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Complete List of Authors:	Herrera, Luis; Universidad de Salamanca; Universidad Central de Venezuela, Fisica Di Prisco, Alicia; Universidad Central de Venezuela Facultad de Ciencias, Physics Ospino, Justo; Universidad de Salamanca, Department of Applied Mathematics
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The transition of a gravitationally radiating, dissipative fluid, to equilibrium

L. Herrera^{1*}, A. Di Prisco^{2†} and J. Ospino^{1,3‡}

¹ Instituto Universitario de Física Fundamental y Matemáticas
Universidad de Salamanca, Salamanca, Spain

² Escuela de Física, Facultad de Ciencias,
Universidad Central de Venezuela, Caracas, Venezuela.

³ Departamento de Matemática Aplicada, Universidad de Salamanca, Salamanca, Spain.

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Abstract

We describe the transition of a gravitationally radiating, axially and reflection symmetric dissipative fluid, to a non-radiating state. It is shown that very shortly after the end of the radiating regime, at a time scale of the order the thermal relaxation time, the thermal adjustment time and the hydrostatic time (whichever is larger), the system reaches the equilibrium state. This result is at variance with all the studies carried out in the past, on gravitational radiation outside the source, which strongly suggest that after a radiating period, the conditions for a return to a static case, look rather forbidding. As we shall see, the reason for such a discrepancy resides in the fact that some elementary, but essential, physical properties of the source, have been overlooked in these latter studies.

*e-mail: lherrera@usal.es

†e-mail: alicia.diprisco@ciens.ucv.ve

‡email: j.ospino@usal.es

1 Introduction

Very powerful methods to study gravitational radiation, beyond the well known linear approximation, were put forward in the sixties [1, 2, 3, 4]. Besides the many fundamental results obtained from these methods, their main merit consists in including non-linear effects, which are known to play a very important role in general relativity. All these approaches describe the gravitational radiation outside its source, generally very far from it, to avoid the appearance of caustics and similar pathologies.

Among the wealth of relevant results obtained by means of the methods mentioned above, there is one which has attracted the attention of researchers for many decades, namely: the existence of gravitational wave tails, which implies that the Huygens's principle does not apply to gravitational waves, if non-linearities are taken into account. The bibliography on this issue is overly lengthy huge, and just as a very restricted and incomplete sample, we shall refer the reader to [5, 6, 7, 8, 9, 10, 11, 12].

One immediate consequence of the violation of the Huygens's principle, is the fact that once the system stops to radiate, it does not return back to the static regime, but instead, enters into what Bondi calls "time dependent systems without news", i.e. there is no gravitational radiation, but the spacetime variables depend on time.

In a recent work [13], we have studied the transition from a static situation to a non-equilibrium (radiative) one. In this work we shall study the inverse problem, i.e. the possible transition from a radiative to a static regime. However, unlike the references mentioned above, we shall focus on the source of the gravitational radiation, instead of the asymptotic structure of spacetime.

Gravitational wave tails are supposed to come from scattering of outgoing waves off the spacetime, this could lead to the conclusion that the late-time scattered gravitational waves will presumably invalidate the assumption of perfect equilibrium, once the radiation process has ceased. However, as we shall see here, this is not so. Indeed, the above conclusion is true **only** if the physical processes within the source are neglected. Equilibrium implies static, if one takes into consideration the source, this is the main point of our research.

Thus, we shall consider an evolving system, consisting of a fluid distribution,

which due to the changes of its multipole moments, produces gravitational radiation. We shall next assume that it ceases to radiate at some time (say $t = 0$). For this to happen it is necessary that at $t = 0$, any fluid element reaches the equilibrium, implying that the hydrostatic equilibrium equations (Eqs.(21,22) in [14]) are satisfied. Shortly after $t = 0$, at a time scale of the order of, or larger than the thermal relaxation time, the thermal adjustment time and the hydrostatic time [15],[16], all transient effects have vanished. It is from this time and on, that we shall evaluate our system, to check if there is any potential impediment, for the transition from a radiative regime to the equilibrium situation, to occur (putting aside the short time required for the transient effects to vanish).

As our main result, we obtain that such a transition is indeed possible, after small time scale of the order of the thermal relaxation time, the thermal adjustment time and the hydrostatic time. Obviously, a static source emerging after the end of the radiation period, would produce a static exterior spacetime, which is in contradiction with the result mentioned above, about the non validity of the Huygens's principle. As we shall discuss in the last section, such discrepancy is related to the fact that the frictional forces existing within the source, and responsible for the rapid decay of transient effects, are not taken into account in the studies focusing in the spacetime far from the sources.

In our study we shall heavily rely on a general formalism developed in [17] using a framework based on the $1 + 3$ approach [18, 19, 20, 21]. Accordingly, in order to avoid the rewriting of some of the equations, we shall frequently refer to [17] and [13], however we warn the reader of some important changes in the notation with respect to the first of these references.

2 Basic definitions and notation

In this section we shall deploy, without giving details, all the variables required for our study, some details of the calculations are given in [17] and [13], and therefore we shall omit them here.

2.1 The metric, the source, and the kinematical variables

We shall consider, axially (and reflection) symmetric sources. For such a system the line element may be written in “Weyl spherical coordinates” as:

$$ds^2 = -A^2 dt^2 + B^2 (dr^2 + r^2 d\theta^2) + C^2 d\phi^2 + 2Gd\theta dt, \quad (1)$$

where A, B, C, G are positive functions of t, r and θ . We number the coordinates $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$.

We shall assume that our source is filled with an anisotropic and dissipative fluid. We are concerned with either bounded or unbounded configurations. In the former case we should further assume that the fluid is bounded by a timelike surface S , and junction (Darmois) conditions should be imposed there.

The energy momentum tensor may be written in the “canonical” form, as

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \quad (2)$$

The above is the canonical, algebraic decomposition of a second order symmetric tensor with respect to unit timelike vector, which has the standard physical meaning when $T_{\alpha\beta}$ is the energy-momentum tensor describing some energy distribution, and V^μ the four-velocity assigned by certain observer.

With the above definitions it is clear that μ is the energy density (the eigenvalue of $T_{\alpha\beta}$ for eigenvector V^α), q_α is the heat flux, whereas P is the isotropic pressure, and $\Pi_{\alpha\beta}$ is the anisotropic tensor. We are considering an Eckart frame where fluid elements are at rest.

Since we choose the fluid to be comoving in our coordinates, then

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right); \quad V_\alpha = \left(-A, 0, \frac{G}{A}, 0\right). \quad (3)$$

We shall next define a canonical orthonormal tetrad (say $e_\alpha^{(a)}$), by adding to the four-velocity vector $e_\alpha^{(0)} = V_\alpha$, three spacelike unitary vectors (these are denoted by **K**, **L**, **S** respectively, in [17])

$$e_\alpha^{(1)} = (0, B, 0, 0); \quad e_\alpha^{(2)} = \left(0, 0, \frac{\sqrt{A^2 B^2 r^2 + G^2}}{A}, 0\right), \quad (4)$$

$$e_{\alpha}^{(3)} = (0, 0, 0, C), \quad (5)$$

with $a = 0, 1, 2, 3$ (latin indices labeling different vectors of the tetrad)

The dual vector tetrad $e_{(a)}^{\alpha}$ is easily computed from the condition

$$\eta_{(a)(b)} = g_{\alpha\beta} e_{(a)}^{\alpha} e_{(b)}^{\beta}, \quad e_{(a)}^{\alpha} e_{\alpha}^{(b)} = \delta_{(a)}^{(b)}, \quad (6)$$

where $\eta_{(a)(b)}$ denotes the Minkowski metric.

In the above, the tetrad vector $e_{(3)}^{\alpha} = (1/C)\delta_{\phi}^{\alpha}$ is parallel to the only admitted Killing vector (it is the unit tangent to the orbits of the group of 1-dimensional rotations that defines axial symmetry). The other two basis vectors $e_{(1)}^{\alpha}, e_{(2)}^{\alpha}$ define the two *unique* directions that are orthogonal to the 4-velocity and to the Killing vector.

It can be shown that the anisotropic tensor may be expressed through three scalar functions defined as (see [13]):

$$\Pi_{(2)(1)} = e_{(2)}^{\alpha} e_{(1)}^{\beta} T_{\alpha\beta}, \quad , \quad (7)$$

$$\Pi_{(1)(1)} = \frac{1}{3} \left(2e_{(1)}^{\alpha} e_{(1)}^{\beta} - e_{(2)}^{\alpha} e_{(2)}^{\beta} - e_{(3)}^{\alpha} e_{(3)}^{\beta} \right) T_{\alpha\beta}, \quad (8)$$

$$\Pi_{(2)(2)} = \frac{1}{3} \left(2e_{(2)}^{\alpha} e_{(2)}^{\beta} - e_{(3)}^{\alpha} e_{(3)}^{\beta} - e_{(1)}^{\alpha} e_{(1)}^{\beta} \right) T_{\alpha\beta}. \quad (9)$$

We may write the heat flux vector in terms of the two tetrad components $q_{(1)}$ and $q_{(2)}$:

$$q_{\mu} = q_{(1)} e_{\mu}^{(1)} + q_{(2)} e_{\mu}^{(2)} \quad (10)$$

or, in coordinate components (see [17])

$$q^{\mu} = \left(\frac{q_{(2)} G}{A \sqrt{A^2 B^2 r^2 + G^2}}, \frac{q_{(1)}}{B}, \frac{A q_{(2)}}{\sqrt{A^2 B^2 r^2 + G^2}}, 0 \right), \quad (11)$$

$$q_{\mu} = \left(0, B q_{(1)}, \frac{\sqrt{A^2 B^2 r^2 + G^2} q_{(2)}}{A}, 0 \right). \quad (12)$$

Of course, all the above quantities depend, in general, on t, r, θ .

For the kinematical variables we have the following expressions (see [17, 13]).

For the four acceleration we have

$$a_\alpha = V^\beta V_{\alpha;\beta} = a_{(1)} e_\mu^{(1)} + a_{(2)} e_\mu^{(2)}, \quad (13)$$

with

$$a_{(1)} = \frac{A'}{AB}; \quad a_{(2)} = \frac{A}{\sqrt{A^2 B^2 r^2 + G^2}} \left[\frac{A_{,\theta}}{A} + \frac{G}{A^2} \left(\frac{\dot{G}}{G} - \frac{\dot{A}}{A} \right) \right], \quad (14)$$

where the dot and the prime denote derivatives with respect to t and r respectively.

For the expansion scalar

$$\begin{aligned} \Theta &= V^\alpha_{;\alpha} = \frac{1}{A} \left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) \\ &+ \frac{G^2}{A(A^2 B^2 r^2 + G^2)} \left(-\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G} \right). \end{aligned} \quad (15)$$

Next, the shear tensor

$$\sigma_{\alpha\beta} = \sigma_{(a)(b)} e_\alpha^{(a)} e_\beta^{(b)} = V_{(\alpha;\beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta}, \quad (16)$$

may be defined through two independent tetrad components (scalars) $\sigma_{(1)(1)}$ and $\sigma_{(2)(2)}$, which may be written in terms of the metric functions and their derivatives as (see [13, 17]):

$$\begin{aligned} \sigma_{(1)(1)} &= \frac{1}{3A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \\ &+ \frac{G^2}{3A(A^2 B^2 r^2 + G^2)} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{G}}{G} \right), \end{aligned} \quad (17)$$

$$\begin{aligned} \sigma_{(2)(2)} &= \frac{1}{3A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \\ &+ \frac{2G^2}{3A(A^2 B^2 r^2 + G^2)} \left(-\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G} \right). \end{aligned} \quad (18)$$

Finally, for the vorticity tensor

$$\Omega_{\beta\mu} = \Omega_{(a)(b)} e_{\beta}^{(a)} e_{\mu}^{(b)}, \quad (19)$$

we find that it is determined by a single basis component:

$$\Omega_{(1)(2)} = -\Omega_{(2)(1)} = -\Omega, \quad (20)$$

where the scalar function Ω is given by

$$\Omega = \frac{G(\frac{G'}{G} - \frac{2A'}{A})}{2B\sqrt{A^2 B^2 r^2 + G^2}}. \quad (21)$$

Now, from the regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry, and in particular at the center (see [22], [23], [24]), we should require that as $r \approx 0$

$$\Omega = \sum_{n \geq 1} \Omega^{(n)}(t, \theta) r^n, \quad (22)$$

implying, because of (21) that in the neighborhood of the center

$$G = \sum_{n \geq 3} G^{(n)}(t, \theta) r^n. \quad (23)$$

2.2 The electric and magnetic part of the Weyl tensor and the super-Poynting vector

Let us now introduce the electric ($E_{\alpha\beta}$) and magnetic ($H_{\alpha\beta}$) parts of the Weyl tensor ($C_{\alpha\beta\gamma\delta}$), defined as usual by

$$\begin{aligned} E_{\alpha\beta} &= C_{\alpha\nu\beta\delta} V^{\nu} V^{\delta}, \\ H_{\alpha\beta} &= \frac{1}{2} \eta_{\alpha\nu\epsilon\rho} C_{\beta\delta}{}^{\epsilon\rho} V^{\nu} V^{\delta}. \end{aligned} \quad (24)$$

The electric part of the Weyl tensor has only three independent non-vanishing components, whereas only two components define the magnetic part. Thus we may write these two tensors, in terms of five tetrad components ($\mathcal{E}_{(1)(1)}$, $\mathcal{E}_{(2)(2)}$, $\mathcal{E}_{(1)(2)}$, $H_{(1)(3)}$, $H_{(3)(2)}$), respectively as (see [17] for details):

$$\begin{aligned}
E_{\alpha\beta} = & \left[(2\mathcal{E}_{(1)(1)} + \mathcal{E}_{(2)(2)}) \left(e_{\alpha}^{(1)} e_{\beta}^{(1)} - \frac{1}{3} h_{\alpha\beta} \right) \right] + \left[(2\mathcal{E}_{(2)(2)} + \mathcal{E}_{(1)(1)}) \left(e_{\alpha}^{(2)} e_{\beta}^{(2)} - \frac{1}{3} h_{\alpha\beta} \right) \right] \\
& + \mathcal{E}_{(2)(1)} \left(e_{\alpha}^{(1)} e_{\beta}^{(2)} + e_{\beta}^{(1)} e_{\alpha}^{(2)} \right), \tag{25}
\end{aligned}$$

and

$$\begin{aligned}
H_{\alpha\beta} = & H_{(1)(3)} \left(e_{\beta}^{(1)} e_{\alpha}^{(3)} + e_{\alpha}^{(1)} e_{\beta}^{(3)} \right) \\
& + H_{(2)(3)} \left(e_{\alpha}^{(3)} e_{\beta}^{(2)} + e_{\alpha}^{(2)} e_{\beta}^{(3)} \right). \tag{26}
\end{aligned}$$

Next, in the well known irreducible decomposition of the Riemann tensor, in terms of the Weyl tensor, the Ricci tensor and the curvature scalar, we shall replace these two latter geometrical quantities, by their expressions in terms of the energy momentum tensor, as implied by the Einstein equations. Doing so, the obtained expression for the Riemann tensor embodies the Einstein equations (see [13] for details).

Also, from the Riemann tensor we may define three tensors $Y_{\alpha\beta}$, $X_{\alpha\beta}$ and $Z_{\alpha\beta}$ as

$$Y_{\alpha\beta} = R_{\alpha\nu\beta\delta} V^{\nu} V^{\delta}, \tag{27}$$

$$X_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\nu}{}^{\epsilon\rho} R_{\epsilon\rho\beta\delta} V^{\nu} V^{\delta}, \tag{28}$$

and

$$Z_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\epsilon\rho} R_{\delta\beta}{}^{\epsilon\rho} V^{\delta}, \tag{29}$$

where $R_{\alpha\beta\nu\delta}^* = \frac{1}{2} \eta_{\epsilon\rho\nu\delta} R_{\alpha\beta}{}^{\epsilon\rho}$ and $\epsilon_{\alpha\beta\rho} = \eta_{\nu\alpha\beta\rho} V^{\nu}$.

The above tensors in turn, may be decomposed, so that each of them is described through four scalar functions known as structure scalars [25]. These are (see [17] for details)

$$\begin{aligned}
Y_T &= 4\pi(\mu + 3P), & X_T &= 8\pi\mu, \\
Y_I &= 3\mathcal{E}_{(1)(1)} - 12\pi\Pi_{(1)(1)}, & X_I &= -3\mathcal{E}_{(1)(1)} - 12\pi\Pi_{(1)(1)} \\
Y_{II} &= 3\mathcal{E}_{(2)(2)} - 12\pi\Pi_{(2)(2)}, & X_{II} &= -3\mathcal{E}_{(2)(2)} - 12\pi\Pi_{(2)(2)}, \\
Y_{III} &= \mathcal{E}_{(2)(1)} - 4\pi\Pi_{(2)(1)}, & X_{III} &= -\mathcal{E}_{(2)(1)} - 4\pi\Pi_{(2)(1)}.
\end{aligned} \tag{30}$$

and

$$\begin{aligned} Z_I &= (H_{(1)(3)} - 4\pi q_{(2)}); & Z_{II} &= (H_{(1)(3)} + 4\pi q_{(2)}) \\ Z_{III} &= (H_{(2)(3)} - 4\pi q_{(1)}); & Z_{IV} &= (H_{(2)(3)} + 4\pi q_{(1)}). \end{aligned} \quad (31)$$

2.3 The variables

To summarize, the whole set of variables fully describing our system are:

- The metric variables A, B, C, G (or the tetrad vectors).
- The kinematical variables $a_{(1)(2)}, \Theta, \sigma_{(1)(1)}, \sigma_{(2)(2)}, \Omega$.
- The physical variables describing the fluid distribution $\mu, P, \Pi_{(1)(1)}, \Pi_{(2)(2)}, \Pi_{(2)(1)}$.
- The dissipative fluxes $q_{(1)}, q_{(2)}$.
- The three scalars defining the electric part of the Weyl tensor $\mathcal{E}_{(1)(1)}, \mathcal{E}_{(2)(2)}, \mathcal{E}_{(2)(1)}$ and the two scalars defining the magnetic part of the Weyl tensor $H_{(1)(3)}, H_{(2)(3)}$.

3 The equations

To determine the evolution of our system, we shall need a set of differential equations, which in the context of the 1+3 approach, is a system of first order equations for the variables listed above. Besides we shall need a transport equation for the dissipative processes. Obviously, for any specific model we should also have available equations of state linking different fluid variables. However, since our study is not related to any specific model, we shall not need to refer to any specific equation of state.

In what follows, we shall point out the origin of different set of equations, without specifying them further, since they are explicitly written in [17], from where we shall import them, as required.

3.1 The transport equation

In the presence of dissipative processes, and in order to avoid the drawbacks generated by the standard (Landau–Eckart) irreversible thermodynamics [26], [27], (see [28]–[31] and references therein) we need a transport equation derived from a causal dissipative theory. In the past (see [13, 17] and references therein) we have resorted to Müller-Israel-Stewart second order phenomenological theory for dissipative fluids [32, 33, 34, 35]). However, in this work, the study of the system begins once the thermal equilibrium has been established, and therefore we have no need of any specific transport equation.

3.2 The differential equations for all the variables

First, we have a first order differential equation system, relating the metric variables with the kinematical variables, these are the equations (14, 15, 17, 18, 21), or in a condensed form

$$V_{\alpha;\beta} = \sigma_{\alpha\beta} + \Omega_{\alpha\beta} - a_{\alpha}V_{\beta} + \frac{1}{3}h_{\alpha\beta}\Theta. \quad (32)$$

Next, the integrability conditions of (32) read

$$V_{\alpha;\beta;\nu} - V_{\alpha;\nu;\beta} = R^{\mu}_{\alpha\beta\nu}V_{\mu}. \quad (33)$$

These last equations provide evolution equations for Θ , $\sigma_{\alpha\beta}$, and Ω . Besides, they provide differential constraints relating dissipatives fluxes with derivatives of the kinematical variables, as well as constraints relating the magnetic part of the Weyl tensor with derivatives of the kinematical variables. These are the equations (B1)–(B9) in [17].

The integrability conditions of (33) are just the Bianchi identities, which provide evolution equations for $X_I, X_{II}, X_{III}, Y_I, Y_{II}, Y_{III}, H_{(1)(3)}, H_{(2)(3)}$, as well as differential constraints for the spatial derivatives of the above quantities. These are the equations (B10)–(B18) in [17].

Finally, it could be useful to write the Bianchi identities as “the conservation laws” $T^{\mu}_{\nu;\mu} = 0$. These are the equations (A6),(A7) in [17].

4 Leaving the radiative regime

From very simple physical considerations, it should be obvious that for the radiation regime to stop, the fluid distribution within the source must reach the hydrostatic and thermal equilibrium. Such a state, requires the fulfillment of hydrostatic equilibrium equations, as well as the fulfillment of Tolman conditions for thermal equilibrium [36], and $H_{(1)(3)} = H_{(2)(3)} = 0$.

Then, shortly after the end of the radiation regime, where shortly means in a time of the order of (or larger than) the hydrostatic time, the thermal relaxation time and the thermal adjustment time, transient effects should have been faded away.

At this point it is worth recalling, that:

- The hydrostatic time is the typical time in which a fluid element reacts on a slight perturbation of hydrostatic equilibrium, it is basically of the order of magnitude of the time taken by a sound wave to propagate through the whole fluid distribution.
- The thermal relaxation time is the time taken by the system to return to the steady state in the heat flux (whether of thermodynamic equilibrium or not), after it has been removed from it.
- Finally, the thermal adjustment time is the time it takes a fluid element to adjust thermally to its surroundings. It is, essentially, of the order of magnitude of the time required for a significant change in the temperature gradients.

From the above comments it is clear that, once the radiation process has ended, and after a time period larger than the three time scales defined before, we have that :

- The kinematical quantities $\Omega(G)$, Θ , $\sigma_{(1)(1)}$, $\sigma_{(2)(2)}$ vanish, as well as the dissipative fluxes $q_{(1)}$, $q_{(2)}$. The vanishing of kinematical variables imply at once that first order time derivatives of the metric variables B , C vanish.
- From the above conditions, it follows at once from (A6) in [17], that $\dot{\mu} = 0$.

- Since we want the equilibrium state to hold for a finite period of time after it was reached, we need to impose $\dot{G} = 0$ (the vanishing of the “source” news function), which as shown in [13], ensures that equilibrium is maintained. Otherwise, the system shall leave the equilibrium.

Then, we have for the four acceleration

$$a_{(1)} = \frac{A'}{AB}; \quad a_{(2)} = \frac{A_{,\theta}}{ABr}, \quad (34)$$

and

$$\dot{\Theta} = \dot{\sigma}_{(1)(1)} = \dot{\sigma}_{(2)(2)} = \dot{\sigma}_{(2)(1)} = \dot{H}_{(1)(3)} = \dot{H}_{(2)(3)} = 0. \quad (35)$$

All this implies in its turn (see Eqs.(56–60, 67) in [13]), that

$$\mu = \mu_{(eq)}, \quad P = 8\pi P_{(eq)}, \quad (36)$$

$$\Pi_{(1)(1)} = \Pi_{(1)(1)(eq)}, \quad \Pi_{(2)(2)} = \Pi_{(2)(2)(eq)}, \quad (37)$$

$$\Pi_{(2)(1)} = \Pi_{(2)(1)(eq)}, \quad (38)$$

where eq stands for the value of the quantity at equilibrium.

Also, (see eqs.(68–71) in [13])

$$\mathcal{E}_{(1)(1)} = \mathcal{E}_{(1)(1)(eq)}, \quad \mathcal{E}_{(2)(2)} = \mathcal{E}_{(2)(2)(eq)}, \quad (39)$$

$$\mathcal{E}_{(2)(1)} = \mathcal{E}_{(2)(1)(eq)}, \quad (40)$$

which imply, because of (30)

$$X_I = X_{I(eq)}, \quad X_{II} = X_{II(eq)}, \quad X_{III} = X_{III(eq)}, \quad (41)$$

and

$$Y_I = Y_{I(eq)}, \quad Y_{II} = Y_{II(eq)}, \quad Y_{III} = Y_{III(eq)}. \quad (42)$$

In the above we have written down all the consequences emerging from the condition that the system ceases to radiate, keeping this equilibrium state for at least a finite period. We shall next scrutinize all the equations listed in Section III, to check that the described transition, is not prohibited.

5 Checking the compatibility of the return to the equilibrium state with the field equations

Once the hydrostatic and thermal equilibrium of the source is assumed, in order to stop the generation of gravitational radiation, one should check if any of the equations described in Sec.III, forbids the transition to the static situation.

Let us start with the equations (32), which, as has already been seen, implies the vanishing of the first time derivatives of the metric functions.

Next we have the equations (33), which are the equations (B1–B9) in [17]. They lead to the following constraints:

- (B1) $\rightarrow Y_T = a_{;\alpha}^\alpha$.
- (B2) $\rightarrow -a_{;\delta}^\delta + 3(e_{(1)}^\mu e_{(1)}^\nu a_{\nu;\mu} + a_{(1)}^2) = Y_I$.
- (B3) $\rightarrow a_{(1)} a_{(2)} + e_{(1)}^{(\mu} e_{(2)}^{\nu)} a_{\nu;\mu} = Y_{III}$.
- (B4) $\rightarrow -a_{;\delta}^\delta + 3(e_{(2)}^\mu e_{(2)}^\nu a_{\nu;\mu} + a_{(2)}^2) = Y_{II}$.
- (B5) $\rightarrow e_{(1)}^{[\mu} e_{(2)}^{\nu]} a_{\mu;\nu} = 0$.

On the other hand, (B6–B7) relate the kinematical variables to the dissipative fluxes, while (B8, B9) relate those variables to the two scalars that define the magnetic part of the Weyl tensor. Since, both, the dissipative fluxes and the magnetic part of the Weyl tensor, as well as the kinematical variables, vanish, these four equations are identically satisfied.

Next we have the Bianchi identities, which are the equations (B10)–(B18) in [17]), they imply:

- (B10) $\rightarrow \dot{X}_I = 0$.
- (B11) $\rightarrow \dot{X}_{III} = 0$.
- (B12) $\rightarrow \dot{X}_{II} = 0$.

- (B14) $\rightarrow -\frac{1}{3}X_{I,\beta}e_{(1)}^\beta - X_{III,\beta}e_{(2)}^\beta - \frac{1}{3}(2X_I + X_{II})(e_{(1);\beta}^\beta - a_\nu e_{(1)}^\nu) - \frac{1}{3}(X_I + 2X_{II})e_{\mu;\beta}^{(2)}e_{(2)}^\beta e_{(1)}^\mu - X_{III}(e_{\mu;\beta}^{(2)}e_{(1)}^\mu + e_{(2);\beta}^\beta - a_\beta e_{(2)}^\beta) = \frac{8\pi}{3}\mu_{;\beta}e_{(1)}^\beta.$
- (B15) $\rightarrow -\frac{1}{3}X_{II,\beta}e_{(2)}^\beta - X_{III,\beta}e_{(1)}^\beta - \frac{1}{3}(X_I + 2X_{II})(e_{(2);\beta}^\beta - a_\beta e_{(2)}^\beta) - \frac{1}{3}(2X_I + X_{II})e_{\mu;\beta}^{(1)}e_{(2)}^\mu e_{(1)}^\beta - X_{III}(e_{\mu;\beta}^{(1)}e_{(2)}^\mu + e_{(1);\beta}^\beta - a_\beta e_{(1)}^\beta) = \frac{8\pi}{3}\mu_{;\beta}e_{(2)}^\beta.$
- (B17) $\rightarrow -2a_{(2)}\mathcal{E}_{(1)(1)} + 2a_{(1)}\mathcal{E}_{(2)(1)} - E_{2;\delta}^\delta e_{(2)}^2 - \frac{Y_{II,\theta}}{3Br} + \frac{Y'_{III}}{B} - [\frac{1}{3}(2Y_I + Y_{II})e_{\beta;\delta}^{(1)} + \frac{1}{3}(2Y_{II} + Y_I)e_{\nu(1)}^\nu e_{\nu;\delta}^{(2)}e_{(2)}^\beta + Y_{III}(e_{\nu;\delta}^{(2)}e_{(1)}^\nu e_{(1)}^\beta + e_{\beta;\delta}^{(2)})]\epsilon^{\gamma\delta\beta}e_\gamma^{(3)} = -\frac{4\pi}{3}\mu_{,\theta}e_{(2)}^2.$
- (B18) $\rightarrow 2a_{(1)}\mathcal{E}_{(2)(2)} - 2a_{(2)}\mathcal{E}_{(2)(1)} + E_{\beta;\delta}^\delta e_{(1)}^\beta + \frac{Y'_{II}}{3B} - \frac{Y_{III,\theta}}{Br} - [-\frac{1}{3}(2Y_I + Y_{II})e_{\nu;\delta}^{(2)}e_{(1)}^\nu e_{(1)}^\beta + \frac{1}{3}(2Y_{II} + Y_I)e_{\beta;\delta}^{(2)} + Y_{III}(e_{\beta;\delta}^{(1)} - e_{(1)}^\nu e_{(1)}^\beta e_{\nu;\delta}^{(2)})]\epsilon^{\gamma\delta\beta}e_\gamma^{(3)} = \frac{4\pi}{3}\mu_{,\beta}e_{(1)}^\beta.$

Equations (B13, B16) in [17] are trivial identities, whereas (B10–B12) also become identities, due to (41). Finally, a detailed inspection of B1–B4, B14, B15, B17, B18, shows that these equations are identically satisfied.

6 Conclusions

The emission of gravitational radiation from axially symmetric sources in the context of general relativity has been described in detail in [13], [17]. Using the above formalism, we have shown that the transition from a state of radiation (gravitational) to an equilibrium state, is not forbidden, after a small time interval of the order of magnitude of the hydrostatic time, the relaxation time, and the thermal adjustment time. This result is in contradiction with previous results [1, 5, 6, 7, 8, 9, 10, 11], that suggest that such a transition is forbidden.

The reason for such a discrepancy becomes intelligible, if we recall that in the above references, the physical properties of the source were not explicitly taken into account. Indeed, it is reasonable to expect that any transient effect within the source, will disappear in the time scale indicated above, due to the presence of frictional forces. Of course such phenomena are not included in any analysis of the field outside the source. Our result strengthens further

the relevance of the physical properties of the source, in any discussion about the physical properties of the field. Also, it emphasizes the need to resort to global solutions, whenever important aspects about the behaviour of the gravitational field are discussed. In other words, the coupling between the source and the external field may introduce important constraints on the physical behaviour of the system, implying that details of the source fluid cannot be left out, because they may be relevant to distant GW scattering.

The obtained result might have been guessed from the analysis of the space-time outside the source carried out recently in [37]. Indeed as the main result in that reference, it is found that the absence of vorticity implies that the exterior space-time is either static or spherically symmetric (Vaidya). Since we are evaluating the system after the thermal equilibrium has been attained, we are left with the static situation.

In spite of the above arguments, it should be clear that the apparent decay of the wave tail, within the time scale considered here, must be confirmed (or denied) by the experiment. Unfortunately though, the corresponding effects are at least 5-6 orders of magnitude below the current detectability of existing gyroscopes (I greatly appreciate private communication by A. Di Virgilio and Wei-Tou Ni on this issue) [38]. Perhaps, for the GW tails, it would be much easier to be detected by the current groundbased interferometers (maybe in the third generation detectors) .

It should be kept in mind that we are dealing here, exclusively, with the gravitational radiation produced by a source represented by a fluid distribution, due to changes in its multipolar moments. The gravitational radiation (of the “synchrotron” type) produced by accelerated massive particles, or the two body problem, do not belong to the class of sources considered here.

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