

(ii) for all \( s \in \mathbb{C} \) the rectangular matrix \( \Psi_c(s) \) is full row rank.

**Proof:** First, we will show that the controllability of \((A_i, b_i), \ i = 1, 2, \ldots, N\) is necessary for the controllability of \((A, B)\).

We will proceed by contradiction. Suppose that there exists a \( j \in \{1, 2, \ldots, N\} \), such that the pair \((A_j, b_j)\) is not controllable. Let \((\lambda_{uc}, \omega_{uc})\) be an uncontrollable left eigenvalue-eigenvector pair of \((A_j, b_j)\). Thus, we have \( \omega_{uc}^* A_j = \lambda_{uc} \omega_{uc}^* \) and \( \omega_{uc}^* b_j = 0 \).

Define \( \omega = e_j \otimes \omega_{uc} \). We claim that \( \omega \) is an uncontrollable left eigenvector of \( A \) associated with the eigenvalue \( \lambda_{uc} \) and \( \omega^* B = 0 \). Consider \( \omega^* A \); By the definition of \( A \) in (3.4) we have

\[
\omega^* A = \omega^* \text{diag}(A_1, A_2, \ldots, A_N) + \omega^* B_d \Sigma C_d.
\]

Based on the definition of \( \omega, \omega_{uc} \) and \( B_d \) we have

\[
(e_j^* \otimes \omega_{uc}^*) B_d = e_j^* \otimes (\omega_{uc}^* b_j) = 0,
\]

since \( \omega_{uc}^*, b_j = 0 \). Also we can check that

\[
\omega^* \text{diag}(A_1, A_2, \ldots, A_N) = \lambda_{uc} \omega^*.
\]

Thus, we conclude that \( \omega^* A = \lambda_{uc} \omega^* \).

Now, we will show that \( \omega^* B = 0 \). By the definition of \( B \) and \( \omega \), one can easily check that \( \omega^* B = \Gamma_j(\omega_{uc}^* b_j) \) which is zero, by the P.B.H test.

In conclusion, we showed that \((\lambda_{uc}, \omega)\) is an uncontrollable eigenvalue-eigenvector.
pair of \((A, B)\), which is a contradiction.

Now, we will assume that condition (i) holds, and, by using Lemma 3, we will show that condition (ii) is equivalent to controllability of (3.3).

First, we will assume that \(\Psi_c(s)\) is full rank for all \(s \in \mathbb{C}\). Suppose \(\left[ P(s) - Z(s)\Sigma \right] \) is singular at \(s = s_0\) and let \(\omega^* \left[ P(s_0) - \Sigma Z(s_0) \right] = 0\), \(\omega_r \neq 0\). By defining

\[
\omega^* = \omega^*_r \zeta(s_0),
\]

we obtain \(\omega^* \Psi(s_0) = 0\). By Lemma 3, we conclude that \((s_0, \omega)\) is a left eigenvalue-eigenvector pair of \(A\). Since \(\Psi_c(s)\) is full row rank (ii), then \(\omega^*_r \Psi_c(s_0) \neq 0\) and \(\omega^*_r Z(s_0) \Gamma \neq 0\). Since \(Z(s_0) = \zeta(s_0) B_d\), we obtain

\[
\omega^*_r \zeta(s_0) B_d \Gamma \neq 0.
\]

Based on (5.10), and definition of \(B\), we conclude that \(\omega^* B \neq 0\) which means (3.3) is controllable.

All the previous steps are reversible and the proof is complete.

\(\blacksquare\)

The above theorem provides a sufficient and necessary condition for the controllability of (3.3). Condition (i) implies that controllability of the sub-systems is necessary for controllability of (3.3). Matrix \(\Psi_c(s)\) encapsulates information about the input-output dynamics of the isolated systems and the extended digraph. In terms of computational efficiency, checking the rank of \(\Psi_c(s)\) is less computationally expensive than running the P.B.H test for (3.3).
As a special case, when the sub-systems are single-integrators \( f_i(s) = 1/s \), checking the rank of
\[
\Psi_c(s) = \begin{bmatrix} sI_N - \Sigma & \Gamma \end{bmatrix}
\]
is equivalent to the P.B.H test for \((\Sigma, \Gamma)\). Also, in case \(\Sigma\) is a Laplacian matrix, our condition simplifies to a well known condition for consensus networks of single-integrators [15]-[18].

The following corollary shows how the above theorem generalizes our previous result regarding the characterization of the controllability of homogenous networks.

**Corollary 5** Consider an HNL (3.3). Suppose \((A_i, c_i)\) is observable, \(p_i(s) = p(s)\) and \(z_i(s) = z(s)\), for \(i = 1, 2, \ldots, N\). Then (3.3) is controllable if and only if

1. \((A_i, b_i)\) is controllable for all \(i \in \{1, 2, \ldots, N\}\),
2. The linear system \((\Sigma, \Gamma)\) associated with the extended digraph \(G_e = \{V_e, E_e\}\) is controllable.

**Proof:** We will prove this theorem based on the result of Theorem [17]. We just need to show that the part \([3]\) of Theorem [17] is the same as controllability of \((\Sigma, \Gamma)\). Thus, for the rest of this proof we will assume that all the sub-systems are controllable.

Consider the matrix \(\Psi_c(s)\) defined as (5.9). Since all the sub-systems have the same \(z(s), p(s)\), we can rewrite \(\Psi_c(s)\) as
\[
\Psi_c(s) = \begin{bmatrix} p(s)I - z(s)\Sigma & z(s)\Gamma \end{bmatrix}.
\] (5.11)

Consider
\[
\Psi_r(s) = \begin{bmatrix} p(s)I - z(s)\Sigma \end{bmatrix}.
\] (5.12)
Chapter 5. Network Controllability: Main Results

One can easily check that if \( z(s_0) = 0 \), then \( \det(\Psi_r(s_0)) \neq 0 \). As a result, we can consider the rank of

\[
\frac{1}{z(s)} \Psi_c(s) = \left[ \frac{p(s)}{z(s)} I - \Sigma \begin{bmatrix} \Gamma \end{bmatrix} \right],
\]

which is equivalent to the P.B.H test for the linear system \((\Sigma, \Gamma)\). As a result, the proof is complete.

In Section 5.1, we assumed that all the sub-systems have the same linear dynamic. The above corollary shows that even when the sub-systems do not have the same dynamic (i.e. \((A_i, b_i, c_i) \neq (A_j, b_j, c_j)\)) but they have the same transfer function, the controllability of the sub-systems and the controllability of \((\Sigma, \Gamma)\) are necessary and sufficient for the controllability of the whole network.

The following corollary provides a condition for the controllability of a network with sub-systems having the same transfer function and where the interconnection digraph is a Cartesian product of digraphs.

**Corollary 6** Consider \((3.3)\) associated with the extended digraph \(G_e = \{V_e, E_e\}\). Assume that the sub-systems are observable and have the same transfer function. Suppose the interconnection digraph is \(G = G_1 \square G_2\) and \(\Gamma = \Gamma_1 \otimes \Gamma_2\). Then the HNL is controllable if and only if

1. for all \( i \in \{1, 2, \ldots, N\} \), the pair \((A_i, b_i)\) is controllable,

2. The pairs \((\Sigma_1, \Gamma_1)\), \((\Sigma_2, \Gamma_2)\) are controllable.

**Proof:** Based on the result of Corollary 5, we know that the controllability of \((3.3)\) is equivalent to controllability of \((\Sigma, \Gamma)\). An application of Proposition 6 completes the proof.
5.2.2 Graph Theoretic Conditions

In this section, we will use Theorem 17 to derive graph theoretic conditions that imply uncontrollability of \((A, B)\).

**Assumption 1** Consider (3.3). Controllability of sub-systems is necessary for controllability of network. Also, we assume that each sub-system sends its full state information to its out-neighbors. Thus, we will assume that \((A_i, b_i)\) and \((A_i, c_i)\) are controllable and observable, respectively, for \(i = 1, 2, \ldots, N\).

We start with a result that links the input-symmetry of the extended digraph to the uncontrollability of \((A, B)\).

**Corollary 7** Consider an HNL (3.3). Suppose Assumption 1 holds and systems \((A_i, b_i, c_i)\), for \(i = 1, 2, \ldots, N\) have the same transfer function. If the extended digraph is input-symmetric, \((A, B)\) is uncontrollable.

**Proof:** Based on the result of Corollary 5, we know that the controllability of (3.3) is equivalent to controllability of \((\Sigma, \Gamma)\). Again, by Proposition 5, we know that if the extended digraph is input-symmetric, then \((\Sigma, \Gamma)\) is not controllable.

\[
\blacksquare
\]
Corollary 7 is a generalization of Theorem 12 in chapter 4. Figure 5.1 shows an input-symmetric extended digraph.

**Theorem 18** Suppose Assumption 1 holds. If \((A, B)\) in (3.3) is controllable; then the extended digraph \(G_e = \{V_e, E_e\}\) is input-connected.

**Proof:** We will proceed by contradiction. Assume that \(G_e = \{V_e, E_e\}\) is not input-connected. From Proposition 3 there exists a permutation matrix \(T \in \mathbb{R}^{N \times N}\) such that

\[
T^T \Sigma T = \begin{bmatrix}
\Sigma_1 & 0_l \\
0_l & \Sigma_3
\end{bmatrix},
T^T \Gamma = \begin{bmatrix}
0_l \\
\Gamma_1
\end{bmatrix},
\]

and \(l > 0\).

Let

\[
\hat{T} = \begin{bmatrix}
T & 0 \\
0 & 1
\end{bmatrix},
\]

and premultiply and post-multiply \(\Psi_c(s)\) by \(T\) and \(\hat{T}\), respectively, to obtain:

\[
T \Psi_c(s) \hat{T} = T^T P(s) T - T^T Z(s) T T^T \Sigma T \: T^T Z(s_0) T T^T \Gamma.
\]

(5.13)

Since \(P(s)\) and \(Z(s)\) are diagonal matrices, we can write

\[
T^T P(s) T = \begin{bmatrix}
P_1(s) & 0 \\
0 & P_2(s)
\end{bmatrix},
T^T Z(s) T = \begin{bmatrix}
Z_1(s) & 0 \\
0 & Z_2(s)
\end{bmatrix}.
\]

Then, (5.13) can be written as:

\[
T^T \Psi_c(s) \hat{T} = \begin{bmatrix}
P_1(s) - Z_1(s) \Sigma_1 & 0 & 0 \\
-Z_2(s) \Sigma_2 & P_2(s) - Z_2(s) \Sigma_3 & Z_2(s) \Gamma_1
\end{bmatrix}.
\]
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Figure 5.2: In Figure 5.2a all the nodes are reachable from the exogenous input. But in Figure 5.2b node $n_2$ is not reachable from the origin.

Let the pair $(s_0, \omega_1)$, $\omega_1 \neq 0$ be such that

$$\omega_1^*[P_1(s_0) - Z_1(s_0)\Sigma_1] = 0.$$  

Let $\omega = \text{col}(\omega_1, 0)$, we can easily check that $\omega^*T^t\Psi_c(s_0)\hat{T} = 0$, which implies that $\Psi_c(s_0)$ is singular. By Theorem 17 we conclude that (3.3) is uncontrollable. This is a contradiction.

Let the connectedness of the extended digraph be a necessary condition for controllability of (3.3), no matter what the dynamics of the sub-systems are. Figure 5.2a shows an example of an input-connected extended digraph while Figure 5.2b shows an extended digraph which is not input-connected.

The following proposition studies the controllability of (3.3) where $[\Sigma|\Gamma]$ is a sin-
Proposition 9 Consider an HNL (3.3) with the extended digraph \( G_e = \{V_e, E_e\} \). Suppose Assumption 1 holds. Let \( W_u \) denote the uncontrollable subspace of (3.3).

Suppose there is a subset \( N \) of nodes of \( G_e = \{V_e, E_e\} \), such that

\[
\bar{\Sigma} = E(I_N)|\Sigma|\Gamma
\]

is singular and \( |\sigma_c(I_N)| > 0 \). Then the HNL associated with the \( G_e = \{V_e, E_e\} \) is uncontrollable, and

\[
\dim(W_u) \geq \dim(\ker(\bar{\Sigma})),
\]

(5.14)

where \( W_u \) denotes the uncontrollable subspace of (3.3).

Proof: The proof is based on Theorem 17. We will prove (5.14) by constructing \( \dim(\ker(\Sigma)) \) linearly independent vectors lying in \( W_u \).

Let \( \nu_1, \nu_2, \ldots, \nu_t \) be a base for \( \ker(\bar{\Sigma}^*) \). Pick arbitrary \( s_0 \in \sigma_c(I_N) \). Construct the vector \( \omega_i \in \mathbb{C}^N \) such that:

a) if \( n_j \in N \): then \( \omega_i^T e_j = \frac{1}{z_j(s_0)} \nu_i \) (we know that \( z_j(s_0) \neq 0 \) since \( p_j(s_0) = 0 \)),

b) if \( n_j \notin N \): then \( \omega_i^T e_j = 0 \),

for \( i = 1, 2, \ldots, t \). One can easily check that \( \omega_i^* P(s_0) = 0 \) and \( \omega_i^* Z(s_0) = \nu_i^* \). Thus, based on singularity of \( \bar{\Sigma} \), we conclude that \( \omega_i^* \Psi_c(s_0) = \omega_i^* [\Sigma|\Gamma] = 0 \), for \( i = 1, 2, \ldots, t \). By construction, \( \omega_1, \omega_2, \ldots, \omega_t \) are linearly independent. Thus, by Theorem 17 we obtain the bound (5.14) and the proof is complete.

The following theorem addresses the effect of dilation sets on the controllability of (3.3).
Figure 5.3: Input-connected digraph $\mathcal{G}$ with dilation set $\mathcal{S} = \{n_1, n_2, n_3\}$ and $T(\mathcal{S}) = \{n_4, o\}$.

**Theorem 19** Consider an HNL (3.3) with the extended digraph $\mathcal{G}_e = \{\mathcal{V}_e, \mathcal{E}_e\}$. Suppose Assumption 1 holds. Suppose the extended digraph $\mathcal{G}_e = \{\mathcal{V}_e, \mathcal{E}_e\}$ is dilated and contains a dilation set $\mathcal{S}$ and $|\sigma_c(\mathcal{V}_e)| > 0$. Then the (3.3) is uncontrollable and

$$\dim(\mathcal{W}_u) \geq (|S| - |T(S)|),$$

where $\mathcal{W}_u$ denotes the uncontrollable subspace of (3.3).

**Proof:** Based on Proposition 4, we know when the extended digraph contains a dilation set $\mathcal{S}$, then the matrix $[\Sigma|\Gamma]$ is singular and $\dim(\ker([\Sigma|\Gamma]^*)) \leq |S| - |T(S)|$. Since all the sub-systems share a common eigenvalue, if $\mathcal{N} = \mathcal{V}_e$, an application of Proposition 9 completes the proof.

The above theorem simply states that if an HNL is controllable, then either the extended digraph contains no dilation set or the linear sub-systems associated with the nodes contained in the dilation set do not share any common eigenvalue.

The following corollary states the relation between the structural controllability of $(\Sigma, \Gamma)$ and controllability of (3.3).
Corollary 8 Consider a controllable HNL (3.3), where linear sub-systems share a common eigenvalue. Suppose Assumption 1 holds. Then linear system \((\Sigma, \Gamma)\) associated with the extended digraph \(G_e = \{V_e, E_e\}\) is structurally controllable.

**Proof:** Based on the results of Theorem 18 and 19 we conclude that the extended digraph \(G_e = \{V_e, E_e\}\) is input-connected and contains no dilation set. Following the result of Theorem 8 if \(G_e\) is input-connected and not dilated, then \((\Sigma, \Gamma)\) is structurally controllable.

\[\blacksquare\]

In chapter 4 Section 4.2 we studied the controllability of network of single-integrators. The above corollary states that all those results can be applied to heterogeneous networks if the sub-systems share a common eigenvalue.

The following theorem studies the controllability of (3.3) where interconnection digraph contains a NINP.

**Theorem 20** Consider the network (3.3). Suppose Assumption 1 holds. Assume that \(\pi = \{C_1, C_2, \ldots, C_l\}\) is a NINP of the interconnection digraph \(G\) and for all \(C_k \in \pi\), the subgraph \(G_{C_k}\) with the interconnection matrix \(\Sigma_k\) is in-regular and \(|\sigma_c(C_k)| > 0\).

Let \(\Gamma_k = E(I_{C_k})\Gamma\).

If (3.3) is controllable, then extended digraph \(G_{e_k}\) associated with the pair \((\Sigma_k, \Gamma_k)\) contains no dilation, for \(k = 1, 2, \ldots, l\).

**Proof:** We will proceed by contradiction.

Let \(C_1 = \{n_1, n_2, \ldots, n_{|C_1|}\}\). Suppose the extended digraph \(G_{e_1}\) associated with the pair \((\Sigma_1, \Gamma_1)\) is dilated. By Proposition 4, we conclude that \(|\Sigma_1|\Gamma_1\) is singular. Pick a nonzero \(v \in \ker([\Sigma_1|\Gamma_1]^*)\). Let \(\bar{v} = \text{col}(v, 0)\).
First, we want to show that $\bar{v}^* \Sigma = 0$. Recall the characterization of $\Sigma$ in equation (2.5). It follows that $\bar{v}^* \text{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_l) = e_1^T \otimes (v^* \Sigma_1) = 0$.

Since $G_1$ is in-regular, $\Sigma_1 1 = \sigma_1^{\text{in}} 1$, where $\sigma_1^{\text{in}}$ denotes the in-degree of the nodes in $G_1$. We know that $v^* \Sigma_1 = 0$, and $\ker(\Sigma_1^*) \oplus \text{Img}(\Sigma_1) = \mathbb{R}^{|C_1| \times |C_1|}$. Consequently, we have $v^* 1 = 0$, which means $\bar{v}^* T = 0$ ($T$ is the characteristic matrix of $\pi$). As a result, following (2.5), we have $\bar{v}^* \Sigma = 0$. Similarly, we can check that $\bar{v}^* \Gamma = 0$.

In summary, we showed that when $G_{e_1}$ is dilated, then $[\Sigma | \Gamma]$ is singular. Let $\mathcal{N} = C_1$. By Proposition 9, we conclude that (3.3) is uncontrollable, which is a contradiction.

In general, finding the dilation sets and checking the result of Theorem 19 can be difficult. The above theorem says that if the interconnection digraph has a NINP, we can check if some of the subgraphs associated with the cells of the NINP are dilated. This is sufficient to show that (3.3) is uncontrollable. Figure 5.4 shows an extended
digraph with the interconnection digraph \( G \) which has a NINP \( \pi \).

The effect of NEP on controllability of consensus network of single-integrators has been studied extensively (see for example [16], [18] and [43]). In the following theorem, we will show that previous results relating the presence of an equitable partitioning to uncontrollability of consensus networks can be generalized to general networks of LTI systems.

**Theorem 21** Consider (3.3) associated with the extended digraph \( G_e = \{V_e, E_e\} \).

Suppose Assumption 1 holds. Suppose there exists an NEP \( \pi = \{C_1, C_2, \ldots, C_l\} \) on \( G = \{V, E\} \) such that for any cell \( C_p \in \pi \) the linear systems associated with the nodes contained in \( C_p \) have the same transfer function. Also, suppose for any \( n_i, n_j \in C_p \), \( \gamma_{ik} = \gamma_{jk} \) for \( k = 1, 2, \ldots, m \).

Then, (3.3) is uncontrollable and

\[
\dim(W_u) \geq n \sum_{i=1}^{\lfloor |\pi| \rfloor} (|C_i| - 1),
\]

where \( W_u \) denotes the uncontrollable subspace.

**Proof:** Consider the extended digraph \( G_e = \{V_e, E_e\} \) and the NEP \( \pi \). Let \( T \) be the characteristic matrix of \( \pi \). Let \( p_i(s), z_i(s) \) and \( \Psi_c(s) \) defined as in Theorem 17.

Consider \( C_1 \in \pi \). Without any loss of generality, suppose \( C_1 = \{n_1, n_2, \ldots, n_r\} \).

Let \( s_1, s_2, \ldots, s_n \in \mathbb{C} \) be the roots of \( p_{c_1}(s) \). Pick an arbitrary \( v \in \mathbb{C}^r \) and \( v^*1_r = 0 \).

By defining \( \bar{v} = \text{col}(v, 0) \in \mathbb{C}^N \), we get \( \bar{v}^*T = 0 \). Let \( \hat{T} = \text{diag}(T, I_m) \). Consider \( \bar{v}^*\Psi_c(s_i)\hat{T}, i = 1, 2, \ldots, n \), which can be written as

\[
\bar{v}^*\Psi_c(s_i)\hat{T} = \begin{bmatrix}
p_{c_1}(s_i)\bar{v}^*T - z_{c_1}(s_i)\bar{v}^*\Sigma T & z_{c_1}(s_i)\bar{v}^*\Gamma
\end{bmatrix},
\]
since $\bar{v}^*P(s_i) = p_{c_1}(s_i)\bar{v}^*$ and $\bar{v}^*Z(s_i) = z_{c_1}(s_i)\bar{v}^*$. By Proposition 2, $\Sigma T = T\Sigma^\pi$. Since all the nodes in the same cell are connected to the exogenous input with the same weight, we conclude that $\Gamma \in \text{Img}(T)$, which implies $\tilde{\varphi}\Gamma = 0$. As a result, $\tilde{\varphi}\Psi(s_i)\hat{T} = 0$, $i = 1, 2, \ldots, n$. Since $\hat{T}$ is full row rank, $\text{rank}(\Psi_c(s)) = \text{rank}(\Psi_c(s)\hat{T})$, $s \in \mathbb{C}$. From Lemma 3 and Theorem 17, we conclude that $(s_i, \zeta(s_i)^*\bar{v})$ is an uncontrollable eigenvalue-eigenvector of $A$, for $i = 1, 2, \ldots, n$.

Similarly, choose a different cell $C_j \in \pi$. Since

$$\text{dim} (\text{ker}(t_j^*)) \leq |C_j| - 1,$$

we can choose $|C_j| - 1$ different linearly independent vector $v_k$ orthogonal to $1_{|C_j|}$. This results in $n(|C_j| - 1)$ uncontrollable eigenvectors of $A$, and the proof is complete.

The above theorem says that given an NEP on the interconnection digraph, if all the sub-systems contained in each cell have the same transfer function and they are connected to each exogenous input with the same weight, then the HNL is uncontrollable. Also, the theorem provides a lower bound for the dimension of the uncontrollable subspace.

Figure 5.5 shows the extended digraph that satisfies the conditions of Theorem 21.
Figure 5.5: Digraph $G$ with a NEP $\Pi = \{C_1, C_2, C_3\}$ and reduced adjacency matrix

\[ \Sigma_{\Pi} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \]
and the characteristic matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5.2.3 Controllability of Diffusively-coupled Networks

Consider linear systems

\[
\dot{x}_i = A_i x_i + b_i u_i, \\
y_i = c_i x_i,
\]

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$ and

\[
u_i = \sum_{j=1}^{N} a_{ij}(y_j - y_i) + \gamma_i v,
\]

where $a_{ij}$ are nonnegative coefficients, $a_{ii} = 0, i = 1, 2, \ldots, N$, $A = [a_{ij}]_{i,j=1,2,\ldots,N} \in \mathbb{R}^{N \times N}$, $\gamma = \text{col}(\gamma_1, \gamma_2, \ldots, \gamma_N) \in \mathbb{R}^N$ and $v \in \mathbb{R}$ denotes the exogenous input. Diffusive digraph $G_c = \{\mathcal{V}_c, \mathcal{E}_c\}$ associated with matrix $A$ is defined similar to (4.4) and
We can write \((5.15)\) and \((5.16)\) in a compact form as

\[
\dot{x} = Ax + bv,
\]  

(5.17)

where \(x = \text{col}(x_1, x_2, \ldots, x_N)\) and

\[
A = A_d - B_d LC_d,
\]

\[
b = B_d \gamma,
\]

and \(L\) is the Laplacian matrix of \(G_c\).

Network \((5.17)\) is \((3.3)\) where \(\Sigma\) is substituted by \(-L\). The interconnection digraph and the extended digraph are as defined in Chapter 5. The resulting system is a generalization of the consensus network defined in Chapter 4 where instead of single-integrators we consider a general SISO linear system. Notice that in this subsection, we are making the assumption of a single exogenous input.

Consider \((5.17)\) with the coupling digraph \(G_c\). \((5.17)\) asymptotically synchronizes if for any initial state \(x_0 = \text{col}(x_1(0), x_2(0), \ldots, x_N(0))\), we have

\[
\lim_{t \to \infty} \|y_i(t) - y_j(t)\| = 0,
\]  

(5.18)

for all \(i, j \in \{1, 2, \ldots, N\}\).

**Theorem 22** \([44]\) Consider \((5.17)\). If \((5.17)\) asymptotically synchronizes then the sub-systems share a common eigenvalue.

The above theorem states a necessary condition for asymptotic synchronization of \((5.17)\).
The following theorem presents necessary conditions for controllability of (5.17), where the sub-systems share a common eigenvalue.

**Theorem 23** Consider (5.17) associated with the extended digraph $\mathcal{G}_e = \{V_e, E_e\}$ and diffusive digraph $\mathcal{G}_c$. Suppose Assumption 1 holds. Suppose (5.17) is controllable and the sub-systems share a common eigenvalue. Then we have the following:

1. The pair $(L, \gamma)$ is structurally controllable,
2. The extended digraph $\mathcal{G}_e$ is spanned by cacti,
3. The interconnection digraph $\mathcal{G}$ has a perfect matching,
4. The diffusive digraph $\mathcal{G}_c$ is strongly connected.

Moreover, the above conditions are equivalent.

**Proof:** From Theorem (18), we know that the extended digraph is input-connected. Since (5.17) synchronizes, then from Theorem 23, all the sub-systems share a common eigenvalue. As a result, by Theorem (19), the extended digraph is not dilated. From Theorem 8, we conclude that $(L, \gamma)$ is structurally controllable. Structurally controllability of $(L, \gamma)$ implies that The extended digraph is spanned by cacti, and The interconnection digraph has a perfect matching (Theorem 9 and 10). By Theorem 14, the structural controllability of $(L, \gamma)$ implies that the diffusive digraph is strongly connected.

\[ \square \]

**Remark 2** Considering Theorem 22, the above theorem states a set of necessary conditions for controllability of asymptotically synchronized (5.17). Specifically, if
is controllable and asymptotically synchronizes, then the extended digraph $G_e$ is spanned by cacti and diffusive digraph $G_c$ is strongly connected.
Example

In this chapter, we will investigate the controllability of a network of 9 biological neurons. We use the FitzHugh–Nagumo model for each neuron, which is a nonlinear model. As our results in the thesis are based on linear models, we linearize the nonlinear model and we will study the controllability properties of the linearized model.

6.1 FitzHugh–Nagumo model

FitzHugh–Nagumo model has been proposed to describe the dynamics of excitable systems such as neurons. The equations are

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1^3 - x_2 + u \\
\dot{x}_2 &= r(x_1 + \rho - \epsilon x_2) \\
y &= x_1
\end{align*}
\] (6.1)
where \( x_1 \) is the membrane potential, \( x_2 \) is the recovery variable, \( u \) is the external current and \( r, \rho, \epsilon \in \mathbb{R} \) are constants. This model is a simplification of the Hodgkin–Huxley model which models in a detailed manner activation and deactivation dynamics of a spiking neuron.

The linearization of this model at the origin is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
1 & -1 \\
r & -er
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0
\end{bmatrix} u,
\]

\( y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). \tag{6.2}

### 6.2 Network of Neurons

We consider 9 FitzHugh–Nagumo model

\[
\begin{aligned}
\dot{x}_i &= x_i - x_i^3 - z_i + u_i \\
\dot{z}_i &= r_i (x_i + \rho_i - \epsilon_i z_i) \\
y_i &= x_i
\end{aligned}
\]

\( \tag{6.3} \)

represented by a set of nodes \( \{n_1, n_2, \ldots, n_9\} \), connected as

\[
u_i = \sum_{j=1}^{N} a_{ij} (y_j - y_i) + \gamma_i v. \tag{6.4}\]
for \( i = 1, 2, \ldots, 9 \), where \( A = [a_{ij}]_{i,j=1,2,\ldots,9} \) denotes the adjacency matrix of the diffusive digraph \( G_c \) shown in Figure 6.1.

Let

1. \( r_i = 0.074, \rho_i = 0, \epsilon_i = 9.73 \), for \( i = 1, 2, 3 \),
2. \( r_i = 0.08, \rho_i = 0, \epsilon_i = 7 \), for \( i = 4, 5, 6 \),
3. \( r_i = 0.09, \rho_i = 0, \epsilon_i = 8.3 \), for \( i = 7, 8, 9 \).

As \( \rho_i = 0, i = 1, 2, \ldots, 9 \), we conclude that \((0, 0), (\sqrt{\frac{3\epsilon_i-3}{\epsilon_i}}, \sqrt{\frac{3\epsilon_i-3}{\epsilon_i^2}})\) and \((-\sqrt{\frac{3\epsilon_i-3}{\epsilon_i}}, -\sqrt{\frac{3\epsilon_i-3}{\epsilon_i^2}})\) are equilibrium points of (6.1). Consequently, origin is an equilibrium of the network (6.3) and (6.4), as well. Given this consensus network of neurons, we want to study which neurons must be perturbed by an external stimulus so that the network is locally controllable at the origin. By the next theorem, local controllability is equivalent to the controllability of the linearized system.

**Theorem 24 ([46])** Consider a nonlinear dynamical system

\[
\dot{x} = f(x) + g(x)u, \quad (6.6)
\]

where \( u \in \mathbb{R} \) and \( f(x), g(x) \) are smooth vectors fields on \( \mathbb{R}^n \). Let \( A = \frac{\partial f}{\partial x}|_{(x,u)=0}, b = \frac{\partial f}{\partial u}|_{(x,u)=0} \).

Suppose \((A, b)\) is a controllable pair. Then (6.6) is locally controllable at the origin.

---

Consider a nonlinear system

\[
\dot{x} = f(x) + g(x)u, \quad (6.5)
\]

where \( u \in \mathbb{R} \) and \( f(x), g(x) \) are smooth vectors fields in \( \mathbb{R}^n \). Let \( \phi_t(x_0) \) denote the trajectory of (6.5) at time \( t \) starting from \( x(0) = x_0 \). Let \( U \subseteq \mathbb{R} \) be an open connected set, and \( T \in \mathbb{R}^+ \). A point \( x_T \in U \) is accessible from \( x_0 \in \mathbb{R}^n \) at time \( T \) if there exists a bounded measurable control \( u(t) \), generating a trajectory \( \phi_t(x_0) \), such that \( \phi_t(x_0) \subset U \) for \( t \in [0, T] \) and \( \phi_T(x_0) = x_T \). The set of all \( x_T \) which are \( U \) accessible from \( x_0 \) at time \( T \), is denoted by \( A(x_0, T, U) \). Let \( A(x_0, U) = \bigcup_{T \geq 0} A(x_0, T, U) \). A control system is said to be controllable on \( U \) if \( A(x_0, U) = U \), for every \( x_0 \in U \) and locally controllable at \( x_0 \) if the system is controllable on some neighborhood of \( x_0 \).
The above theorem is one of the earliest results in local controllability of nonlinear systems. For more details about nonlinear local controllability, see, e.g., [47].

Based on the result of Theorem 24, we will consider the linearization of (6.3) and (6.4) at the origin (6.2). Thus, the network becomes

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{z}_i 
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
r_i & -\epsilon_i r_i
\end{bmatrix} \begin{bmatrix}
x_i \\
z_i
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} u_i, \\
y_i = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_i \\
z_i
\end{bmatrix},
\]

(6.7)

\[
u_i = \sum_{j=1}^{N} a_{ij} (y_j - y_i) + \gamma_i v,
\]

(6.8)

for \(i = 1, 2, \ldots, 9\). Our goal is to find \(\gamma_i \in \{0, 1\}, i = 1, 2, \ldots, 9\), such that the network (6.7), (6.8) is controllable.
Figure 6.2: Interconnection digraph of networks of 9 neurons. Neurons with different colors have different parameters.

We can check that (6.2) is controllable and observable for all the nodes. As the equation
\[
\det(P(s) - Z(s)L) = 0
\]
does not have any repeated root, based on Corollary \[x\] and P.B.H test, we need only one exogenous input to make the network controllable.

First, we can check that all the systems matrices share a common eigenvalue \(\lambda = 0.9469\) and \(G_e\) is strongly connected (Figure 6.1). Thus, no matter how we connect the exogenous input, Theorem 22 is satisfied and \((L, \gamma)\) is structurally controllable, i.e., the extended digraph \(G_e\) is input-connected and not dilated (Theorem 8).

Now, consider the interconnection digraph \(G = \{V, E\}, V = \{n_1, n_2, \ldots, n_9\}\) as shown in Figure 6.2. Consider the partitioning \(\pi = \{C_1, C_2, C_3, \ldots, C_8\}\), where \(C_1 = \{n_1, n_2\}, C_i = \{n_{i+1}\}, i = 2, \ldots, 8\). Based on Definition 7, \(\pi\) is an NEP. Since neurons \(n_1, n_2\) have the same transfer function, based on Theorem 21, if we connect \(n_1, n_2\) to the exogenous input with the same weights, the network becomes
uncontrollable. For example, if we connect \( \{n_1, n_2, n_4\} \) (or only \( n_4 \)) to the exogenous input, the network is uncontrollable.

Now, we will check the controllability by choosing a node, connecting it to the exogenous input and checking if the assumptions of Theorem 17 are satisfied. If we connect the exogenous input only to \( n_1 \) (or only \( n_2 \)) by checking the rank of \( \Psi_c(s) \), we conclude that the network is controllable. Similarly, for any \( i = 3, 4, \ldots, 9 \), if we connect the exogenous input to \( \{n_1, n_i\} \) or \( \{n_2, n_i\} \), the network becomes controllable. The extended digraph of controllable network where the exogenous input is connected to \( n_2, n_4 \) is shown in Figure 6.3.

This example shows the application of controllability analysis in Deep brain stimulation (DBS). By making the brain’s network controllable, we can increase the success rate of DBS.
Chapter 7

Concluding Remarks and Future Work

In this thesis, we studied the controllability of networks of SISO linear time-invariant systems. We derived a necessary and sufficient condition for the controllability and we provided graph-theoretic conditions for uncontrollability. In particular, we studied the effect of input-connectedness, dilation, identical neighbor partitioning and equitable partitioning on the controllability of networks. Finally, we specialized our results to analyze diffusively-coupled heterogeneous networks, thus extending previous work on controllability of consensus networks. We illustrated the results by studying the controllability of a simple neuronal network model.

There are several open questions we plan to investigate in the future. First, we aim to extend our results to heterogeneous networks of MIMO linear systems. Finding sufficient conditions for controllability based on the properties of the interconnection digraph is still an open problem. Investigating applications of our results to biological and social networks is an interesting topic for future work.
Bibliography


