A Model for the Mass and Distribution of Particles in Dark Matter Halos

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A Model for the Mass and Distribution of Particles in Dark Matter Halos
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Abstract

This model is intended for dark-matter-dominated galaxies and galaxy clusters for which the centrifugal force caused by system rotation is negligible. Such systems, ostensibly dark matter halos, would tend to be spherical. Consider a uniform sphere of identical, massive particles in equilibrium (not contracting or expanding). In the quantum model, gravitation pulls the particles together and quantum uncertainty pushes them apart. In the corresponding classical model, gravitation pulls the particles together and thermal motion pushes them apart. This model provides an expression for particle mass as a function of the total mass and density of the system and its quantum state or temperature. Using the measured total mass and density of our dark-matter-dominated galaxy, and assuming the system is in the ground state, the particle mass is found to be 10.5 eV and the temperature 0.042 K. This represents the lowest possible system temperature and particle mass. If, on the other hand, the system is in equilibrium with the cosmic microwave background, the particle mass is found to be 693 eV. This range of inferred particle masses supports the hypothesis of “low-mass dark matter” with approximate mass 100 eV. However, the system temperature is not presently known so it is possible that the temperature is higher and, consequently, the particles are heavier. The average speed of the particles is found to be approximately 1/1000th the speed of light in our galaxy. Remarkably, this result does not depend on the system temperature and, therefore, does not depend on the particle mass. The extension of this model to variable density provides a straightforward solution to the “core-cusp problem” because the distribution of dark matter that minimizes the system energy has a flat central dark matter density profile.

1. Introduction

The present model provides the particle mass as a function of the total mass and density of the system and its quantum state or temperature. We then use the measured total mass and density of our dark-matter-dominated galaxy, along with a range of assumptions about its quantum state or temperature, to infer the range of particle masses. This range of particle masses will be found to support the hypothesis of “low-mass dark matter” with approximate mass 100 eV [1].

The quantum model is extended to account for non-uniform density. We will find that accounting for radially variable density does not substantially change the inferred particle mass. However, the non-uniform density analysis will show that this model provides a straightforward solution to the “core-cusp problem” [2] because the distribution of dark matter that minimizes the system energy has a flat density profile at the core.

2.1 Quantum Model

Let the energy, $E$, of the system be the sum of the potential energy, $U$, and kinetic energy, $K$. 

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The virial theorem provides the relationship between the kinetic and potential energy for a system in equilibrium.

\[ K = -\frac{1}{2} U \]  

(2)

The potential energy of a gravitationally bound spherical system is given by

\[ U = -\frac{3GM^2}{5R} \]  

(3)

where \( G \) is the universal gravitational constant, \( M \) is the total mass of the system, and \( R \) is the radius of the system. Following the approach of Feynman [3], the non-relativistic kinetic energy is given by

\[ K = \frac{N p^2}{2m} \]  

(4)

where \( N \) is the number of particles, \( p \) is the momentum of each particle, and \( m \) is the mass of each particle. Like Feynman [3], the particle momentum is constrained by quantum mechanics such that

\[ p = mv = \frac{Q\hbar}{a}, Q = 1, 2, 3, \ldots \]  

(5)

where \( \hbar \) is the reduced Planck constant and \( a \) is the particle spacing. The quantum number, \( Q \), is like that of the Bohr atomic model and \( Q = 1 \) represents the ground state. The total number of particles is related to the system mass and particle mass according to

\[ N = \frac{M}{m} \]  

(6)

The relationship between the system radius, \( R \), and the particle spacing is given by

\[ R^3 = N a^3 \]  

(7)

Combining equations (1) to (7) and solving for the particle spacing, we obtain

\[ a = \frac{5\hbar^2 Q^2}{3G N^3 m^3} \]  

(8)

As an alternative to using the virial theorem of Equation (2), the same expression for \( a \) has been obtained by minimizing the total energy, \( i.e., \) finding the value of \( a \) for which \( dE/da = 0 \). The density, \( \rho \), of the system is given by

\[ \rho = \frac{M}{\frac{4}{3}\pi R^3} \]  

(9)

Substituting Equations (6), (7), and (8) into (9) and solving for \( m \), we obtain

\[ m = \left( \frac{500\pi \hbar^6 \rho Q^6}{81G^3 M^2} \right)^{\frac{1}{7}} \]  

(10)
The system energy is then given by

\[ E = \frac{-2}{3} \frac{1}{\pi^3} \frac{1}{5} \frac{1}{G} \frac{5}{M^3} \frac{1}{\rho^3} \]  
(11)

which is not a function of the quantum state, \( Q \). Rearranging Equation (5) so that \( v = Q h/(ma) \) and substituting Equations (6), (8), and (10), we obtain

\[ v = 2^{\frac{1}{3}} 3^{\frac{1}{5}} 5^{\frac{1}{2}} \pi^{\frac{1}{6}} G^{\frac{1}{2}} M^{\frac{1}{3}} \rho^{\frac{1}{9}} \]

(12)

which also does not depend on the quantum state, \( Q \). The Appendix provides a validation exercise in which this model is used to infer the mass of the neutron from the nominal properties of neutron stars.

### 2.2 Classical Model

Equations (1), (2), (3), (6), and (9) are again used. In the classical model, however, the kinetic energy is given by

\[ K = N \frac{1}{2} m \langle v^2 \rangle = N \frac{3}{2} k T \]

(13)

where \( T \) is the system temperature and \( k \) is the Boltzmann constant.

Equation (13) may be rearranged to isolate the quantity, \( m/T \). Then, combining Equations (1), (2), (3), (6), (9) and (13), we may solve for \( m/T \), obtaining

\[ \frac{m}{T} = 5 \left( \frac{3}{4\pi} \right)^{\frac{1}{3}} \frac{k}{M^3 G \rho^3} \]  
(14)

Substituting Equation (14) into (13), we may solve for \( \sqrt{\langle v^2 \rangle} \), obtaining

\[ \sqrt{\langle v^2 \rangle} = 2^{\frac{1}{3}} 3^{\frac{1}{5}} 5^{\frac{1}{2}} \pi^{\frac{1}{6}} G^{\frac{1}{2}} M^{\frac{1}{3}} \rho^{\frac{1}{9}} \]

(15)

which is the same as the quantum result of Equation (12). The quantum and classical models give the same average particle speed, which is only a function of \( M \) and \( \rho \) (and not a function of \( h, Q, \) or \( k, T \)).

Equating the classical and quantum expressions for particle mass, \( m \), we find that,

\[ T = 2^{\frac{11}{12}} 3^{\frac{5}{6}} 5^{\frac{5}{8}} \pi^{\frac{11}{24}} G^{\frac{3}{8}} \frac{k}{M^{\frac{5}{12}} \rho^{\frac{5}{12}} Q^{\frac{3}{4}}} \]

(16)

This provides the relationship between temperature, \( T \), and quantum state, \( Q \), of the system given its total mass, \( M \), and density, \( \rho \).

### 3. Results

We assume our galaxy to be dark-matter-dominated. Several authors have estimated the mass and local density of the Milky Way. McMillan [4] estimated the Milky Way mass to be \((1.30 \pm 0.30) \times 10^{12}\)
solar masses and the local dark matter density to be $0.4 \pm 0.04 \text{ GeV/cm}^3$. Kafle et al. [5] estimated the mass to be $(0.80^{+0.31}_{-0.16}) \times 10^{12}$ solar masses and the local dark matter density to be $0.35^{+0.08}_{-0.07} \text{ GeV/cm}^3$. Here, we assume the mass to be $10^{12}$ solar masses and the density to be $0.4 \text{ GeV/cm}^3$ where the local density provided by McMillan [4] is, in this section, taken to be an average density. Therefore, $M = 2 \times 10^{42} \text{ kg}$ and $\rho = 7.12 \times 10^{-22} \text{ kg/m}^3$. Results for the quantum and classical models are provided in Table 1. For each quantum state, there is a corresponding system temperature. The average particle speed is found to be approximately $1/1000$ the speed of light.

The particle number density, $N/V$, is related to the particle mass and system density by

$$\frac{N}{V} = \frac{M/m}{M/\rho} = \frac{\rho}{m}$$

(17)

The particle number density is provided in Table 1. The de Broglie wavelength, $\lambda = h/(mv)$ where $h$ is the Planck constant, is also provided in Table 1. The de Broglie wavelength is larger than the particle spacing, $a$, for the low quantum states.

Table 1: Results of the quantum and classical models. The rows in bold typeface correspond to the ground state and the temperatures of the cosmic microwave and neutrino backgrounds.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$E [J]$</th>
<th>$m [kg]$</th>
<th>$a [\mu m]$</th>
<th>$T [K]$</th>
<th>$v [m/s]$</th>
<th>$N/V [cm^{-3}]$</th>
<th>$\lambda [\mu m]$</th>
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<tr>
<td>1</td>
<td>-9.14x10^{32}</td>
<td>1.88x10^{-35}</td>
<td>18.5</td>
<td>0.042</td>
<td>302,396</td>
<td>3.79x10^{7}</td>
<td>117</td>
</tr>
<tr>
<td>2</td>
<td>-9.14x10^{32}</td>
<td>3.16x10^{-35}</td>
<td>22.0</td>
<td>0.070</td>
<td>302,396</td>
<td>2.25x10^{7}</td>
<td>69.3</td>
</tr>
<tr>
<td>3</td>
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<td>4.28x10^{-35}</td>
<td>24.3</td>
<td>0.095</td>
<td>302,396</td>
<td>1.66x10^{7}</td>
<td>51.2</td>
</tr>
<tr>
<td>10</td>
<td>-9.14x10^{32}</td>
<td>1.06x10^{-34}</td>
<td>32.8</td>
<td>0.233</td>
<td>302,396</td>
<td>6.72x10^{6}</td>
<td>20.7</td>
</tr>
<tr>
<td>100</td>
<td>-9.14x10^{32}</td>
<td>5.94x10^{-34}</td>
<td>58.4</td>
<td>1.313</td>
<td>302,396</td>
<td>1.20x10^{6}</td>
<td>3.69</td>
</tr>
<tr>
<td>170</td>
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<td>66.7</td>
<td>1.954</td>
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<td>8.05x10^{5}</td>
<td>2.48</td>
</tr>
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<td>265</td>
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<td>1.235x10^{-33}</td>
<td>74.5</td>
<td>2.726</td>
<td>302,396</td>
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<td>1.77</td>
</tr>
<tr>
<td>1000</td>
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<td>3.343x10^{-33}</td>
<td>103.9</td>
<td>7.380</td>
<td>302,396</td>
<td>2.13x10^{5}</td>
<td>0.6554</td>
</tr>
<tr>
<td>1001</td>
<td>-9.14x10^{32}</td>
<td>3.345x10^{-33}</td>
<td>103.9</td>
<td>7.385</td>
<td>302,396</td>
<td>2.129x10^{5}</td>
<td>0.6551</td>
</tr>
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</table>

If the system of particles is in the ground state, then the particle mass is found to be $1.88 \times 10^{-35}$ kg (1/48,452 the mass of the electron or 10.5 eV). If, on the other hand, the system is in thermal equilibrium with the cosmic microwave background (with temperature, $T = 2.725 \text{ K}$), then the particle mass is found to be $1.235 \times 10^{-33}$ kg (1/738 the mass of the electron or 693 eV). If the system is in thermal equilibrium with the cosmic neutrino background (thought to have temperature, $T = 1.95 \text{ K}$), then the particle mass is found to be $8.85 \times 10^{-34}$ kg (1/1029 the mass of the electron or 496 eV). This range of particle masses supports the hypothesis of "low-mass dark matter" with approximate mass 100 eV [1].

4. Extension to Variable Density

Let $\rho = \rho(r)$, i.e., $a = a(r)$, where $r$ is the distance from the galaxy center.

The increase in gravitational potential energy of the system between $r$ and $r + dr$ is given by
\[dU = -\frac{G M_{\text{interior}} M_{\text{shell}}}{r} = -\frac{G}{r} \left( \int_0^r 4\pi \xi^2 \rho(\xi) d\xi \right) \left( \int_0^r 4\pi r^2 \rho(r) dr \right) = -16\pi^2 G r \rho(r) \left( \int_0^r \xi^2 \rho(\xi) d\xi \right) dr \quad (18)\]

We now develop the expression for kinetic energy. Like Equation (4),

\[dK = dN \frac{p^2}{2m} \quad (19)\]

Substituting Equation (5) into (19),

\[dK = dN \frac{Q^2 h^2}{2ma^2}, \quad Q = 1, 2, 3, ... \quad (20)\]

where \(a = a(r)\). Like Equation (6),

\[dN = \frac{dM}{m} = \frac{4\pi r^2 dr \rho(r)}{m} \quad (21)\]

Like Equation (9),

\[\rho = \frac{m}{3\pi a^3} \quad (22)\]

where \(\rho = \rho(r)\) and \(a = a(r)\).

Combining Equations (20), (21), and (22), the increase in kinetic energy of the system between \(r\) and \(r + dr\) is given by

\[dK = \frac{2\left(\int_0^r \frac{\xi^2}{4\pi^2} \rho(\xi) d\xi \right) \frac{5}{2} \rho(r) \frac{5}{2} r^2 dr}{m^3} \quad (23)\]

Adding Equations (18) and (23),

\[dU + dK = -16\pi^2 G r \rho(r) \left( \int_0^r \xi^2 \rho(\xi) d\xi \right) dr + \frac{2\left(\int_0^r \frac{\xi^2}{4\pi^2} \rho(\xi) d\xi \right) \frac{5}{2} \rho(r) \frac{5}{2} r^2 dr}{m^3} \quad (24)\]

The total energy, \(E\), from \(r = 0\) to \(r = R\) is then given by

\[U + K = -16\pi^2 G \int_0^R r \rho(r) \left( \int_0^r \xi^2 \rho(\xi) d\xi \right) dr + \frac{2\left(\int_0^R \frac{\xi^2}{4\pi^2} \rho(\xi) d\xi \right) \frac{5}{2} \rho(r) \frac{5}{2} r^2 dr}{m^3} \int_0^R r^2 \left( \rho(r) \right)^3 dr \quad (25)\]

which reduces to the constant density expression when \(\rho \neq \rho(r)\), i.e., \(a \neq a(r)\).

The problem is to find the function, \(\rho(r)\), and particle mass, \(m\), that minimize the total energy of the system with mass, \(M\), and a given density at some location, \(r\). In the following case, \(R \to \infty\).

Presume an Einasto profile [6] of the form

\[\rho \propto e^{-\left(\frac{r}{a}\right)^\alpha} \quad (26)\]
where $\sigma$ and $\alpha$ are parameters. Suppose for demonstration we let $\alpha = 2$. Then a galaxy with total mass, $M$, has density given by

$$\rho = \frac{M}{\pi^2 \sigma^3} e^{-\left(\frac{\sigma}{2}\right)^2} \quad (27)$$

Also suppose the system is in the ground state, i.e., $Q = 1$. Then the total energy of the system is a function of the particle mass, $m$, and the parameter, $\sigma$. For any particle mass, we may find the value of the parameter, $\sigma$, that minimizes the total energy. The correct particle mass is that which satisfies the additional constraint that the local dark matter density be $7.12 \times 10^{-22} \text{ kg/m}^3$ [4] at the sun’s distance from the galactic centre, $2.7 \times 10^{20} \text{ m}$. This calculation was performed and Figures 1 and 2 show that the resulting particle mass is $1.6 \times 10^{-35} \text{ kg}$.

![Figure 1](https://mc06.manuscriptcentral.com/cjp-pubs)

Figure 1: Total energy, $E \text{ [J]}$, versus the parameter, $\sigma \text{ [m]}$ for particle mass, $m = 1.6 \times 10^{-35} \text{ kg}$. The minimum energy occurs at $\sigma = 7.7 \times 10^{20} \text{ m}$.
Figure 2: Density, \( \rho \ [kg/m^3] \), versus distance, \( r \ [m] \), from the galactic center for particle mass, \( m = 1.6 \times 10^{-33} \ kg \) and the parameter, \( \sigma = 7.7 \times 10^{20} \ m \). This satisfies the constraint that the local dark matter density be \( 7.12 \times 10^{-22} \ kg/m^3 \) at the sun’s distance from the galactic centre, \( 2.7 \times 10^{20} \ m \).

Table 2 contains the inferred particle masses for several values of the parameter, \( \alpha \). Figure 3 shows that the distribution of mass is increasingly uniform and less concentrated at the galactic center with increasing \( \alpha \).

Table 2: Inferred particle masses for several values of the parameter, \( \alpha \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( m \ [kg] )</th>
<th>( \sigma \ [m] )</th>
</tr>
</thead>
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<td>1/2</td>
<td>( 1.5 \times 10^{-35} )</td>
<td>( 4.2 \times 10^{19} )</td>
</tr>
<tr>
<td>1</td>
<td>( 1.5 \times 10^{-35} )</td>
<td>( 4 \times 10^{20} )</td>
</tr>
<tr>
<td>3/2</td>
<td>( 1.55 \times 10^{-35} )</td>
<td>( 6.5 \times 10^{20} )</td>
</tr>
<tr>
<td>2</td>
<td>( 1.6 \times 10^{-35} )</td>
<td>( 7.7 \times 10^{20} )</td>
</tr>
<tr>
<td>3</td>
<td>( 1.65 \times 10^{-35} )</td>
<td>( 8.7 \times 10^{20} )</td>
</tr>
<tr>
<td>6</td>
<td>( 1.73 \times 10^{-35} )</td>
<td>( 9.1 \times 10^{20} )</td>
</tr>
</tbody>
</table>
Figure 3: Density, $\rho \ [kg/m^3]$, versus distance, $r \ [m]$, from the galactic center for the values of the parameter, $\alpha$, in Table 2. The uniform density system is also shown.

The remaining problem is that the value of the parameter, $\alpha$, is unknown; that is, we do not know the optimal shape of the density distribution. To estimate the optimal mass distribution, we fix the particle mass and then find the value of the parameter, $\alpha$, that minimizes the total energy. Table 3 provides the results for two nominal particle masses.

Table 3: Total energies for several values of the parameter, $\alpha$, and two particle masses. Recall that for the uniform density system, where $m = 1.88 \times 10^{-35} \ kg$, we found that $E = -9.14 \times 10^{52} \ J$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sigma \ [m]$</th>
<th>$E \ [J]$</th>
<th>$\alpha$</th>
<th>$\sigma \ [m]$</th>
<th>$E \ [J]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>$2.4 \times 10^{19}$</td>
<td>$-7.0 \times 10^{52}$</td>
<td>1/2</td>
<td>$3.3 \times 10^{19}$</td>
<td>$-4.97 \times 10^{52}$</td>
</tr>
<tr>
<td>1</td>
<td>$2 \times 10^{20}$</td>
<td>$-9.5 \times 10^{52}$</td>
<td>1</td>
<td>$3.1 \times 10^{20}$</td>
<td>$-6.70 \times 10^{52}$</td>
</tr>
<tr>
<td>3/2</td>
<td>$4 \times 10^{20}$</td>
<td>$-1.03 \times 10^{53}$</td>
<td>3/2</td>
<td>$5.5 \times 10^{20}$</td>
<td>$-7.23 \times 10^{52}$</td>
</tr>
<tr>
<td>2</td>
<td>$5 \times 10^{20}$</td>
<td>$-1.05 \times 10^{53}$</td>
<td>2</td>
<td>$7.1 \times 10^{20}$</td>
<td>$-7.43 \times 10^{52}$</td>
</tr>
<tr>
<td>3</td>
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<td>$-1.06 \times 10^{53}$</td>
<td>3</td>
<td>$8.7 \times 10^{20}$</td>
<td>$-7.50 \times 10^{52}$</td>
</tr>
<tr>
<td>6</td>
<td>$7.7 \times 10^{20}$</td>
<td>$-1.04 \times 10^{53}$</td>
<td>6</td>
<td>$1.02 \times 10^{21}$</td>
<td>$-7.32 \times 10^{52}$</td>
</tr>
</tbody>
</table>

According to Table 3, the lowest energy system is achieved with the parameter, $\alpha = 3$, for both particle masses. Then, from Table 2, we conclude that the particle mass is $1.65 \times 10^{-35} \ kg$. Recall that the uniform density result was $1.88 \times 10^{-35} \ kg$, which is not substantially different. However, this non-uniform density analysis provides a straightforward solution to the “core-cusp problem” [2] because the minimum system energy is attained with a dark matter density profile that is flat at the core.
This calculation has been repeated for the energy state corresponding to the temperature of the cosmic microwave background, \( Q = 265 \), with \( m = 1.235 \times 10^{-33} \) kg, the mass inferred in Table 1. The results for \( \sigma \) and \( E \) are identical to those of \( Q = 1 \) with \( m = 1.88 \times 10^{-35} \) kg in Table 3. This appears to be a more general version of the result expressed in Equation (11) that the system energy does not depend on the quantum state, \( Q \). In any case, the flat central density profile applies beyond just the ground state.

**Conclusion**

The present model for dark-matter-dominated galaxies assumed that quantum uncertainty or thermal motion provides the repulsion that prevents indefinite gravitational collapse of the system. This model provided the particle mass as a function of the total mass and density of the system. The inferred particle mass also depended on the system temperature or quantum state, which is unknown but might lie between the ground state and the temperature of the cosmic microwave background. Using the measured total mass and density of our dark-matter-dominated galaxy, and assuming the system temperature is between the ground state and the temperature of the cosmic microwave background, the inferred particle mass range was shown to support the hypothesis of “low-mass dark matter”. However, the system temperature is not presently known so it is possible that the temperature is higher than that of the cosmic microwave background, which would imply heavier particles. The extension of this model to variable density provides a straightforward solution to the “core-cusp problem” because the density distribution that minimizes the system energy has a flat profile at the core.

**Appendix: Application of the Model to Neutron Stars**

We apply our quantum analysis to the neutron star, which is largely a balance between gravitation and quantum uncertainty. We use a nominal mass of 1.5 solar masses and radius of 10 km (refer to Figure 4 in [7]), and assume the system is in the ground state and composed entirely of neutrons. From Equation (10), with \( Q = 1 \), we infer the neutron mass to be \( 1.27 \times 10^{-27} \) kg, which is within 25% of the actual mass of the neutron. From Equation (8), we find (using the inferred neutron mass) the spacing, \( a \), to be \( 7.52 \times 10^{-16} \) m, which is comparable to the size of the atomic nucleus.

Rearranging Equation (14) for temperature, we find (again, using the inferred neutron mass) the system temperature to be \( 3.70 \times 10^{11} \) K. From Equation (12) or Equation (15), we find the average particle speed to be \( 1.10 \times 10^8 \) m/s, implying that relativistic effects should be accounted for. Although the model is simple—accounting for limited physics, assuming uniform density, not taking into account relativistic effects, and using nominal mass and radius—it reasonably estimates the neutron mass from the properties of the neutron star.

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References


