Revenue Management in the Sharing Economy

by

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Abstract

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In this thesis, we study revenue management problems in the sharing economy.

In Chapter 2, we study contingent stimulus policies in crowdfunding. We consider a model where backers arrive sequentially at a crowdfunding project. Upon arrival, a backer makes her pledging decision by taking into account the expected project success rate. We characterize the dynamics of a project’s pledging process. To boost success, we propose and characterize three contingent stimulus policies, namely seeding, feature upgrade and limited-time offer. We show that the benefit of contingent policies is greatest in the middle of crowdfunding campaigns. Testing with the data set of Kickstarter, we obtain empirical evidence that the projects’ success rates improve by 14.6% on average with updates in the middle of the campaign and when the pledging progress is lagging.

In Chapter 3, we study the revenue management problem with an all-or-nothing constraint. We consider a seller who would receive her rewards only when her total sales exceed a threshold by the end of the selling period. We derive the optimal sales intensity and prove that the intensity is contingent on the sales progress and is not monotone. We analyze the deterministic heuristics and show that their performances are compromised because of the all-or-nothing constraint. We then propose a two-stage modified resolving heuristic that greatly improves the performance and has provable optimality gaps.

In Chapter 4, we study platform competition of on-demand matching platforms. Customers and service providers chooses the platforms according to nested logit models. We show that as the users gain access to multiple platforms, the platforms can match supply
and demand more efficiently, thus increasing in the utilization of the services. However, access to multiple platforms also increases the overall attractiveness of the service. We show that if the supply side is stringent enough, the price will increase when the platforms compete on the demand side. Similarly if the demand side is stringent enough, the wage will decrease when the platforms compete on the supply side.
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Chapter 1

Introduction

The recent emergence of sharing economy has changed the way we do business. Thanks to the advancement of technology, individuals can be collectively engaged in the business initiative, and that makes ideas such as ridesharing and crowdfunding prosperous. However the new business models in the sharing economy also create new operational challenges. In this thesis, we study three revenue management problems in the sharing economy.

In Chapter 2 we study the RM problem in crowdfunding. Crowdfunding has become a popular means of financing in recent years thanks to the establishment of platforms including Kickstarter and Indiegogo. Despite the popularity, according to the data we collected, over 70% of the projects fail to reach their goals by deadline. In this chapter, we investigate stimulus policies available to the project creators to boost success.

We assume that potential backers arrive at the project webpage according to a non-homogeneous Poisson process. Upon arrival, a backer makes her pledging decision by taking into account the expected success of the project. We characterize the dynamics of a project’s pledging process with a rational expectations equilibrium. In particular, we show that there exists a “cascade effect” on backers’ pledging, which is mainly driven by the all-or-nothing nature of crowdfunding projects. We analyze three contingent stimulus policies, namely, seeding, feature upgrade and limited-time offer. We show that the optimal stimulus policies have a cutoff-time structure. Then we propose simple heuristics derived from the deterministic counterpart of the stochastic model and show that they are asymptotically optimal when the problem is scaled up. However, the profit loss from the static heuristic can have an order of magnitude higher than the square root of the scale parameter—the typical order in loss from deterministic heuristics in other operations settings. This result underscores the importance of implementing contingent policies in crowdfunding. Lastly, we analyze the data of Kickstarter, and demonstrate the benefit
In Chapter 3 we study the management of business performance under an all-or-nothing constraint. In many business problems, the agents would only be rewarded if their performances exceed a predetermined threshold by deadline. For example in crowdfunding, because of the all-or-nothing mechanism, the project creators face a dilemma of how much they should spend on advertising. While increasing advertising intensities improves the project’s likelihood of success, the fact that the project may fail to reach the target discourages the creator from investing more on promotion. In this chapter, we investigate the optimal strategies for businesses that face the all-or-nothing constraint, and provide practical heuristics to mitigate the adverse effect of such constraint.

We consider a seller who would receive her rewards only when her total sales exceeds a threshold by the end of the selling period. The seller can adjust her sales intensity during the time period. However increasing the sales intensity is costly. The reward to reach the threshold encourages the seller to increase the intensity, yet the seller is also deterred as her spending will be wasted if she fails to reach the target. We derive the optimal strategy and show that it is non-monotone in time and the distance to the threshold. We demonstrate that when the evaluation period is long enough, the expected sales must be greater than the threshold for the seller to make a profit. We then provide some practical heuristics inspired by the deterministic version of the problem. While it can still be asymptotically optimal, the performance of the static heuristics is undermined because of the all-or-nothing constraint. We then propose a two stage modified resolving heuristic that greatly improves the performance and has a logarithmic revenue loss.

In Chapter 4 we investigate price and wage decisions of on-demand matching platforms under competition. On-demand matching platforms, such as Uber and Lyft match demand and supply by setting prices for its customers and wages for its suppliers. Since the operations of the platforms are similar to one and another, many users have access to multiple platforms. In this chapter, we study the impact of unloyal users on the prices, wages and the profitability of the platforms.

We model the platform choice decisions of customers and suppliers with nested logit models. We assume that they make decisions sequentially, first deciding whether they would use the service, and then choose the platform. We prove that there exists a unique equilibrium under price and wage competition. We show that as the users gain access to multiple platforms, the platforms enjoy a larger potential supply and demand pool. This results in an increase in the utilization of the service, as well as users' welfare. However, access to multiple platforms also increases the overall attractiveness of the service, which may allow the platforms to charge a higher price and a lower wage. We show that
when the size of the supply pool is small enough, the prices may increase under price
competition as the platform must increase wages aggressively to recruit enough suppliers.
We also show that when the size of the demand pool is small enough, the equilibrium
wage decreases under wage competition, as the platforms must keep the prices low to
recruit enough customers.
Chapter 2

Contingent Stimulus in Crowdfunding

2.1 Introduction

Crowdfunding is a form of innovative financing that has grown enormously in recent years. It is reported that the crowdfunding industry will soon account for more funding than venture capital (Barnett 2015). One of the leading crowdfunding platforms is Kickstarter. On Kickstarter, creators may raise funds from potential backers to start their ventures, and backers are rewarded with variations of the products being produced. In 2014 alone, 22,252 projects were successfully funded there, raising around $529 million from 3.3 million people from nearly every country on the planet.\footnote{Source: https://techcrunch.com/2015/01/06/kickstarter-2014-numbers/ (accessed August 20, 2018.).}

A typical crowdfunding project on Kickstarter has a predetermined monetary goal. The project will be successfully funded only if the goal is reached within a specified time period. Improving chances of successfully raising the required funds lies at the core of the design of crowdfunding projects for project creators as well as for the platforms. Higher success rates benefit all parties: creators receive much-needed funds to initiate their ventures; backers get a chance to support their favorite projects and are rewarded with products being produced; and platforms receive a commission from every successfully funded projects. However, owing to the unpredictability of how many backers will arrive and what their preferences and valuations will be, there is much uncertainty about the outcome of a project, especially since every project has a limited time to meet its target. Using a data set that we collected from Kickstarter from January 30 to June 27, 2015, we found that 63.4% (13,745) of the projects failed to collect more than 20% of their
goals before the deadline. An additional 8.45% (1,831) of projects collected at least 20% of their goals but eventually failed to meet their target.

Traditionally, the effort to improve the success rates of projects concentrates on optimizing the upfront design of project characteristics, such as targeted amount, reward levels and corresponding prices, which are fixed during the campaign horizon (e.g., Hu et al. 2015 and Alaei et al. 2016). However, because of the inherent uncertainty and all-or-nothing mechanism of crowdfunding projects, we advocate that contingently providing incentives or adjusting project features over the course of a crowdfunding campaign is as important as, if not more important than, the upfront design.

Most crowdfunding platforms do allow project creators to update their projects and post related information on projects’ web pages. Updates can range from simple reminders and expressions of appreciation to tangible modifications to the project, such as new designs or extra features. As a matter of fact, both Kickstarter and Indiegogo promote updates as a good way to raise awareness and boost success rates. Our data suggests that, on average, successful projects make 1.1 updates per week, whereas the failed ones make only 0.2.

We use two projects posted on Kickstarter to illustrate the effect of contingent updates on projects’ success. The creators of project “Cuberox” seek to develop a waterproof six-screen computer powered by the Linux operating system. The project was launched on February 24, 2015, aiming to gather $150,000 by March 30, 2015. Figure 2.1(a) displays the cumulative amount pledged to the project during its crowdfunding campaign. As the figure suggests, at first the amount pledged grew steadily; however the increase slowed down significantly in the middle of the campaign. A few backers also expressed concern that the project might not reach its goal. But the creators did not take any action. The pledging almost halted, and the project eventually failed; see Figure 2.1(a).

Another project launched around the same time is “Looking Up, Way Up,” which is a proposed documentary about Burt Rutan, a celebrated aerospace engineer. The project was launched on February 25, 2015, with a deadline of March 28, 2015, and a goal of $80,000. The cumulative amount pledged to the project over time is displayed in Figure 2.1(b). We can see that the first half of the pledging trajectory resembles that of “Cuberox.” However, the number picked up again in the middle of the campaign and eventually reached its target. A closer look at the project timeline shows that project creators announced two raffles for a few free limited-edition items on March 13 and March 17, 2015, which contributed to a significant increase in the pledging number. Whereas the

\[\text{See } \text{https://support.indiegogo.com/hc/en-us/articles/205183587-Post-Updates-to-Raise-Awareness-Funds} \text{ (accessed August 20, 2018).}\]
Motivated by the preceding examples, we study contingent stimulus policies commonly used by creators during their project campaigns to improve their chances of raising the required funds. Specifically, we consider a situation where backers with a heterogeneous, private willingness to pledge (or valuation) arrive sequentially at a crowdfunding project. Upon arriving, a backer makes her pledging decision according to her valuation which depends on project characteristics, as well as according to the expected success of the project which depends on the time of arrival and the amount pledged at that time. We first study, as a base model, the random pledging process without any creators’ contingent stimulus. Specifically, we characterize the dynamics of a project’s pledging process and the structural properties of the project’s success rate, using the concept of rational expectations equilibrium. In particular, due to the all-or-nothing nature of crowdfunding projects, we show that there exists a “cascade effect” on backers’ pledging. That is, a backer’s pledge not only reduces the required number of pledgers by one, but it also boosts the confidence of backers who arrive later, leading to a greater likelihood of pledging by future arrivals. Overall, a backer’s pledge results in a relatively much higher success rate compared to without the pledge. Moreover, the relative benefit of adding one more pledger improves as the chance of success grows dimmer.
Next, we consider three different types of contingent stimulus policies that are costly to implement and focus on the optimal timing of using them. First, we consider a seeding policy, where the project creator has the option to offer one or more free samples to backers at one particular time before the deadline. Owing to the cascade effect, free samples increase the pledging likelihood of future arrivals and thus lead to higher success rates. Second, motivated by a common practice, we consider a feature upgrade policy, where project creators are able to upgrade project features once over the course of the crowdfunding campaign. These two policies are similar in the sense that they are both reactive; i.e., both of them seek to increase the likelihood of future pledging if there are fewer early pledgers than expected. As a result, the optimal policies for these two policies follow a similar structure. That is, for any number of additional pledgers required to reach the target, there exists a cutoff time such that the creator should implement the stimulus if and only if the remaining time is less than or equal to the cutoff. We also show that the cutoff time increases in the number of additional pledgers required, which indicates that the further the total amount pledged is from the goal, the earlier the stimulus policies should be applied. The third policy is a limited-time offer (LTO), where project creators are able to offer extra bonuses, such as free T-shirts, to early adopters. Compared with the other two policies, a limited-time offer is more proactive in the sense that it encourages backers to pledge early with the hope of attracting more backers later on owing to the cascade effect. Because of this difference, the optimal use of the limited-time offer contrasts with that of the other two policies. There is still a cutoff time for any number of additional pledgers required to reach the target; however, the creator should end limited-time offers if and only if the remaining time is greater than or equal to this cutoff. Though all three policies indirectly benefit all backers through the boost in the success rate, seeding and limited-time offer only directly benefit a few of those who get the promotions, whereas feature upgrade directly benefits all, once the project becomes successful.

The cutoff-time structure in the optimal policies suggests that the project creators should wait and apply (or end) the stimulus only when the early pledging trajectory is unsatisfactory (or satisfactory in the case of LTO). In addition, what all three policies share in common is that their benefit in absolute terms vanishes when their duration is either too long or too short. On the one hand, when there is ample time left, a project is likely to be successful without any stimulus. On the other hand, when time is limited, the chance of reaching the funding goal is slim even with stimulus policies. As a result, it is more effective to apply stimulus policies in the middle of the pledging process. This is validated by our empirical analysis of a data set collected from Kickstarter. We show
that, although making updates during the funding campaign always improves a project’s chance of success, updates are most effective in the middle of a campaign, especially when the pledging is lagging. On average, updating under this scenario improves success rates by 14.6%.

Our problem bears a resemblance to traditional revenue management (RM). In both settings, contingent incentives can be provided depending on the time-to-go and the state, which is the pledges needed in crowdfunding or unsold inventory in revenue management. For example, the policy of LTO shares some similarity with the mark-up problem in revenue management (see, e.g., Feng and Gallego [1995]). In both cases, project creators or firms can induce early sales by offering discounts and bonuses to backers or customers. Though the contingent policies improve success rates, they may be hard to compute. A commonly used approach in revenue management is to derive heuristics from the deterministic counterpart of the original stochastic problem. Motivated by such a practice, we also study static heuristics with pre-committed execution time for all three stimulus policies. We show that the heuristics are asymptotically optimal, as the arrival rate and the project target are scaled up by $m$ when $m$ becomes sufficiently large. However, the profit loss using the static heuristic in the LTO case has a magnitude with an order higher than $\sqrt{m}$, i.e., one larger than that typically observed in RM settings (e.g., Gallego and Van Ryzin [1994], Feng and Gallego [1995]). This result underscores the importance of contingent policies in crowdfunding.

We summarize the contributions of our paper as follows. We characterize the “cascade effect” on backers’ pledging. Due to this effect, it is important to monitor the progress from the start of the campaign and use stimulus policies when the pledging progress were slower than expected. We show that the optimal timing to apply stimulus policies has a cut-off structure that is contingent upon the progress of the pledging. We also study heuristic policies and evaluate their performances. A project where the amount pledged grows at a healthy pace does not need interference, whereas one whose pledging progress turns out unsatisfactory would benefit from applying stimuli. We corroborate this finding with the data we collected from Kickstarter. Project updates are shown to offer the greatest boost to success rates when the middle of the campaign is reached and the total amount pledged falls behind.

We apply concepts and tools from RM to the emerging crowdfunding setting. Due to the all-or-nothing nature of the crowdfunding mechanism, we derive a set of theoretical results that are drastically different from those obtained in the traditional RM settings. First, the monotonicity properties such as those of the success likelihood function and profit function often exist in the relative sense (i.e., ratios), rather than in the absolute
sense (i.e., differences). Second, the deterministic problem with certain arrivals does not necessarily provide an upper bound for the stochastic problem with random arrivals. In other words, demand uncertainty may help in the crowdfunding setting. Third, the performance gap between static heuristics solved from the deterministic problem and optimal policies can have an order of magnitude higher than that is commonly observed in RM and other operational settings, due to the cascade effect.

2.2 Literature Review

This paper contributes to the small but growing literature on the crowdfunding mechanism. The origin of crowdfunding can be traced back to the provision point mechanism that is traditionally used in the provision of public goods from private contributions (see, e.g., Bagnoli and Lipman 1989 and Varian 1994). Crowdfunding differs from this stream of literature in that a backer cannot benefit from a crowdfunding project without actually pledging, and thus the free-riding problem that commonly arises in the provision of public goods is not a salient concern in crowdfunding.

The recent emergence of online crowdfunding platforms, such as Kickstarter and Indiegogo, has attracted a wide range of researchers who have studied the phenomenon both empirically and analytically. On the empirical side, researchers have studied many different aspects of the crowdfunding mechanism, including geographic dispersion of investors (Agrawal et al. 2011), backer dynamics over the project funding cycle (Kuppuswamy and Bayus 2013), positive network externalities (Li and Duan 2016), factors that lead to successful projects (Mollick 2014) and the long-term benefit from launching crowdfunding campaigns (Mollick and Kuppuswamy 2014). On the analytical side, Belleflamme et al. (2014) discuss the optimal choices between reward-based and equity-based crowdfunding under various conditions. Hu et al. (2015) study pricing and product design decisions and demonstrate unique benefits of menu pricing in the context of crowdfunding. Chakraborty and Swinney (2016) study how the creators may signal the quality of their projects through funding targets and how the creators’ behavior can be different under the objective of profit-maximization versus success-maximization. Marinesi and Girotra (2016) investigate the rationale behind two types of crowdfunding mechanisms, all-or-nothing versus keep-it-all. Chen et al. (2016) study an entrepreneur who essentially needs venture capital but could use a crowdfunding campaign to learn what the market is. The authors study whether the entrepreneur should launch a crowdfunding campaign and, if so, how to choose the campaign instruments. Chen et al. also study how crowdfunding interacts with more traditional financing sources, such as venture capital and bank fi-
nancing. Veeraraghavan et al. (2016) study and compare mechanisms for overcoming the startup problem and improving the project success. Alaei et al. (2016) seek to unravel the commonly observed phenomenon that crowdfunding projects either succeed or fail by large margins, by modeling the detailed pledging process (see more discussion below). The authors then study the creator’s ex ante decisions of reward pricing and funding target. Unlike the analytical works that mainly address the upfront design of crowdfunding projects in terms of price, target and mechanism, our work focuses on the contingent policies that creators can apply to the dynamic pledging progress after the project design has been determined. We demonstrate the importance of contingent policies, analyze three implementable policies and show their benefits analytically and empirically. In a different setting of new product launching, both Marinesi and Girotra (2013) and Araman and Caldentey (2016) study a firm who learns customer demand through crowdvoting and then contingently decides on product launching decisions. The former focuses on forward-looking customer behavior and the intended use of the acquired information. The latter focuses on the detailed modeling of the voting process and the firm’s optimal timing to stop the voting and start or abandon launching the product, which bears a resemblance to our detailed modeling of the pledging process and the creators’ contingent polices for their optimal stopping problems.

The closest theoretical work to ours is Alaei et al. (2016), because both papers model the dynamic pledging process in which backers anticipate the pledging behavior of later arrivals and take the project’s success rate into account when making pledging decisions. They model the stochastic process as an anticipating random walk. As a base, we model the pledging process with backers’ anticipation, using a different approach, namely, the differential and difference equations, which are a tool commonly used in RM. Moreover, our model works under a more general set of assumptions, namely, that the distribution of backers’ valuations takes a general form and their arrivals follow a non-homogeneous Poisson process, as opposed to a two-point distribution of backer valuations and the assumption of one backer per time period in Alaei et al. (2016). Lastly, as mentioned above, the primary difference is that they consider upfront pricing and target decisions, taking into account the resulting pledging process, whereas we study contingent policies as the pledging process evolves.

The closest empirical work to the theme of our paper is by Li and Duan (2016). They study the pledging process empirically and demonstrate that the portion of funds already raised has a positive effect on investors’ pledging decisions (i.e., positive network externality), and that the time elapsed has a negative effect (i.e., negative time effect). Those empirical findings are consistent with the structural properties of the pledging
process (without stimulus) derived analytically from our model. The authors also briefly study the dynamic promotions based on simulations. For a promotion policy that informs a larger number of investors (similar to seeding in our context), they suggest a heuristic, which is to carry out the promotion when the simulated success likelihood falls under a predetermined threshold. We show analytically that the optimal timing of one-shot promotions has a cutoff-time structure, which is simpler to implement than a policy depending on the simulated likelihood of success. Moreover, we demonstrate theoretically and empirically the effectiveness of contingent policies, whereas their support for dynamic promotions is based on simulated counterfactual analysis.

Crowdfunding shares some similarities with group buying, which also uses the all-or-nothing mechanism with a threshold. Anand and Aron (2003) compare the group-buying mechanism against the listed price mechanism, and illustrate its superiority when the market size is uncertain. Chen et al. (2010) study the optimal design of group-buying mechanisms under quantity discounts. Jing and Xie (2011) explore the role of group buying in facilitating consumer social interactions. More recently, Hu et al. (2013) show analytically the impact of sign-up information disclosure on the success rates of group-buying deals. Using data from Groupon, Wu et al. (2014) find two types of threshold-induced effects. Marinesi et al. (2015) study the benefit of group buying as a means of moderating demand between peaks and troughs. Ming and Tunca (2016) characterize the dynamic sign-up process in group buying by capturing consumer purchase equilibrium with rational expectations of future. Then based on the model, they perform structural estimation and find that consumers do not exhibit large-scale systematic waiting behavior.

On the methodological side, the contingent policies we study in this paper are similar to the dynamic policies in RM (for comprehensive surveys, see, e.g., McGill and Van Ryzin 1999, Bitran and Caldentey 2003, Elmaghraby and Keskinocak 2003). In traditional revenue management, firms seek to maximize the revenue from selling limited inventory over a fixed time horizon by changing prices dynamically depending on the progress of sales. In our work, we adopt the rational expectations equilibrium (REE) framework that has been used in the RM literature to analyze forward-looking behavior of customers (see, e.g., Su 2007, Liu and van Ryzin 2008, Zhang and Cooper 2008, Levin et al. 2009 and Liu and Zhang 2013). Our work differs from studies of traditional RM in that, because of the all-or-nothing nature of crowdfunding projects, backers’ pledging decisions are temporally linked in a direct way as captured in the cascade effect, whereas in RM they are typically moderated by prices alone (though earlier prices may be indirectly linked with later prices owing to the capacity constraint). Because of that important difference, we show that a contingent policy plays a much more important role in crowdfunding in the sense of a
higher magnitude of profit loss when implementing the deterministic heuristic, than in RM.

In the RM settings, Levin et al. (2008) consider a risk-averse objective that takes into account the probability of meeting a revenue target. Besbes and Maglaras (2012) study financial milestone constraints on the revenues and sales that are imposed at different time points along the sales horizon. Those constraints are soft in the sense that the constraints can be violated with a penalty. In contrast, in all-or-nothing crowdfunding, a successfully funded project requires the predetermined funding goal to be achieved within a given time, as a hard constraint. This situation is similar to the setting of Besbes et al. (2016) in which the firm under debt would earn nothing if the generated revenues are not more than the debt at the end of the sales horizon. The difference is that there the firm would only collect the residual revenues after paying the debt, whereas crowdfunding creators collect all revenues if the project is successful. Lastly, Swinney et al. (2011) consider a start-up who maximizes the survival probability in an investment timing game. In our setting, the creators not only need to consider the project’s success probability, but also take into account the cost of stimuli. Given a healthy growth of the pledging process, the creators may not want to offer the stimulus even though doing so can increase the success probability.

2.3 The Model

We consider a crowdfunding platform where creators (such as entrepreneurs or artists) are able to raise funds from potential backers to start their ventures. Initially, the creator posts its crowdfunding project, which is characterized by a targeted amount \( X \), a fixed time horizon \( T \), and prices for rewards. A project is deemed to be successful only when the total pledged amount reaches or exceeds the target \( X \) by the end of the time horizon.

Although creators are allowed or even advised to choose a price menu for rewards on most crowdfunding platforms (Hu et al. 2015), we make a simplification assumption that there is only one price tier \( p \) in our model. Each backer who contributes the amount of \( p \) will be rewarded with a copy of the final product at the end of the crowdfunding campaign. This assumption allows us to characterize precisely the pledging dynamics. Indeed, most analytical works in the crowdfunding literature adopt this single-tier-pricing assumption (see, e.g., Alaei et al. 2016), and our key insights on contingent stimuli are not expected to change even with the presence of a price menu. As we focus on the contingent policies during the campaign, the upfront design of the project, including the target \( X \), the duration \( T \), and the price \( p \) is assumed to be exogenously given.
2.3.1 Individuals’ Pledging Decisions

We start by analyzing individual backers’ optimal pledging decisions. To facilitate our discussion, we denote by $t$ the time remaining until the end of the crowdfunding project, i.e., the time-to-go. Backers patronize the project sequentially according to a non-homogeneous Poisson process with a time-varying rate $\lambda_t$. Upon arrival, they are able to observe the cumulative amount pledged. This information assumption is consistent with the common practice by most crowdfunding platforms such as Kickstarter and Indiegogo. In making her pledging decision, a backer takes into account her valuation of the project, which is dependent on the project’s characteristics, and her expectation of the success of the crowdfunding project, which is dependent on the elapsed time and the cumulative amount pledged when she arrives. We assume that backers form a rational expectations equilibrium. That is, backers act on their rational expectations of the project’s success when making pledging decisions and the final outcome is consistent with their expectations. A backer decides to contribute to the project if and only if she expects her utility from contributing to the project higher than that of not contributing. We do not allow backers to wait strategically. That is, upon arrival, backers either make a pledge or leave the system. For group buying, an all-or-nothing mechanism as well, Ming and Tunca (2016) empirically show that customers’ strategic waiting behavior is not significant. The implications of considering strategic waiting will be discussed in Section 2.6.

The willingness to pledge of backers is private information. In the eye of creators, the pledging behavior can be characterized through pledging likelihood functions defined as follows.

**Definition 1 (Individual’s Pledging Likelihood)** $H(q)$ denotes the probability that a backer pledges to the project upon arrival, given her expectation of the success rate of the crowdfunding project being $q$.

By using this notation, we emphasize the dependence of a backer’s pledging likelihood on the success probability of the project. But we keep in mind that a backer’s pledging likelihood depends on the project’s characteristics as well. We will discuss policies that involve contingent control of those characteristics later in the paper. We further assume that $H(q)$ satisfies the following properties throughout the rest of the paper.

**Assumption 1 (Properties of Individual’s Pledging Likelihood)**

(i) $H(q)$ increases in $q$. 
(ii) For any \( q > 0 \), \( H(q) > 0 \).

(iii) \( \frac{H(\alpha q)}{H(q)} \) increases in \( q \) for any \( 0 < \alpha < 1 \).

Assumption (i) is consistent with the intuition that a backer is more likely to pledge when the project is more likely to succeed. The same assumption is also made in previous works, such as Hu et al. (2013), in the context of online group buying. Assumption (ii) says that, as long as the success rate of the crowdfunding project is not zero, there will be some backers who are willing to pledge. Assumption (iii) implies that the influence of the project’s success rate on backer’s pledging decisions becomes less salient when the likelihood of success is higher. In other words, a backer’s pledging decision becomes less sensitive to success-rate perturbations when the success likelihood is higher. We use the following example to illustrate the generality of Assumption 1.

Example 2.1 To gain granularity on how exactly backers’ pledging decisions may depend on the success likelihood, we consider an example where the creator chooses the quality of the project as \( \theta \). For a given quality level \( \theta \), a type-\( v \) backer has a willingness-to-pledge \( v \cdot \theta \) for the project, where \( v \) is assumed to be the realization of a continuous random variable, drawn from an unbounded distribution with cumulative distribution function (cdf) \( F(\cdot) \) and probability density function (pdf) \( f(\cdot) \). If the backer chooses to pledge but the project fails eventually, an “inconvenience penalty” \( c \) will be incurred, where \( 0 \leq c < p \).

Therefore, the expected surplus from pledging for the crowdfunding project includes two components: if the project turns out to be successful, at the end of the campaign, the backer enjoys a payoff of \( v \theta - p \); otherwise, a cost of \( c \) is incurred. Any backer whose belief in the project’s success likelihood is \( q \) will pledge if and only if

\[
(v \theta - p) \cdot q - c \cdot (1 - q) > 0 \quad \Rightarrow \quad H(q) = \bar{F}\left(\frac{1}{\theta} \left( p + c \cdot \left( \frac{1}{q} - 1 \right) \right) \right). \quad (2.1)
\]

Lemma 2.1 \( H(q) \) in (2.1) satisfies Assumption 1 if the distribution of backers’ types has an increasing generalized failure rate (IGFR), i.e., \( v \cdot \frac{f(v)}{F(v)} \) is an increasing function in \( v \).

Lemma 2.1 gives a sufficient condition for Assumption 1 for the specific form of \( H(q) \) in (2.1). The IGFR is a very general assumption as it captures many commonly used distributions, such as normal and uniform distributions.

\[ \text{The cost may consist of psychological frustration in backers who failed to get the product or service they desired. It may also stem from economic losses. When a crowdfunding project fails to reach its goal, backers will not be charged. However, since they will not know that and be able to use the money for other purposes until the time expires, they will have experienced a loss because of the time value of money.} \]
Chapter 2. Contingent Stimulus in Crowdfunding

Example 2.1 specifies an individual utility-maximization model where the pledger has full information about the project. The general form of the pledging likelihood function can also accommodate observational learning behavior in which a pledger may not have complete information about the project but can rationally anticipate the future arrivals’ pledging behaviors.

2.3.2 Pledging Dynamics

The previous discussion of individuals’ pledging decisions sets the stage for our characterization of the dynamics of the pledging process. Since backers’ pledging decisions are determined by the expected success through the individual’s pledging likelihood function, the pledging dynamics can be captured by the evolution of the project’s likelihood of success over time. Recall that the crowdfunding project needs to gather $X$ dollars before the end of a fixed time horizon. Given the price $p$ charged to each backer, the project requires at least $N \equiv \lceil \frac{X}{p} \rceil$ pledgers before time expires. We denote by $n$, where $0 \leq n \leq N$, the additional number of pledgers required to reach the project’s target, i.e., the pledges needed. The funding progress of the project towards reaching the goal is uniquely captured by the state space $\{(t, n) : 0 \leq t \leq T, 0 \leq n \leq N\}$.

2.3.2.1 Success Rate.

Let $M(t)$ denote the number of backers who have pledged with time-to-go $t$. The project is successfully funded if and only if $M(0) \geq N$. For a backer who arrives at time-to-go $t$ and pledges needed $n$, her expected project’s success rate, conditional on her pledging, is denoted by $Q_t(n-1)$. Under the rational expectations equilibrium, her expectation will be fulfilled by backers who arrive later and act on their rational expectations. Then the dynamics of the project’s success likelihood in equilibrium can be summarized as follows.

Proposition 2.2 (Rational Expectations Equilibrium (REE)) There exists a unique REE, such that the probability $Q_t(n)$ of the project being successfully funded at state $(t, n)$, is given by

$$\frac{\partial Q_t(n)}{\partial t} = \lambda_t \cdot H(Q_t(n-1)) \cdot (Q_t(n-1) - Q_t(n)), \quad (2.2)$$

with boundary conditions $Q_t(0) = 1$ for all $t$ and $Q_0(n) = 0$ for all $n > 0$.

The success likelihood at any state $(t, n)$ can be solved by backward induction. However, in general, obtaining the closed form of $Q_t(n)$ is extremely difficult, if not impossi-
ble, even for special forms of $H(\cdot)$. Nevertheless, we are able to show a set of structural properties of $Q_t(n)$.

**Theorem 2.3** (Structural Properties of Equilibrium Success Likelihood)

(i) $Q_t(n)$ strictly increases in $t$ for any $n \geq 1$ and strictly decreases in $n$ for any $t > 0$.

(ii) $\frac{Q_t(n-1) - Q_t(n)}{Q_t(n)} \geq \frac{1}{e^{n-1}}$, where $\bar{\lambda} \equiv \sup\{\lambda_t : 0 \leq t \leq T\}$.

(iii) For any $n \geq 1$ and $t > 0$, both $\frac{Q_t(n-1)}{Q_t(n)}$ and $\frac{H(Q_t(n-1))}{H(Q_t(n))}$ decrease in $t$ and increase in $n$. Moreover, $\lim_{t \to 0} \frac{Q_t(n-1)}{Q_t(n)} = \infty$.

(iv) For any $h > 0$, $\frac{Q_{t+h}(n)}{Q_t(n)}$ strictly increases in $n$.

(v) For any $n \geq 1$, there exists a threshold $\hat{t}(n) = \sup\{t : H(1) \int_0^t \lambda_s ds < n\}$ such that for all $0 < t < \hat{t}(n)$, the success likelihood $Q_t(n)$ in the stochastic problem is strictly higher than that in the deterministic counterpart where potential backers arrive in a fluid fashion with rate $\lambda_t$.

Theorem 2.3(i) shows that the chance of the project being successful increases with more time remaining and fewer pledgers required. Theorem 2.3(ii) gives a lower bound on the relative change in the success likelihood by adding one more pledger. The guaranteed relative improvement in the likelihood of success with one more pledger is larger if the arrival rates are smaller.

The most interesting property of $Q_t(n)$ is shown in Theorem 2.3(iii). The effect of backers’ pledging decisions on a project’s success likelihood is twofold: (1) On one hand, a backer’s pledging reduces the required number of pledgers by one and thus leads to a higher likelihood of success; (2) On the other hand, the backer’s pledging also boosts the confidence of backers who arrive later, leading to a higher likelihood future arrivals will pledge. These two factors add up to what we referred to as the cascade effect of an individual’s pledging on future backers’ pledging decisions. Theorem 2.3(iii) shows that this compounding cascade effect is more salient when the time is closer to the deadline and/or the number of additional pledgers required is larger. It would also be interesting to contrast this property with results from a typical RM setting, where the firm has to sell a limited amount of inventory over a finite horizon. There a customer’s valuation of the product is not affected by the purchase decisions of other customers. However, in our crowdfunding situation, any individual backer’s pledging decision would directly and positively affect subsequent backers’ decisions.

Theorem 2.3(iv) shows the impact of time-to-go on the project’s success likelihood for a fixed pledges needed. A longer time remaining results in a higher likelihood of success.
for the project as shown in Theorem 2.3(i). Theorem 2.3(iv) shows that this effect is more significant when the number of additional pledgers required is larger.

Theorem 2.3(v) says that a crowdfunding project with random arrivals can have a strictly higher success rate than its deterministic counterpart. This happens when the deterministic setting is doomed to fail with the success likelihood being zero, but the random counterpart still has a positive success likelihood due to the possibility that a large number of backers will arrive. This observation may also partially explain why the majority of Kickstarter projects fail (71.85% in our Kickstarter data): the creators want to give their project a shot, knowing that it is likely not successful but its failure may render not much to lose under the all-or-nothing mechanism. For those projects that have poor upfront prospect, the contingent policies we will propose may be even more effective (see, e.g., Example 2.2).

2.3.2.2 Upfront Design.

Given the cascade effect on backers’ pledging decisions, it is important to carefully consider the project’s characteristics before launching the crowdfunding campaign. Consider two designs of a project, namely, design \( a \) and design \( b \), which can differ in various project characteristics, such as price and quality. Suppose that design \( b \) is more attractive in the sense that \( H^a(q) < H^b(q) \) for any \( q > 0 \). We have the following structural results from the comparisons of the project’s success likelihood and backers’ pledging likelihood between the two projects.

**Proposition 2.4** (Upfront Design of Crowdfunding Projects) Consider two pledging likelihood functions \( H^a(q) \) and \( H^b(q) \). If \( H^a(q) < H^b(q) \) for any \( q > 0 \), and \( \frac{H^a(q)}{H^b(q)} \) increases in \( q \), then both the ratios of success likelihoods, \( \frac{Q^a_t(n)}{Q^b_t(n)} \), and pledging likelihoods, \( \frac{H^a_t(Q^a_t(n))}{H^b_t(Q^b_t(n))} \), increase in \( t \) and decrease in \( n \).

Proposition 2.4 underscores the importance of the design of project characteristics. A small difference in backers’ pledging likelihood may lead to a huge difference in the project’s success likelihoods because of the cascade effect. Proposition 2.4 states that, given two different project designs, the relative difference in the project’s success likelihoods is more significant when the time is closer to the deadline and/or the number of additional pledgers required is larger. The same applies to backers’ pledging likelihood as well.

Recall that design \( a \) is less attractive. The assumption that \( \frac{H^a(q)}{H^b(q)} ( < 1 ) \) is an increasing function of \( q \) requires that the relative difference in the pledging likelihood under two designs increase when the project’s likelihood of success decreases. That is, the inferior
design hurts backers’ pledging likelihood more significantly when the success likelihood of the project is lower. We revisit Example 2.1 and investigate when this assumption is satisfied. Two sufficient conditions are summarized below. It turns out that the assumption can be easily satisfied when the project can be configured with different prices or qualities.

**Lemma 2.5 (Properties of Pledging Likelihood)** Consider the pledging likelihood function derived in Example 2.1.

(i) For two quality levels $\theta_a < \theta_b$, the ratio of pledging likelihoods, $\frac{H^{p_a}(q)}{H^{p_b}(q)}$, is an increasing function of $q$.

(ii) If the distribution of backers’ valuations in Assumption 2.2 has an increasing failure rate (IFR), then for two prices $p_a > p_b$, the ratio of pledging likelihoods, $\frac{H^{p_a}(q)}{H^{p_b}(q)}$, is an increasing function of $q$.

### 2.3.2.3 Expected Revenue.

All of the above structural properties are about the success rates and pledging likelihood. Next we derive those for the expected revenue of a crowdfunding project. In practice, many crowdfunding platforms allow backers to pledge even after the target is reached, a practice which is referred to as *overfunding*. In our base model, we do not consider overfunding. (We show in the online supplement that the following proposition still holds in the case of overfunding.) The reasons are the following: (1) Ex ante, most creators are primarily concerned about the probability of collecting the targeted amount rather than overfunding. They would have sufficient funds to start their project once the target is reached. The creators could, and are most likely to, choose to continue to accept pledges; however the overfunding will not make or break the project. (2) The timing of backers’ pledging may make a difference to what products they receive. For instance, because of production capacity constraints, those who pledge after the target has been reached may not receive their products until months after the initial release. Such differences are not incorporated in our pledging model. (3) Lastly, the process of crowdfunding can be broken down into two stages. We model the pledging process in the first stage before the target is met. In the stage after the target is reached, the pledging dynamics become simple: as the success likelihood stays constant at 1, backer’s decisions solely depend on their valuations and project characteristics. We focus only on the nontrivial pledging process before the funding goal is achieved.

Without considering overfunding, we denote the expected revenue at state $(t, n)$ by $J^b_t(n) = Np \cdot Q_t(n)$. It is obvious that $J^b_t(n)$ increases in $t$ and decreases in $n$. The impact
of an additional pledger on the expected revenue is summarized in the proposition below, which is derived from Theorem 2.3(iii).

**Proposition 2.6 (Value of Pledgers)** The marginal increase in the expected revenue with one more pledger at state \((t, n)\), \(\frac{J^1_t(n-1) - J^1_t(n)}{J^1_t(n)}\), decreases in \(t\) and increases in \(n\).

Like Theorem 2.3(iii), Proposition 2.6 shows that an additional pledger is more valuable when the time is closer to the deadline and/or the number of additional pledgers required is larger. In the traditional RM literature, monotonicity properties are derived for the absolute difference between the expected revenues. However, because of the cascade effect demonstrated in Theorem 2.3 in the context of crowdfunding, analogous properties exist but they are for the relative difference.

### 2.4 Contingent Stimulus Policies

We have illustrated the importance of the upfront design of projects’ characteristics in Proposition 2.4. Given the stochastic nature of backer arrivals and their willingness to pledge to the project, the pledging process may still fail to meet the creator’s expectations even if the project’s characteristics are optimized ex ante. In such cases, the creator can be better off taking ex post actions to influence backers during the campaign. In this section, we consider three different types of contingent stimulus policies from the perspective of project creators, namely, seeding, feature upgrade and limited-time offer. They are different in their effect on cost structure and pledging, but they share the common feature that the associated costs to the creators do not materialize unless the project is successful. We discuss the optimal ways of applying these three policies, and quantify potential benefits. In the following analysis, with a slight abuse of notation, we denote the pledging likelihood at state \((t, n)\) by \(H_t(n) \equiv H(Q_t(n-1))\).

#### 2.4.1 Seeding Policy

We start with the simplest stimulus policy, where the creator has the option to offer \(n_0\) number of free samples \((1 \leq n_0 < N)\) to backers exactly once during the campaign. A special case of the stimulus is to decrease the target level from \(N\) to \(N - n_0\) upfront. However, the superiority of this seeding policy over the manipulation of the target level is obvious. The creator would choose to offer free samples only along certain sample paths in which the early pledging progress is not satisfactory. When the pledging process materializes in a way that favors the creator, the free samples could be saved, allowing
the creator to obtain a higher profit. We limit our discussion to the case where the free samples are offered once at most. In practice, too many rounds of free samples can raise fairness concerns from backers and hurt the image of the creator.

We assume that the free samples will be claimed immediately and that backers do not expect a future offer of free samples when they make their pledging decisions. If they do, under our assumption of no strategic waiting, the incentive for backers to pledge now will be even higher, thus leading to a higher value of contingent seeding. This is because backers will be more confident in the project success since they expect an intervention by the creator when progress stalls.

**Theorem 2.7 (Optimal Cutoff for Seeding)** For each \( n \geq 1 \), there exists a cutoff time \( \tau^*(n) \), such that the creator will offer free samples if and only if \( t \leq \tau^*(n) \).

Theorem 2.7 sheds light on the conditions under which the creator is better off offering the free samples. For any current pledges needed \( n \), there exists a cutoff \( \tau^*(n) \) such that the creator should offer the free samples if and only if the time-to-go is no more than this cutoff. Although details of the proof are more involved and can be found in the appendix, we describe the intuition as follows. The creator makes the optimal stopping decision by comparing the optimal expected revenues with and without offering free samples. In particular, from Theorem 2.3, we show that the relative improvement in the success likelihood by offering free samples decreases in \( t \). Thus, when there is ample time left, the cost of free samples outweighs the improvement in the likelihood of success, and the project creator will choose to hold out as a result. On the other hand, when the time-to-go is short enough, it is optimal to offer free samples immediately to boost the chances of success.

We present the monotonicity properties of the cutoffs as follows. It is not surprising that the cutoff \( \tau^*(n) \) is increasing in the pledges needed \( n \). This implies that the seeding policy is more likely to be used at a time when the pledging number is further away from the target.

**Corollary 2.8** \( \tau^*(n) \) increases in \( n \), i.e., \( \tau^*(N) \geq \tau^*(N - 1) \geq \cdots \geq \tau^*(n_0) = \cdots = \tau^*(1) = 0 \).

In general, it is very hard to derive the closed form solution of \( \tau^*(n) \). To see how \( \tau^*(n) \) may look, we consider two special cases: Case (i) \( H(q) \equiv H \), i.e., a backer’s pledging decision is based solely on the project’s characteristics, rather than the likelihood of success. From the threshold characterization (see the proof in the appendix), for this case, we have \( \tau^*(n) = 0 \) for all \( n \geq 1 \). That is, the creator will never offer the free
samples. This is sensible considering that the benefit of the seeding policy is driven by the cascade effect of backers’ pledging decisions. The seeding policy has no influence when backers are not affected by the decisions of others. Case (ii) \( H(q) = \begin{cases} 1 & \text{if } q > \bar{q} \\ 0 & \text{if } q \leq \bar{q} \end{cases} \), as a result of that backers have homogeneous willingness to pledge. Then the creator will offer the free samples if and only if backers’ perceived project success likelihood drops to \( \bar{q} \) for the first time; otherwise, backers are expected to pledge upon arrival, rendering the free samples unnecessary.

Denote by \( J_{s,T,N} \) the optimal expected revenue with the option of seeding when the deadline is \( T \) and the goal is \( N \). We compare \( J_{s,T,N} \) with the expected revenue under no stimulus \( J_{b,T,N} \), and obtain the following structural properties:

**Theorem 2.9**

(i) For any \( N \geq 1 \), \( \frac{J_{s,T,N}}{J_{b,T,N}} \) decreases in \( T \).

(ii) For any \( N > n_0 \), \( \lim_{T \to \infty} J_{s,T,N} - J_{b,T,N} = \lim_{T \to 0} J_{s,T,N} - J_{b,T,N} = 0 \).

The seeding policy always benefits the project because it gives extra flexibility to the project creator, allowing him to keep the pledging process at a healthy pace by giving out free samples if necessary. From Theorem 2.9, we can see that the relative benefit of seeding becomes more significant as the time remaining gets shorter. However, its absolute benefit vanishes as \( T \) approaches either infinity or zero. When the time is long enough, having few pledgers at the beginning of the process will not have a huge negative impact because future arrivals may still reverse the trend, resulting in a low value of seeding. On the other end of the spectrum, when the time is very short, few backers will come to the project, leading to the ineffectiveness of the cascade effect, as well as the seeding policy. Consequently, the benefit of seeding is significant when time is limited but not impossibly short. We further confirm this finding numerically in Section 4.5 and empirically in Section 2.5.

### 2.4.2 Feature Upgrade

In the second policy, we allow the creator to upgrade project features once during the campaign. This policy is motivated by the common practice of popular crowdfunding platforms, such as Kickstarter and Indiegogo, on which project creators can update project features over the course of the pledging process. The new feature could be, for example, a new color for a fashion product or a bonus soundtrack for an album. With
the upgrades the project creator hopes that backers will be more willing to pledge. However, upgrading project features could be costly. Consequently, the key question here is whether and when the project creator should offer an upgraded version of their project.

To answer this question, we enrich the base model as follows. Assume that the cost of an upgrade is $K$. As a result of the upgraded project, backers’ pledging likelihood increases to $\tilde{H}(q)$, where $\tilde{H}(q) \geq H(q)$ for any $q$. The corresponding likelihood of success is denoted by $\tilde{Q}_t(n)$. In the context of Example 2.1, the feature upgrade is that the quality level of the project increases from $\theta$ to $\tilde{\theta}$.

**Theorem 2.10 (Optimal Cutoff for Feature Upgrade)** For each $n$, there exists a cutoff time $\tau_u(n)$, such that the creator will upgrade if and only if $t \leq \tau_u(n)$.

The policy of feature upgrade differs from the seeding policy in that it does not directly interfere with the pledging number. However, both of them rely on the cascade effect of backers' pledging decisions to be effective. As a result, the optimal policy of feature upgrade is similar to the seeding policy. That is, for any pledges needed $n$, there exists a cutoff in time $\tau_u(n)$ such that the creator should upgrade the project features if and only if the remaining time towards the end of the time period is less than or equal to this cutoff. We also show that $\tau_u(n)$ is increasing in $n$, which is summarized below.

**Corollary 2.11** $\tau_u(n)$ increases in $n$, i.e., $\tau_u(N) \geq \tau_u(N-1) \geq \cdots \geq \tau_u(1)$.

Corollary 2.11 implies that the feature upgrade policy is more likely to be used at a time when the pledging number is further away from the target.

Lastly, denote by $J_{T,N}^u$ the optimal expected revenue with the option of feature upgrade when the duration is $T$ and the goal is $N$. Following a similar proof as that of Theorem 2.9, we show that the relative difference in expected revenues with and without feature upgrade increases in the duration $T$, but the absolute benefit vanishes as $T$ approaches infinity or zero.

**Theorem 2.12**

(i) For an $N \geq 1$, $\frac{J_{T,N}^u}{J_{T,N}^b}$ decreases in $T$.

(ii) For any $N \geq 1$, $\lim_{T \to \infty} J_{T,N}^u - J_{T,N}^b = \lim_{T \to 0} J_{T,N}^u - J_{T,N}^b = 0$.

When the duration is sufficiently long, the chance that the project will be successfully funded is high, and that eliminates any incentive for the project creator to upgrade the project features. When the duration is very short, a project upgrade will affect decisions by only a negligible fraction of backers. Consequently, the stimulus will bring only a limited benefit. The implication of Theorem 2.12(ii) is that the benefit of a feature upgrade is greatest when the project duration is moderate. We further confirm this finding numerically in Section 4.5 and empirically in Section 2.5.
2.4.3 Limited-Time Offer (LTO)

Because of the cascade effect on backers’ pledging decisions, it is important to encourage backers to pledge early in the process. One way to achieve this is to introduce a limited-time offer (LTO), such as free T-shirts, to those who pledge early. In the context of Example 2.1, the creator may offer products of higher quality $\hat{\theta}$ for the same price $p$ to early arrivals. The creator may choose to end the LTO and switch back to normal quality $\theta$ whenever the momentum is established. The use of limited-time offers is prevalent in a wide range of industries, especially when new products are being introduced to the market. In this subsection, we seek to quantify the value of LTOs in the context of crowdfunding, and discuss related issues.

LTO differs from the preceding two policies, namely seeding and feature upgrade, in one important aspect: LTO is a proactive policy in which the creator induces early pledging by making the project more attractive at the beginning, whereas seeding and feature upgrade policies are reactive in the sense that the creator responds to the progress of the pledging, and chooses to apply the policies only if the number of early pledgers is low. As a result, the optimal use of an LTO differs inherently from that of those two policies.

For the creator, there is an increase in the marginal cost for each unit purchased by backers during an LTO, which we denote by $k$. Compared with feature upgrade, the promotional product being offered during an LTO is typically a standard version of the product plus some extra gifts. Thus, the creator can conveniently stop the LTO and switch back to the standard product. During an LTO, backers’ pledging likelihood increases to $\hat{H}(q)$, whereas that corresponding to the normal quality level is $H(q)(\leq \hat{H}(q))$ for any likelihood of success $q$. In contrast, feature upgrade typically involves a permanent upgrade of certain characteristics of the product, e.g., making a proposed smart watch waterproof. Thus a fixed cost is incurred for producing the superior product.

**Theorem 2.13 (Optimal Cutoff for LTO)** For any $n$, there exists a cutoff time $\tau^l(n)$, such that the creator will end the limited-time offer if and only if $t \geq \tau^l(n)$.

Theorem 2.13 shows that, for any pledges needed $n$, there exists a cutoff in time $\tau^l(n)$ such that the creator should end the LTO if and only if the time remaining before the end of the project is greater than or equal to this cutoff. In other words, if the project has already attracted a large number of pledgers while the remaining time is long, the creator can end the LTO immediately to enjoy a lower unit cost without jeopardizing the project’s success. However, if the remaining time is short, in particular if it is less than the cutoff time $\tau^l(n)$, the creator is better off continuing the LTO. The profit margin for
each backer is lower in such circumstances; however it is compensated for by a higher chance of reaching the target. It is not surprising that the benefit of LTOs, another type of stimulus policy, also vanishes as \( t \) approaches either infinity or zero, as does the benefit of the other two policies. The result is summarized as follows, where \( J_{T,N}^l \) is the optimal expected revenue with the option of an LTO when the duration is \( T \) and the goal is \( N \).

\[ \text{Theorem 2.14} \text{ For any } n \geq 1, \lim_{T \to \infty} J_{T,N}^l - J_{T,N}^b = \lim_{T \to 0} J_{T,N}^l - J_{T,N}^b = 0. \]

### 2.4.4 Heuristics and Asymptotic Properties

Due to the complexity involved in computation of time thresholds, it is desirable to have a heuristic for the optimal stopping time that is easy to implement and also performs well. In RM settings when there are a finite number of allowable prices, Gallego and Van Ryzin (1994) and Feng and Gallego (1995) show that a simple static heuristic, derived from the deterministic counterpart of the problem, could be asymptotically optimal, as the arrival rates or the sales horizon, and the number of capacity are scaled up. In the same spirit, we investigate whether a heuristic of a predetermined, fixed stopping time, derived from the deterministic counterpart, would perform well in the context of crowdfunding.

In particular, we consider a series of problems where, in the \( m \)-th problem, the demand rate \( \lambda_t^{(m)} = m \lambda_t \) and the goal \( N^{(m)} = mN \). We denote by \( \tau_{\sigma,h} \) the static heuristic stopping time for stimulus policy \( \sigma \), where \( \sigma \) denotes one of the three stimuli, with \( s \) for seeding, \( u \) for feature upgrade, and \( l \) for LTO. More specifically, for seeding and feature upgrade, \( \tau_{s,h} \) and \( \tau_{u,h} \) are the predetermined times to execute the two stimulus policies, and \( \tau_{l,h} \) is the predetermined time to end the LTO. In addition, for the seeding policy, we study the case where the creator has the option to hand out \( mn_0 \) free samples once in the \( m \)-th problem. For feature upgrade, we study the case where the fixed upgrade cost is \( mK \) in the \( m \)-th problem. With a slight abuse of notation, we denote the expected revenue under the static heuristic and the optimal expected revenue with the option of the corresponding contingent stimulus in the \( m \)-th problem by \( J_{m}^{\sigma,h} \) and \( J_{m}^{\sigma,*} \), \( \sigma = \{s,u,l\} \), respectively.

For seeding and feature upgrade, we construct the following heuristic, which is derived from the deterministic counterpart. Denote the total cumulative arrival rate by \( \Lambda \equiv \int_0^T \lambda_s ds \). The creator will use stimulus policy \( \sigma \) with the cutoff time \( \tau_{\sigma,h} = \begin{cases} 0 & \text{if } \Lambda H(1) > N \\ T & \text{if } \Lambda H(1) \leq N \end{cases} \), where \( \sigma \in \{s,u\} \). That is, under such a heuristic, the policies are implemented immediately if the expected demand is less than or equal to the goal
In the context of the seeding policy there can be a slight modification to this heuristic that is more beneficial. That is, if in the last minute the goal has not yet been reached, the creator should offer free samples, thereby effectively lowering the target. In the deterministic setting, if \( \Lambda H(1) > N \), no stimulus is needed. However, in the stochastic setting, if there are not enough pledgers before the time expires, offering free samples in the last minute may turn the project from failure to success.
immediately, i.e., not use LTO at all, because, even if the LTO leads to a higher pledging likelihood from backers, the revenue with the LTO is equal to zero due to \( p = k = 1 \). Therefore, the optimal expected revenue in the stochastic problem is given by \( 1 - e^{-H(1)T} \). For the deterministic problem, the optimal stopping time for the LTO is given by \( x^* = \frac{\hat{H}(1)T - 1}{H(1) - H(1)} \) and the corresponding revenue is \( 1 - e^{-\frac{\hat{H}(1)T - 1}{H(1) - H(1)}} \).

With \( T = \frac{1}{H(1)} \), we have \( J_{l,d}^* \leq J_{l,d}^* \), for any \( m \), may not be true if the condition \( (p - k)\hat{H}(1) > pH(1) \) fails to hold. This is in stark contrast with the classic RM settings.

The reason why the optimal revenue of the deterministic problem may be lower than its stochastic counterpart is that, for small-scale projects with a low funding target, the uncertainty in backers’ arrivals and valuations may actually help the project, especially when LTOs are costly. In the stochastic setting, due to the high cost of LTOs, the creator never uses it; even though the success likelihood is low, the revenue could be high should the project indeed succeed. On the contrary, in the deterministic problem, the creator has no choice but to use LTOs in order to reach the target, even if it is costly. As a result, \( J_{l,s}^* \leq J_{l,d}^* \), for any \( m \), may not be true if the condition \( (p - k)\hat{H}(1) > pH(1) \) fails to hold. This is in stark contrast with the classic RM settings.

Like the case of seeding and feature upgrade, we show that the optimal revenue from the stochastic problem is asymptotically the same as that from the deterministic problem for LTO (see Proposition 2.19 in the appendix). Now we examine the performance of the static heuristic by comparing its expected revenue \( J_{l,h}^* \) in the stochastic problem with the optimal revenue \( J_{l,d}^* \) of the deterministic problem. In fact, we obtain a result for any asymptotically optimal static heuristic.

**Theorem 2.15** (Asymptotically Optimal Heuristics for Limited Time Offer) Consider \( \Lambda H(1) < N < \Lambda \hat{H}(1) \). For any static heuristic \( J_{l,h}^* \) in which LTO ends at a fixed time, if it is asymptotically optimal, i.e., \( \lim_{m \to \infty} J_{l,h}^* = 1 \), then \( \lim_{m \to \infty} \sqrt{m} \cdot \left( 1 - \frac{J_{l,h}^*}{J_{l,d}^*} \right) = \infty \).

It is interesting to compare Theorem 2.15 with the asymptotically optimal heuristics in RM. Gallego and Van Ryzin (1994) show that a simple heuristic that changes the price at a fixed time, independently of the scale \( m \), could be asymptotically optimal for the classic RM problem. However, because of the all-or-nothing nature of crowdfunding, we show in the proof of Theorem 2.15 that a well-performing static heuristic must add buffer time to the deterministic heuristic; i.e., LTOs should generally be ended at a time later than the optimal stopping time \( x^* \) derived from the deterministic problem. The difference

\[ 5 \] If this condition fails, the design of the static heuristic becomes trivial: either LTOs are never used or they used from the beginning to the end.
between \( x^* \) and the optimal stopping time in the stochastic problem also depends on the scale parameter \( m \). When \( m \) increases, demand variability decreases, and thus the safety buffer required to guarantee a high success rate for a project also decreases. The optimal stopping time in the stochastic problem converges to \( x^* \) as \( m \) approaches infinity.

More importantly, profit losses resulting from deterministic heuristics also differ considerably between the settings of crowdfunding and RM. The asymptotically optimal heuristic that changes prices at a fixed time in the setting of RM yields a profit loss with an order of \( \sqrt{m} \) (e.g., see Gallego and Van Ryzin 1994, Feng and Gallego 1995). However, Theorem 2.15 shows that the profit loss has an order of magnitude higher than \( \sqrt{m} \) in crowdfunding, which is also driven by the all-or-nothing mechanism of crowdfunding and the resulting cascade effect. This result underscores the importance of using contingent policies and implies that the benefit has an order higher than \( \sqrt{m} \). In summary, on one hand, static heuristics, which are easy to compute, can perform well for large-scale systems, e.g., when the arrival rates are large. On the other hand, in small-scale systems, static heuristics can fall short and contingent policies are more desired.

2.4.5 Numerical Examples

We further investigate the effectiveness of various stimulus policies for small scale systems with numerical experiments. We consider the setup as described in Example 2.1, where the creator can make the project more attractive by improving the quality of the project. The parameters in the numerical experiments are specified as follows. A backer’s valuation \( v \) is drawn from an exponential distribution with mean of $100. The contribution \( p \) required from each backer is $120, the quality level \( \theta \) of the project is 1, and the penalty cost \( c \) for each consumer if the project fails to reach its target is $30. The goal \( X \) of the project is set to be $1,500, which is equivalent to requiring at least \( N = 15 \) pledgers. The duration of the campaign is 30 days, and the arrival rate \( \lambda_t \) is assumed to be 2 potential pledgers per day.

Using Proposition 2.2 and backward induction, we can compute the success likelihood \( Q_t(n) \) without any contingent stimulus policy. The result is displayed in Figure 2.2(a), which is consistent with our empirical observation (see Figure 2.4(a) below). The expected success rate right after the project launch is 34.3%. Of course, whether this project indeed succeeds by the end of the campaign depends on the realized sample path, especially the number of pledgers appearing in the early stage of the crowdfunding campaign, due to the cascade effect. For instance, if 5 backers pledge during the first 5 days, then the project’s likelihood of success increases to over 87%. On the other hand, that
drops to nearly zero if nobody pledges during the first 5 days. The latter case is when the project creator may be able to save the project with stimulus policies. The thresholds $\hat{t}(n)$ in Theorem 2.3(v) can also be computed. For example, $\hat{t}(5) = 16.6$. This implies, with 5 pledgers needed, if the time-to-go is less than 16.6 days, the success likelihood in the stochastic problem with random arrivals is strictly larger than that in its deterministic counterpart with certain arrivals.

![Graph](image_url)

Figure 2.2: A Numerical Result illustrating Benefits of Stimulus Policies

Next, we evaluate the optimal expected profit under each of the three policies referred to in the preceding subsections. The results are shown in Figure 2.2(b). Here, we assume that the creator is able to improve the project’s quality level to $\hat{\theta} = 1.5$ with a cost of $K = $500 under the feature upgrade policy. Alternatively, the creator is able to offer products at the higher quality level $\hat{\theta} = 1.5$ to early arrivals with an LTO at an additional cost of $k = $30 per unit. From Figure 2.2(b), we first observe that benefits of stimulus policies are not monotone in the duration of projects, given the same target $N = 15$. Consistent with our theoretical results, when the project duration is short (i.e., $T < 15$), the benefit of stimulus policies is marginal because projects are likely to fail no matter what policies the project creator uses to attract backers. On the other end of the spectrum, when there is ample time (i.e., $T > 35$), project are highly likely to succeed even without stimulus. The benefit of stimulus policies is most salient with a moderate project duration (i.e., $15 \leq T \leq 35$ for this particular example). In other words, for those projects that have potential but are not overwhelmingly popular, offering stimulus at the right time could help tremendously. For instance, let us compare the results with
and without stimulus when $T = 30$, which is the duration of the crowdfunding campaign in our initial setup. The expected profit without any stimulus policy is $618. With the optimal seeding policy, the expected profit increases to $1,012$, i.e., a 63.8% increase benchmarked with the expected profit without stimulus. Similarly, the expected profits increase by 115.9% and 139.6% with the optimal feature upgrade and LTOs respectively.

Table 2.1: Statistics of Simulated Profit from 385 Scenarios (in $)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base model without stimulus</td>
<td>1,631</td>
<td>2,448</td>
<td>0</td>
<td>9,391</td>
</tr>
<tr>
<td>Optimal contingent policy</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeding</td>
<td>2,875</td>
<td>2,837</td>
<td>0</td>
<td>9,978</td>
</tr>
<tr>
<td>Feature upgrade</td>
<td>3,911</td>
<td>2,154</td>
<td>0</td>
<td>9,529</td>
</tr>
<tr>
<td>LTO</td>
<td>4,518</td>
<td>2,506</td>
<td>0</td>
<td>10,028</td>
</tr>
<tr>
<td>Static heuristics</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeding</td>
<td>1,634</td>
<td>2,447</td>
<td>0</td>
<td>9,389</td>
</tr>
<tr>
<td>Feature upgrade</td>
<td>2,190</td>
<td>2,513</td>
<td>0</td>
<td>9,391</td>
</tr>
<tr>
<td>LTO</td>
<td>1,696</td>
<td>2,421</td>
<td>0</td>
<td>9,408</td>
</tr>
<tr>
<td>Static heuristics with 5% buffer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeding</td>
<td>1,644</td>
<td>2,441</td>
<td>0</td>
<td>9,391</td>
</tr>
<tr>
<td>Feature upgrade</td>
<td>2,478</td>
<td>2,494</td>
<td>0</td>
<td>9,410</td>
</tr>
<tr>
<td>LTO</td>
<td>1,855</td>
<td>2,366</td>
<td>0</td>
<td>9,408</td>
</tr>
</tbody>
</table>

We further verify the effectiveness of the optimal contingent policies and the static heuristics through a comprehensive simulation. We set the arrival rate $m\lambda$ and the goal $mN$, where $\lambda$ ranges from 1.5 and 2.5, $N$ ranges from 12 and 18 and $m$ ranges from 1 to 5. We generate 385 scenarios and compare the profits with stimulus against that in the base model without stimulus. The results are summarized in Table 2.1. Compared with the base model, the static heuristics lead to an increase in profit of 13% on average. However, consistent with Theorem 2.15, contingent stimulus policies are substantially more effective, averaging over 117% increase in profit. In addition, we consider a modified static heuristic that requires the same amount of computational effort as the static heuristic. More specifically, we add a buffer to the cutoff time to increase the likelihood of applying the stimulus policies. That is, for the scenario where the goal is $N$, we use the cutoff time corresponding to the goal $N(1+\alpha)$, where $\alpha > 0$ is a buffer parameter. In Table 2.1, we also show the profits of static heuristics with a buffer parameter of $\alpha = 5\%$. This modified static heuristic improves profit by 8% on average compared to static heuristics without a buffer. Nevertheless, contingent policies still
outperform this modified static heuristics by a significant margin. To summarize, both our analytical and simulation results underscore the importance of contingent policies for small-scale crowdfunding settings.

### 2.5 Empirical Evidence

We built a data crawler on the Google App Engine platform to collect data from Kickstarter between January 30 and June 27, 2015. Whenever a new project was posted, the data crawler extracted static project information, such as the project name, goal and campaign duration. It also kept track of the pledging in terms of the intertemporal number of pledgers, cumulative pledged amount, project creators’ updates and backers’ comments whenever there was any change to the project. This real-time data set allows us to uncover the pledging patterns, as well as the impact of the creators’ updates.

![Figure 2.3: Funding Ratio Distribution](image)

In total, our data includes 21,657 Kickstarter projects. Table 2.2 shows the summary

<table>
<thead>
<tr>
<th>Project Attributes</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal ($)</td>
<td>67,009</td>
<td>1,401,462</td>
<td>1</td>
<td>100,000,000</td>
</tr>
<tr>
<td>Funding ratio</td>
<td>1.90</td>
<td>99.93</td>
<td>0</td>
<td>12,984</td>
</tr>
<tr>
<td>Duration (days)</td>
<td>33.63</td>
<td>11.66</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td># of updates per week</td>
<td>0.69</td>
<td>1.40</td>
<td>0</td>
<td>27.30</td>
</tr>
</tbody>
</table>
statistics for all of those projects. The average project target in the sample was $67,009.\(^6\)

The average crowdfunding campaign duration was 33.63 days. We compute the funding ratio as the total pledged amount to the target. As shown in Figure 2.3, although 1,110 projects managed to collect over 200% of the goals, the majority of successful projects collected no more than 120% of their goals. (This is consistent with our treatment of not focusing on overfunding in the analytical model.) Project creators are allowed to make changes to their projects over the course of their crowdfunding campaign. On average, project creators updated their projects 0.69 times per week.\(^7\) We also observe significant variations in project update frequencies in our sample, ranging from 0 to 27.30 times per week. This variation allows us to study the effect of project updates on the project’s likelihood of success.

![Figure 2.4](image)

(a) Success Rate by Time-to-Go

(b) Success Rate by Time-to-Go and Pledges Needed

Figure 2.4: Average Project Success Rate as a Function of Time-to-Go and Pledges Needed

We first display the project’s success rate as a function of time-to-go and pledges needed by investigating the trajectories of all projects in the sample. Specifically, we break down time-to-go and pledges needed of each project into 10 stages, i.e., 0 – 10%, 10% – 20%, …, 90% – 100%, and compute the average success rate for projects that fall into the same time stage and pledge stage. The results are summarized in Figure 2.4. The

\(^6\)Project targets may be in different currencies depending on where the project creators were located. We ignore the differences and assume that they were all measured in dollars.

\(^7\)Updates may or may not be one of the stimulus strategies considered in the paper. We do not differentiate them in this empirical study.
first observation is that, on average, a project is less likely to succeed with either a shorter remaining time given the same pledges needed, or a higher amount required to reach the target given the same time-to-go. This is consistent with our theoretical results on the pledging likelihood function $Q_t(n)$, as shown in Theorem 2.3. The empirical evidence also shows the importance of maintaining the momentum of the pledging, especially at the beginning of a campaign. For instance, Figure 2.4(b) shows that over 94% of projects will fail if they do not secure at least 10% of their goal after one-fifth of the time has passed. Secondly, we see from Figure 2.4(a) that the probability that a newly launched project will eventually reach its goal is around 33%, which is about the same as the expected success of the project shown in the numerical example in Section 4.5. In other words, in terms of the funding probability, our numerical example is a “typical” project, and the effectiveness demonstrated in the numerical experiments further lends some credibility to the importance of stimulus policies in practical settings.

Table 2.3: Number of Updates in Successful and Failed Projects Per Week

<table>
<thead>
<tr>
<th></th>
<th>Project Count</th>
<th>Mean</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Successful projects</td>
<td>6089</td>
<td>1.136</td>
<td>0.0179</td>
</tr>
<tr>
<td>Failed projects</td>
<td>15568</td>
<td>0.186</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

Next we study the effect of the creator’s updates on the project’s likelihood of success. The effectiveness of creators’ updates is supported by our data as well. We find that, on average, successful projects made 1.136 updates per week, while failed ones made only 0.186 and the difference is statistically significant (see Table 2.3).

Figure 2.5: Difference in Success Rates with and without Updates
It is extremely complicated to quantify the exact benefit of updates because of data and identification issues. On the data side, the nature of updates, whether it is seeding, feature upgrade or LTO, may be hard to classify. The identification could also be challenging because the difference in the number of updates may be a reflection of the creators’ intrinsic motivation, which also affects campaign outcomes. A rigorous full-scale econometric model is beyond the scope of this paper. However, we provide some model-free evidence which demonstrates the importance of update timings. We divide campaigns along the time dimension into three stages of equal length: early, middle and late. Similarly, using the ratio of the pledged amount to the project’s target, we divide campaigns along the pledging-ratio dimension into three different stages, namely, initial, middle and final. We then investigate the effect of updates by comparing the outcomes of the projects for which creators made updates and the projects without updates in each of the nine categories. The results are summarized in Figure 2.5. In general, projects with updates have, on average, higher success likelihood across all nine categories. The difference is greatest in the middle of a crowdfunding campaign and in the initial stage when the pledging amount is falling behind. In this scenario, the average success rate increases from 11.9% to 26.5% with updates. This scenario is consistent with our theoretical results in Theorems 2.7 and 2.10, where we show that when applying stimuli, it is optimal to do so only if the pledging slows down but not when the pledging is going smoothly. Moreover, the benefit in this particular scenario as the greatest is consistent with our results in Theorems 2.9 and 2.12, where we show that the benefit of stimuli is most significant when the time-to-go is in an intermediate range.

2.6 Conclusion

Archimedes once said “give me a fulcrum, and I shall move the world.” In this paper, we study the optimal timing of contingently placing a “fulcrum” in the context of crowdfunding, with the potential of tilting the random pledging process from failure to success. In particular, we evaluate three different policies in detail, namely, seeding, feature upgrade and limited-time offer. The three policies seek to encourage backers’ pledging in different ways. Seeding directly interacts with the pledging process by reducing the number of pledgers needed to reach the target and making the project more promising for future arrivals. With feature upgrade, project creators offer a superior version of the product with the hope of attracting more backers. This upgraded product is offered to future arrivals, as well as those who have already pledged. On the other hand, limited-time offer seeks to exploit the cascade effect in the pledging process by using promotional prod-
ucts to encourage backers to pledge early. However, unlike feature upgrade, promotional products are offered only during the LTO period.

Our analysis provides useful guidance on when and how project creators should apply these policies. First, we show that the potential benefits of the three policies vanish when the remaining time approaches either infinity or zero. It implies that these policies would be most effective in the middle of the pledging process. This is also consistent with the contingent nature of these policies. That is, project creators may want to “wait and see” and implement them only when the pledging trajectory is unsatisfactory at the beginning. Second, in practice, it is tempting to use static heuristics when it comes to the implementation of these policies, because of their simplicity. However, we use the limited-time offer policy as an example, and show that profit losses from static heuristics in crowdfunding are much larger than losses from deterministic heuristics in RM. Thus, a static heuristic may not perform well in the context of crowdfunding, especially when the scale of the problem (e.g., the predetermined funding goal) is small. This underscores the necessity of implementing contingent policies.

Our study serves as the first step towards an understanding of the dynamics of crowdfunding projects. Future research may take into account possible strategic waiting by pledgers. Though Ming and Tunca (2016) empirically show that strategic waiting behavior is not significant in group buying, backers’ behaviors may be different in crowdfunding. In the context of LTO, the pledgers do not have an incentive to wait because the early-bird bonus may not be available later. However, in the case of seeding and feature upgrading, the pledgers do have an incentive to wait after they arrive. Strategic waiting behavior may demand an earlier use of seeding and feature upgrading than what we characterize, to deter strategic waiting. Future research may also consider other types of information uncertainty beyond the project’s likelihood of success and may investigate their influence on the pledging dynamics. For instance, another salient concern from consumers is whether and when project creators will successfully deliver the products (Mollick and Kuppuswamy 2014). This type of information asymmetry and uncertainty may affect backers’ pledging decisions even after the target is reached. To assure backers, it might be beneficial for the creators to deposit part of the funding beforehand, as a way to signal the quality of their products. On the empirical side, we are collaborating with Kickstarter to build a toolbox that predicts the success rate of the projects and make suggestions to project creators of creating updates during the campaign. We will adopt the framework in Proposition 2.2 and empirically estimate the arrival rate and the pledge likelihood.
2.7 Proofs

Proof of Lemma 2.1. (i) Taking derivative of $H(q)$ w.r.t. $q$, we have
\[ \frac{dH(q)}{dq} = \frac{c}{\theta q^2} f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right) > 0. \]

(ii) Assumption (ii) is guaranteed by the fact that the support of the distribution $F(\cdot)$ is unbounded.

(iii) We prove Assumption (iii) by contradiction. Taking derivative of $H(\alpha q)/H(q)$ w.r.t. $q$, we have
\[ \frac{d}{dq} \left( \frac{H(\alpha q)}{H(q)} \right) = \frac{H(\alpha q)}{H(q)} \left[ \frac{c}{\theta \alpha q^2} \frac{f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{\alpha q} - 1 \right)) \right)}{F \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{\alpha q} - 1 \right)) \right)} - \frac{c}{\theta q^2} \frac{f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right)}{F \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right)} \right]. \]

Suppose there exists a $q'$ such that $\frac{d}{dq} \left( \frac{H(\alpha q')}{H(q')} \right) \leq 0$, which implies that $\frac{f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q'} - 1 \right)) \right)}{F \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q'} - 1 \right)) \right)} \leq \frac{f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right)}{F \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right)}$. Coupling with the IGFR property that $1 \frac{d}{dq} \left[ p + c \cdot \left( \frac{1}{q} - 1 \right) \right] f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{q} - 1 \right)) \right) \geq 1 \frac{d}{dq} \left[ p + c \cdot \left( \frac{1}{\alpha q} - 1 \right) \right] f \left( \frac{1}{\bar{\theta}} (p + c \cdot \left( \frac{1}{\alpha q} - 1 \right)) \right)$, we have $p + c \cdot \left( \frac{1}{\alpha q} - 1 \right) \geq \frac{1}{\alpha} \left[ p + c \cdot \left( \frac{1}{q} - 1 \right) \right]$. A direct consequence of the preceding inequality is that $(p - c) \geq \frac{p - c}{\alpha}$, which contradicts with $0 < \alpha < 1$ and $p > c$. Thus, we obtain the desired result. ■

Proof of Proposition 2.2. Suppose that a backer arrives with time-to-go $t > 0$ and pledges needed $n \geq 1$. This focal backer would decide whether or not to pledge based on her expected project's success rate conditional on her pledging, i.e., $Q_t(n - 1)$. Consider what happens in a small time interval $\delta$, and we have
\[ Q_t(n) = (1 - \delta \lambda_t H(Q_t(n - 1))) \cdot Q_{t-\delta}(n) + \delta \lambda_t H(Q_t(n - 1)) \cdot Q_{t-\delta}(n - 1) + o(\delta). \]
Rearranging and taking the limit as $\delta \to 0$, we obtain Equation (2.2). With the boundary conditions, the solution to Equation (2.2), which is an ordinary differential equation solved by induction, is unique. ■

Proof of Theorem 2.3. (i) We prove this by induction. First when $n = 1$, because $Q_t(0) = 1$, it is easy to verify that $Q_t(1) = 1 - \exp \left( - \int_0^t \lambda_s H(1) ds \right)$ is the unique solution of Equation (2.2). Hence $Q_t(1)$ increases in $t$, and $Q_t(1) < Q_t(0)$.
Now assume the statement is true for $n - 1$ ($n \geq 2$), then for $n$:
\[ \frac{\partial}{\partial t} \left[ Q_t(n-1) - Q_t(n) \right] = \lambda_t \left[ H(Q_t(n-2))(Q_t(n-2)-Q_t(n-1)) - H(Q_t(n-1))(Q_t(n-1)-Q_t(n)) \right]. \]
Since $Q_t(n-2) - Q_t(n-1) > 0$, we have
\[
\frac{\partial}{\partial t} \left[ Q_t(n-1) - Q_t(n) \right] > 0.
\]
Based on Grönwall’s Inequality and the fact that $Q_t(n-1) - Q_t(n)|_{t=0} = 0$, we have
\[
Q_t(n-1) - Q_t(n) > 0
\]
for any $t > 0$. This also implies that $\frac{\partial Q_t(n)}{\partial t} > 0$. Therefore
\[
\text{the statement is also true for } n.
\]

(ii) The inequality is equivalent to
\[
\frac{Q_t(n)}{Q_t(n-1)} \leq 1 - e^{-\lambda t}.
\]
Consider the function $e^{\lambda t}Q_t(n)$. Taking the derivative w.r.t. $t$, we have
\[
\frac{\partial(e^{\lambda t}Q_t(n))}{\partial t} = \lambda e^{\lambda t}Q_t(n) + e^{\lambda t}\frac{\partial Q_t(n)}{\partial t} \leq \lambda e^{\lambda t}Q_t(n) + \lambda e^{\lambda t}[Q_t(n-1) - Q_t(n)] = \lambda e^{\lambda t}Q_t(n-1),
\]
where the inequality is due to $\frac{\partial Q_t(n)}{\partial t} > 0$ and $\frac{\partial Q_t(n)}{\partial t} \leq \lambda [Q_t(n-1) - Q_t(n)]$, as implied by Equation (2.2).

Integrating from 0 to $t$ on both sides, we have
\[
Q_t(n) \leq \int_0^t \lambda e^{-\lambda(t-s)}Q_s(n-1)ds \leq \lambda Q_t(n-1) \int_0^t e^{-\lambda(t-s)}ds = (1 - e^{-\lambda t})Q_t(n-1).
\]
where the second inequality is due to the increasing monotonicity of $Q_t(n-1)$ in $t$ as shown in Theorem 2.3(i). Therefore, we conclude that $\frac{Q_t(n)}{Q_t(n-1)} \leq 1 - e^{-\lambda t}$.

(iii) We will prove that $\frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in $t$ and $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in $t$ by induction. Consider first when $n = 1$. Because $\frac{Q_t(1)}{Q_t(0)} = Q_t(1)$ and $\frac{H(Q_t(1))}{H(Q_t(0))} = \frac{H(Q_t(1))}{H(1)}$, the monotonicity is guaranteed by part (i) and Assumption 1(ii). Now assume that the monotonicity in $t$ holds for $n - 1$. We next show that $r_t(n) = \frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in $t$ and $\varphi_t(n) = \frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in $t$. First from part (i), we observe that $0 < r_t(n) < 1$ for $t > 0$. Taking the derivative of $r_t(n)$ w.r.t. $t$, we have
\[
\frac{\partial r_t(n)}{\partial t} = \frac{\lambda_t H(Q_t(n-1)) [Q_t(n-1) - Q_t(n)] - Q_t(n) \lambda_t H(Q_t(n-2)) [Q_t(n-2) - Q_t(n-1)]}{Q_t(n-1)} \leq \lambda_t \left[ \frac{H(Q_t(n-1))}{Q_t(n-1)} \left( \frac{Q_t(n)}{Q_t(n-1)} - 1 \right) - \frac{Q_t(n)}{Q_t(n-1)} \frac{H(Q_t(n-2))}{H(Q_t(n-1))} \left( \frac{Q_t(n-1)}{Q_t(n-2)} - 1 \right) \right]
\]
Suppose that there exists some $t_1$ such that $\frac{\partial r_t(n)}{\partial t} \bigg|_{t=t_1} \leq 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial r_t(n)}{\partial t} \bigg|_{t=t_2} > 0$. Otherwise, if $\frac{\partial r_t(n)}{\partial t} \leq 0$ for all $t < t_1$, then $\lim_{t \to 0} r_t(n) = 0 \geq r_t(n)$, which contradicts with the fact that $Q_t(n) > 0$. Due to the
continuity of $\frac{\partial r_t(n)}{\partial t}$, there exists some $t_3 \in [t_2, t_1]$, such that $\frac{\partial r_t(n)}{\partial t}|_{t=t_3} = 0$. That is,

$$\varphi_{t_3}(n - 1) \left( \frac{1}{r_{t_3}(n)} - 1 \right) - \left( \frac{1}{r_{t_3}(n-1)} - 1 \right) = 0.$$ 

Because $\varphi_t(n-1)$ strictly increases in $t$ and $r_t(n-1)$ increases in $t$, and $r_t(n)$ decreases in $t$ between $[t_3, t_1]$, we have

$$\varphi_{t_1}(n - 1) \left( \frac{1}{r_{t_1}(n)} - 1 \right) - \left( \frac{1}{r_{t_1}(n-1)} - 1 \right) > \varphi_{t_3}(n - 1) \left( \frac{1}{r_{t_3}(n)} - 1 \right) - \left( \frac{1}{r_{t_3}(n-1)} - 1 \right) = 0,$$

which implies that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1} > 0$. However, this contradicts with the preceding statement that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1} \leq 0$. Therefore, we conclude that $\frac{\partial r_t(n)}{\partial t} > 0$ for any $t > 0$.

Next we show that $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in $t$. For any $t' > t$, we have

$$H(Q_{t'}(n)) = H \left( \frac{Q_{t'}(n)}{Q_{t'}(n - 1)} Q_{t'}(n - 1) \right) \geq H \left( \frac{Q_t(n)}{Q_t(n - 1)} Q_t(n - 1) \right),$$

where the inequality is due to the increasing monotonicity of $\frac{Q_t(n)}{Q_t(n - 1)}$ in $t$ and Assumption 1(i). Due to Assumption 1(iii) and Theorem 2.3(i), we have

$$\frac{H(Q_{t'}(n))}{H(Q_{t'}(n - 1))} \geq \frac{H \left( \frac{Q_t(n)}{Q_t(n - 1)} Q_t(n - 1) \right)}{H(Q_t(n - 1))} = \frac{H(Q_t(n))}{H(Q_t(n - 1))}.$$

We hence prove the increasing monotonicity of $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ in $t$.

For the monotonicity in $n$, because we have shown that $\frac{\partial r_t(n)}{\partial t} > 0$ for any $t > 0$, $\varphi_t(n-1) \left( \frac{1}{r_t(n)} - 1 \right) - \left( \frac{1}{r_t(n-1)} - 1 \right) > 0$. Since $\varphi_t(n-1) \leq 1$, we have $r_t(n) < r_t(n-1)$,

i.e., $\frac{Q_t(n)}{Q_t(n-1)} < \frac{Q_t(n-1)}{Q_t(n-2)}$. A direct consequence is that $\frac{H(Q_t(n))}{H(Q_t(n-1))} = \frac{H \left( \frac{Q_t(n)}{Q_t(n-1)} Q_t(n-1) \right)}{H(Q_t(n-1))} < \frac{H \left( \frac{Q_t(n-1)}{Q_t(n-2)} Q_t(n-1) \right)}{H(Q_t(n-1))}$. Due to Assumption 1(iii) and part (i), we have

$$\frac{H(Q_t(n-1))}{H(Q_t(n-2))} \leq \frac{H \left( \frac{Q_t(n-1)}{Q_t(n-2)} Q_t(n-2) \right)}{H(Q_t(n-2))} = \frac{H(Q_t(n-1))}{H(Q_t(n-2))}.$$ 

Therefore, we conclude that $\frac{H(Q_t(n))}{H(Q_t(n-1))} < \frac{H(Q_t(n-1))}{H(Q_t(n-2))}$ for any $t > 0$. 
(iv) For any $n \geq 1$, we have

\[
\frac{Q_{t+h}(n)}{Q_t(n)} = \frac{Q_{t+h}(n)}{Q_{t+h}(n-1)} \cdot \frac{Q_{t+h}(n-1)}{Q_t(n-1)} \cdot \frac{Q_t(n-1)}{Q_t(n)} > \frac{Q_{t+h}(n-1)}{Q_t(n-1)},
\]

where the inequality is due to $\frac{Q_{t+h}(n)}{Q_{t+h}(n-1)} > \frac{Q_t(n)}{Q_t(n-1)}$ as shown in Theorem 2.3(iii). We thus obtain the results.

(v) In the deterministic problem, when $t < \hat{t}$, the project will fail with certainty even if every backer who arrives pledges. Therefore under REE, the success probability is 0. However for the stochastic problem, we have shown that $Q_t(n)$ strictly increases in $t$ for $n \geq 1$. Hence $Q_t(n) > 0$ for any $t > 0$. This completes the proof. ■

Proof of Lemma 2.5 (i) Taking derivative of $\frac{H^a_p(q)}{H^b_p(q)}$ w.r.t. $q$, we have

\[
\frac{\partial}{\partial q} \left( \frac{H^a_p(q)}{H^b_p(q)} \right) = \frac{H^a_p(q)}{H^b_p(q)} \frac{c}{q^2} \left[ f \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \bar{F} \left( \frac{1}{\theta} (pb + c(\frac{1}{q} - 1)) \right) - f \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \right].
\]

Because $\theta_a < \theta_b$ and Assumption 1, we conclude that $\frac{\partial}{\partial q} \left( \frac{H^a_p(q)}{H^b_p(q)} \right) > 0$. Thus, we obtain the announced results.

(ii) Taking derivative of $\frac{H^a_p(q)}{H^b_p(q)}$ w.r.t. $q$, we have

\[
\frac{\partial}{\partial q} \left( \frac{H^a_p(q)}{H^b_p(q)} \right) = \frac{1}{[H^b_p(q)]^2} \left[ \frac{c}{q^2} f \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \bar{F} \left( \frac{1}{\theta} (pb + c(\frac{1}{q} - 1)) \right) - \frac{c}{q^2} \bar{f} \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \right]
\]

\[
= \frac{H^a_p(q)}{H^b_p(q)} \frac{c}{q^2} \left[ f \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \bar{F} \left( \frac{1}{\theta} (pb + c(\frac{1}{q} - 1)) \right) - \bar{f} \left( \frac{1}{\theta} (pa + c(\frac{1}{q} - 1)) \right) \right].
\]

Due to $p_a > p_b$ and that $\frac{\bar{f}(v)}{\bar{F}(v)}$ increases in $v$, we conclude that $\frac{\partial}{\partial q} \left( \frac{H^a_p(q)}{H^b_p(q)} \right) > 0$. ■

Proof of Theorem 2.7. Denote $J^*_t(n)$ as the optimal expected revenue at state $(t, n)$ assuming that the free sample has not been offered. We prove that $\tau^*(n)$ is given by

\[
\tau^*(n) = \sup \left\{ t : H_t((n - n_0)^+) \cdot Q_t((n - n_0 - 1)^+) - \left( H_t((n - n_0)^+) - H_t(n) \right) \cdot Q_t((n - n_0)^+) \right\}
\]

\[
\geq H_t(n) \frac{J^*_t(n-1)}{(N-n_0)p}.
\]

(2.3)

Expected revenue $J^*_t(n)$ at state $(t, n)$ is given by

- when $n \geq 1$ and $t \leq \tau^*(n)$, $J^*_t(n) = (N-n_0)p \cdot Q_t((n-n_0)^+)$;
• when \( t > \tau^s(n) \), \( J^*_t(n) \) is given by

\[
\frac{\partial J^*_t(n)}{\partial t} = \lambda_t H_t(n) \left[ J^*_t(n-1) - J^*_t(n) \right],
\]

(2.4)

with boundary conditions \( J^*_\tau(n) = (N-n_0)p \cdot Q_{\tau^s(n)}((n-n_0)^+) \) and \( J^*_t(0) = Np \).

Denote \( l_t(n) \equiv \frac{J_t^*(n)}{Q_t(n-n_0)^+} \). We add to the statement that \( l_t(n) \) increases in \( t \), and prove by induction. When \( n \leq n_0 \), the optimal expected revenue is given by \( J^*_t(n) = Np \cdot Q_t(n) + (N-n_0)p \cdot (1-Q_t(n)) \). That is, the creator’s optimal policy is to hold off until right before the deadline, and to activate “seeding” if no backer pledges by then. It is not hard to verify that it is the unique solution to the differential equation characterized by Equation (2.4). We thus conclude that \( l_t(n) = J^*_t(n) \) increases in \( t \) for \( n \leq n_0 \).

Assume that the statement is true for \( n - 1 \), where \( n \geq n_0 + 1 \). Next, we seek to derive \( J^*_t(n) \) by showing that the creator’s optimal policy is to “seed” immediately when \( t \leq \tau^s_t(n) \) and to hold off when \( t > \tau^s_t(n) \). We can rewrite the inequality within the curly brackets in Equation (2.3) as follows.

\[
1 + \left( \frac{H_t(n-n_0)}{H_t(n)} - 1 \right) \left( 1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0-1)} \right) \geq \frac{J^*_t(n-1)}{(N-n_0)p \cdot Q_t(n-n_0-1)}.
\]

RHS of the inequality increases in \( t \) because \( l_t(n-1) \) increases in \( t \), while LHS decreases in \( t \) due to Theorem 2.3(iii). Therefore, for any \( t \leq \tau^s(n) \), the inequality within the curly brackets in Equation (2.3) holds; whereas the direction of the inequality is flipped for any \( t > \tau^s(n) \).

Suppose there exists some \( t_1 > \tau^s(n) \) such that the creator’s optimal policy is to offer the free samples immediately, i.e., \( J^*_t(n) = (N-n_0)p \cdot Q_{t_1}(n-n_0) \). Comparing the case without offering the free samples at time \( t_1 \), we have

\[
J^*_t(n) \geq \lambda_{t_1} H_{t_1}(n) \delta \cdot J^*_{t_1-\delta}(n-1) + (1 - \lambda_{t_1} H_{t_1}(n) \delta) \cdot J^*_{t_1-\delta}(n) + o(\delta) \\
\geq \lambda_{t_1} H_{t_1}(n) \delta \cdot J^*_{t_1-\delta}(n-1) + (1 - \lambda_{t_1} H_{t_1}(n) \delta) \cdot (N-n_0)p \cdot Q_{t_1-\delta}(n-n_0) + o(\delta).
\]

Plugging \( Q_{t_1}(n-n_0) = (1 - \lambda_{t_1} H_{t_1}(n-n_0) \delta) \cdot Q_{t_1-\delta}(n-n_0) + \lambda_{t_1} H_{t_1}(n-n_0) \delta \cdot Q_{t_1-\delta}(n-n_0-1) + o(\delta) \) into \( J^*_t(n) \) in the inequality above, rearranging and taking the limit as \( \delta \to 0 \), we have

\[
(N-n_0)p \left[ H_{t_1}(n-n_0)Q_{t_1}(n-n_0-1) - (H_{t_1}(n-n_0)-H_{t_1}(n))Q_{t_1}(n-n_0) \right] \geq H_{t_1}(n)J^*_t(n-1).
\]

This contradicts with the fact that \( t_1 > \tau^s(n) \). Therefore, the creator’s optimal policy
is to hold off when \( t > \tau^s(n) \), i.e., \( J^*_t(n) > (N-n_0)p \cdot Q_t(n-n_0) \). Consider what happens in a small time interval \( \delta \), we have

\[
J^*_t(n) = (1-\delta \lambda_t H_t(n)) \cdot J^*_{t-\delta}(n) + \delta \lambda_t H_t(n) \cdot J^*_{t-\delta}(n-1) + o(\delta).
\]

Rearranging and taking the limit as \( \delta \to 0 \), we obtain Equation (2.4).

We next show that the creator’s optimal policy is to “seed” immediately when \( t < \tau^s(n) \). Suppose that there exists some \( t_2 < \tau^s(n) \), such that \( J^*_t(n) = (N-n_0)p \cdot Q_t(n-n_0) \) for any \( t \leq t_2 \), and \( J^*_t(n) > (N-n_0)p \cdot Q_t(n-n_0) \) when \( t \in (t_2, t_2+h] \). (Because \( J^*_0(n) = 0 \) for any \( n > n_0 \), we can always find some \( t_2 \) such that \( J^*_t(n) = (N-n_0)p \cdot Q_t(n-n_0) \) for any \( t \leq t_2 \).) Then, for any \( t \in (t_2, t_2+h] \)

\[
J^*_{t+\delta}(n) = (1-\lambda_{t+\delta} H_{t+\delta}(n)\delta) \cdot J^*_t(n) + \lambda_{t+\delta} H_{t+\delta}(n)\delta \cdot J^*_t(n-1) + o(\delta).
\]

Let \( \delta \to 0 \), we obtain \( \frac{\partial J^*_t(n)}{\partial t} = \lambda_t H_t(n) \left[ J^*_t(n-1) - J^*_t(n) \right] \) over interval \((t_2, t_2+h)\]. According to Equation (2.3), \( J^*_t(n-1) \leq \frac{(N-n_0)p}{H_t(n)} \left[ H_t(n-n_0)Q_t(n-n_0) - H_t(n)Q_t(n-n_0) \right] \). Also because \( J^*_t(n) > (N-n_0)p \cdot Q_t(n-n_0) \) when \( t \in (t_2, t_2+h] \), we have

\[
\frac{\partial J^*_t(n)}{\partial t} < \lambda_t (N-n_0)p \left[ H_t(n-n_0)Q_t(n-n_0) - H_t(n)Q_t(n-n_0) \right] = \lambda_t (N-n_0)p \cdot H_t(n-n_0) \left[ Q_t(n-n_0) - Q_t(n-n_0) \right].
\]

However, we know from Equation (2.2) that \( \frac{\partial}{\partial t} \left[ (N-n_0)p \cdot Q_t(n-n_0) \right] = \lambda_t (N-n_0)p \cdot H_t(n-n_0) \left[ Q_t(n-n_0) - Q_t(n-n_0) \right] \). Therefore, \( \frac{\partial}{\partial t} \left[ J^*_t(n) - (N-n_0)p \cdot Q_t(n-n_0) \right] < 0 \) for any \( t \in (t_2, t_2+h] \). Since \( \left[ J^*_t(n) - (N-n_0)p \cdot Q_t(n-n_0) \right] \big|_{t=t_2} = 0 \), we obtain that \( J^*_t(n) < (N-n_0)p \cdot Q_t(n-n_0) \) when \( t \in (t_2, t_2+h] \). This contradicts with the assumption we made earlier. Hence, the creator’s optimal policy is to “seed” immediately for any \( t < \tau^s(n) \), i.e., \( J^*_t(n) = (N-n_0)p \cdot Q_t(n-n_0) \) for any \( t < \tau^s(n) \).

Lastly, we show that \( l_t(n) \) is an increasing function of \( t \). This is obvious when \( t \leq \tau^s(n) \), as \( \frac{J^*_t(n)}{Q_t(n-n_0)} = (N-n_0)p \). When \( t > \tau^s(n) \), taking the derivative of \( l_t(n) \) w.r.t. \( t \),
we have

\[
\frac{\partial l_t(n)}{\partial t} = \lambda_t H_t(n) \left( J_t^s(n-1) - J_t^s(n) \right) - \lambda_t H_t(n-n_0) \frac{J_t^s(n) \left( Q_t(n-n_0 - 1) - Q_t(n-n_0) \right)}{Q_t(n-n_0)^2} \]

\[
+ \left( l_t(n) - l_t(n-n_0) \right) \frac{Q_t(n-n_0)}{Q_t(n-n_0 - 1)} - l_t(n) \frac{H_t(n-n_0)}{H_t(n)} \left[ 1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0 - 1)} \right] \]

\[
= \lambda_t H_t(n) Q_t(n-n_0 - 1) \frac{J_t^s(n)}{Q_t(n-n_0)} \left[ l_t(n-1) - l_t(n) - \left( l_t(n) - l_t(n-n_0) \right) \frac{H_t(n-n_0)}{H_t(n-n_0 - 1)} \right] l_t(n). \]

Notice that \( J_t^s(n) > (N-n_0)p \cdot Q_t(n-n_0) \) when \( t > \tau^s(n) \), and thus \( l_t(n) > (N-n_0)p \) when \( t > \tau^s(n) \). Suppose that there exists some \( t_3 > \tau^s(n) \) such that \( \frac{\partial l_t(n)}{\partial t} \bigg|_{t=t_3} < 0 \). Then, there must be some \( t_4 \in (\tau^s(n), t_3) \), such that \( \frac{\partial l_t(n)}{\partial t} \bigg|_{t=t_4} \geq 0 \); otherwise, \( \frac{\partial l_t(n)}{\partial t} < 0 \) for any \( \tau_s(n) < t \leq t_3 \), leading to \( l_{t_4} = l_{\tau^s(n)}(n) = (N-n_0)p \), which contradicts with the result that \( l_t(n) > (N-n_0)p \) when \( t > \tau^s(n) \).

Due to the continuity of \( \frac{\partial l_t(n)}{\partial t} \), there exists some \( t_5 \in [t_4, t_3) \) such that \( \frac{\partial l_t(n)}{\partial t} \bigg|_{t=t_5} = 0 \), and \( \frac{\partial l_t(n)}{\partial t} < 0 \) on \( (t_5, t_3] \). That is,

\[
l_{t_5}(n-1) - l_{t_5}(n) = \left( \frac{H_{t_5}(n-n_0)}{H_{t_5}(n)} - 1 \right) \left( 1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0 - 1)} \right) l_{t_5}(n) = 0.
\]

According to Theorem 2.3, \( H_{t_5}(n-1) \) decreases in \( t \) and \( \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0 - 1)} \) increases in \( t \). Coupling with the result that \( l_t(n) \) strictly decreases within \( (t_5, t_3] \), we have

\[
l_{t_3}(n-1) - l_{t_3}(n) = \left( \frac{H_{t_3}(n-n_0)}{H_{t_3}(n)} - 1 \right) \left( 1 - \frac{Q_{t_3}(n-n_0)}{Q_{t_3}(n-n_0 - 1)} \right) l_{t_3}(n) > l_{t_5}(n-1) - l_{t_5}(n) = \left( \frac{H_{t_5}(n-1)}{H_{t_5}(n)} - 1 \right) \left( 1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0 - 1)} \right) l_{t_5}(n) = 0.
\]

This implies that \( \frac{\partial l_t(n)}{\partial t} \bigg|_{t=t_3} > 0 \) and contradicts with our assumption that \( \frac{\partial l_t(n)}{\partial t} \bigg|_{t=t_3} < 0 \). We thus complete the proof.

**Proof of Corollary 2.8.** We prove by induction. When \( n = n_0 + 1 \), it is straightforward
that \( \tau^s(n_0 + 1) \geq \tau^s(n_0) = \cdots = \tau^s(1) = 0 \). Now assume the statement is true for \( n - 1 \), i.e., \( \tau^s(1) \leq \cdots \leq \tau^s(n - 1) \) for some \( n > n_0 \). We prove \( \tau^s(n - 1) \leq \tau^s(n) \) by showing that for any \( t < \tau^s(n - 1) \), the creator’s optimal action is not to offer the free samples at state \((t, n)\). Suppose this is not true, then \( t > \tau^s(n) \). From Equation (2.3), we have

\[
H_t(n - n_0)Q_t(n - n_0 - 1) - \left( H_t(n - n_0) - H_t(n) \right)Q_t(n - n_0) < H_t(n)J_t^s \frac{(n - 1)}{(N - n_0)p},
\]

Because \( t < \tau^s(n - 1) \), \( J_t^s(n - 1) = (N - n_0)p \cdot Q_t(n - n_0 - 1) \). Plugging \( J_t^s(n - 1) \) into the inequality above, we have

\[
H_t(n - n_0)Q_t(n - n_0 - 1) - \left( H_t(n - n_0) - H_t(n) \right)Q_t(n - n_0) < H_t(n)Q_t(n - n_0 - 1)
\]

\[
\Rightarrow \left( H_t(n - n_0) - H_t(n) \right) \left( Q_t(n - n_0 - 1) - Q_t(n - n_0) \right) < 0.
\]

However, it contradicts with Theorem 2.3(i) and Assumption 1(i). We thus obtain the announced results. ■

**Proposition 2.16 (Asymptotically Optimal Heuristics for Seeding and Feature Upgrade)**

(i) For seeding, if \( \Lambda H(1) \leq N - n_0 \), \( \lim_{m \to \infty} \frac{J_{m}^{s,h}}{m} = \lim_{m \to \infty} \frac{J_{m}^{*,s}}{m} = 0 \); otherwise, \( \lim_{m \to \infty} \frac{J_{m}^{s,h}}{m} = 1 \).

(ii) For feature upgrade, if \( \Lambda \tilde{H}(1) \leq N \), \( \lim_{m \to \infty} \frac{J_{m}^{u,h}}{m} = \lim_{m \to \infty} \frac{J_{m}^{*,u}}{m} = 0 \); otherwise, \( \lim_{m \to \infty} \frac{J_{m}^{u,h}}{m} = 1 \).

**Proof of Proposition 2.16**

- Consider the seeding policy first.

(i) Let \( M \) be the number of pledgers during the entire horizon. We have \( J_{m}^{s,*} \leq mNp \cdot P(M \geq m(N - n_0)) \), because \( mNp \) is the maximum amount of possible fund the creator could possibly collect, and \( P(M \geq m(N - n_0)) \) is the probability of using seeding at the beginning. Therefore, to prove that \( \frac{J_{m}^{s,*}}{m} \to 0 \), it is sufficient to show \( \lim_{m \to \infty} P(M \geq m(N - n_0)) = 0 \). We consider the following two cases:

  (a) \( \Lambda H(1) < N - n_0 \). Since the pledging rate is at most \( H(1) \), it is sufficient to prove the statement under this special case. According to central limit theorem, the random variable \( \xi_m = \frac{1}{\sqrt{m}} \left[ M - m \int_0^T \lambda_s H(1) ds \right] \) \( \to N(0, \int_0^T \lambda_s H(1) ds) \). Hence, we have

\[
P(M \geq m(N - n_0)) = P(\xi_m \geq \sqrt{m} \left( N - n_0 - \int_0^T \lambda_s H(1) ds \right))
\]
Because of pledgers during time
fund collected is used must converge to 1 as $m$.

$\Lambda N$ could possibly collect. The rest of the proof is the same as that in the case when $T$
and the project’s success likelihood at any time $t$
likelihood will still be less than 1. Therefore, when $m \to \infty$, the pledging likelihood at
any time $t$ is less than $H(1)$ with probability 1, implying that $P\left( M \geq m(N - n_0) \right) \to 0$.

(ii) When $N - n_0 < \Lambda H(1) < N$, the probability that the seeding policy would be
used must converge to 1 as $m \to \infty$. Otherwise, if with a positive probability that the
creator does not use the seeding policy, the project would fail almost surely. Hence, the
fund collected is $m(N - n_0)p$ with probability 1 and $\lim_{m \to \infty} \frac{J_{m}^{s,h}}{m} \leq (N - n_0)p$.

We show that with the static heuristic where seeding is applied at the beginning, for
a small $0 < \epsilon < 1$, if the pledging likelihood of a backer that arrives at any time $T - h$ is
$H(1 - \epsilon)$, then the project’s success likelihood will converge to 1 uniformly at any time
almost surely. To see that, we note that due to the law of large numbers, the number of
pledgers between $T$ and $T - h$ is given by $m \int_{T-h}^{T} \lambda_s H(1 - \epsilon)ds + \omega(m)$. $\omega(m)$ is a random variable, where $E[\omega(m)] = 0$ and $\lim_{m \to \infty} \frac{\omega(m)}{m} = 0$. Therefore, the number of pledgers
required to reach the target at time $T - h$ would be $m \left( N - n_0 - \int_{T-h}^{T} \lambda_s H(1 - \epsilon)ds \right) - \omega(m)$. The pledging rate between $t = T - h$ and $t = 0$ is $\lambda_s H(1 - \epsilon)$, and thus the number of
pledgers during time $t = T - h$ and $t = 0$ would be $m \int_{0}^{T-h} \lambda_s H(1 - \epsilon)ds + \omega'(m)$. Because $\int_{0}^{T} \lambda_s H(1 - \epsilon)ds = \Lambda H(1 - \epsilon) > N - n_0$, we have $\lim_{m \to \infty} \frac{m \int_{0}^{T-h} \lambda_s H(1 - \epsilon)ds + \omega'(m)}{m} = \int_{0}^{T-h} \lambda_s H(1 - \epsilon)ds \geq N - \int_{T-h}^{T} \lambda_s H(1 - \epsilon)ds \leq \lim_{m \to \infty} \frac{m \left( N - n_0 - \int_{T-h}^{T} \lambda_s H(1 - \epsilon)ds \right) - \omega(m)}{m}$. Consequently, at equilibrium, backers’ pledging likelihood must be greater than $H(1 - \epsilon)$, and the project’s success likelihood at any time $T - h$ converges to 1 when $m \to \infty$. Let $\epsilon \to 0$, REE pledging likelihood would converge to $H(1)$ at any time $T - h$. Therefore, $\frac{J_{m}^{s,h}}{m(N - n_0)p} = 1$. Since $J_{m}^{s,h} \leq J_{m}^{s,*} \leq m(N - n_0)p$, $\lim_{m \to \infty} \frac{J_{m}^{s,h}}{J_{m}^{s,*}} = 1$.

When $\Lambda H(1) > N$, $J_{m}^{s,*} \leq mNp$, as this is the maximum amount of fund the creator
could possibly collect. The rest of the proof is the same as that in the case when $N - n_0 < \Lambda H(1) < N$. The underlying rationale is that, under static heuristic, the probability that
the project would reach the goal before $t = 0$ would converge to 1 as $m \to \infty$.

- For feature upgrade, the proof is analogous to that for the seeding policy.
2.8 Appendix

2.8.1 “Cascade Effect” with Overfunding

In this section, we extend the base model and consider the overfunding case where backers can continue to pledge after the target is reached. We show that “cascade” effect demonstrated in Proposition 2.6 still holds. First similar to Proposition 2.2, we derive $J_t^b(n)$, the expected revenue at state $(t, n)$, in the following proposition:

**Proposition 2.17** In the case of overfunding, $J_t^b(n)$ is given by:

$$
\frac{\partial J_t^b(n)}{\partial t} = \lambda_t \cdot H(Q_t(n-1)) \cdot (J_t^b(n-1) - J_t^b(n)),
$$

with boundary conditions $J_t^b(0) = (N + \int_0^t \lambda_s H(1)ds)$ for all $t \geq 0$, and $J_0^b(n) = 0$ for $n \geq 1$.

Now that we express the expected revenue with overfunding, we prove that Proposition 2.6 still holds:

**Proposition 2.18** (Value of Pledgers (with Overfunding)) The marginal increase in the expected revenue with one more pledger at state $(t, n)$, $\frac{J_t^{b(n-1)} - J_t^{b(n)}}{J_t^b(n)}$, decreases in $t$ and increases in $n$.

**Proof of Proposition 2.18** We just need to show that $\frac{J_t^b(n)}{J_t^b(n-1)}$ increases in $t$.

First we show that $J_t^b(1) \leq J_t^b(0) - p$ for any $t \geq 0$. When $t = 0$, $0 = J_0^b(1) \leq J_0^b(0) - p = (N - 1)p$. Suppose that there exists a $t > 0$ such that $J_t^b(1) > J_t^b(0) - p$, then because of the continuity of $J_t^b(n)$, we can find $t_1 < t_2$ such that $J_{t_1}^b(1) = J_{t_1}^b(0) - p$ and $J_{t_2}^b(1) > J_{t_2}^b(0) - p$ for any $t_1 < t \leq t_2$. This means that $\frac{\partial J_t^b(1)}{\partial t} = \lambda_t \cdot H(1) \cdot (J_t^b(0) - J_t^b(1)) < \lambda_t \cdot H(1)p$ for $t_1 < t \leq t_2$. However $\frac{\partial (J_t^b(0) - p)}{\partial t} = \lambda_t \cdot H(1)p$, hence $J_{t_2}^b(1) < J_{t_2}^b(0) - p$, and this leads to contradiction. Therefore $J_t^b(1) \leq J_t^b(0) - p$ for any $t \geq 0$.

Now we take derivative of $\frac{J_t^b(1)}{J_t^b(0)}$:

$$
\frac{\partial J_t^b(1)}{\partial t J_t^b(0)} = \frac{\lambda_t H(1) \cdot (J_t^b(0) - J_t^b(1))}{J_t^b(0)} - \frac{J_t^b(1)}{J_t^b(0)} \cdot \frac{\lambda_t H(1)p}{J_t^b(0)}
= \lambda_t H(1) \cdot \left[1 - \frac{J_t^b(0) + p}{J_t^b(0)} \cdot \frac{J_t^b(1)}{J_t^b(0)}\right].
$$

Because $J_t^b(1) \leq J_t^b(0) - p$,

$$
\frac{\partial J_t^b(1)}{\partial t J_t^b(0)} \geq \lambda_t H(1) \cdot \left[1 - \frac{J_t^b(0) + p}{J_t^b(0)} \cdot \frac{J_t^b(0) - p}{J_t^b(0)}\right] > 0.
$$
We have proven that \( \frac{\delta x_t(n)}{\delta t} \) increases in \( t \) for \( n = 1 \). Now suppose that the statement is true for \( n - 1 \). Using the same proof technique in Theorem 2.2, we can show the statement is also true for \( n \).

### 2.8.2 Additional Proofs

**Proof of Proposition 2.4.** Denote \( x_t(n) = \frac{Q^a_t(n)}{Q^b_t(n)} \) and \( \gamma_t(n) = \frac{H^a(Q^a_t(n))}{H^b(Q^b_t(n))} \). We first prove that \( x_t(n) \) and \( \gamma_t(n) \) increase in \( t \) by induction. When \( n = 0 \), \( x_t(0) = 1 \) and \( \gamma_t(0) = \frac{H^a(1)}{H^b(1)} \), and thus the monotonicity holds trivially. Now suppose that the statement is true for \( n - 1 \). Taking the derivative of \( x_t(n) \) w.r.t. \( t \), we have

\[
\frac{dx_t(n)}{dt} = \lambda_t H^a(Q^a_t(n - 1)) \left[ Q^a_t(n - 1) - Q^b_t(n) \right] - \frac{Q^a_t(n) \lambda_t H^b(Q^b_t(n - 1)) \left[ Q^a_t(n - 1) - Q^b_t(n) \right]}{[Q^b_t(n)]^2}
\]

\[
= \lambda_t \left[ Q^a_t(n) \right] \left[ H^a(Q^a_t(n - 1)) \left( \frac{Q^a_t(n - 1)}{Q^b_t(n)} - 1 \right) - H^b(Q^b_t(n - 1)) \left( \frac{Q^b_t(n)}{Q^a_t(n)} - 1 \right) \right]
\]

Denote \( L(t) = x_t(n) - \left[ 1 + \gamma_t(n-1) \left( 1 - \frac{Q^a_t(n)}{Q^b_t(n-1)} \right) \right] x_t(n) \). Next we show that if there exists some \( t_1 \) such that \( L(t_1) < 0 \), there must exist some \( t_2 \in (0, t_1) \) such that \( L(t_2) \geq 0 \). Consider the following two cases.

(1) \( \lim_{t \to 0} \gamma_t(n - 1) = 0 \). Using L’ Hopital’s rule, we have

\[
\lim_{t \to 0} x_t(n) = \lim_{t \to 0} \frac{\partial x_t(n)}{\partial t} = \lim_{t \to 0} \frac{\lambda_t H^a(Q^a_t(n - 1)) \left( Q^a_t(n - 1) - Q^b_t(n) \right)}{\lambda_t H^b(Q^b_t(n - 1)) \left( Q^b_t(n - 1) - Q^a_t(n) \right)}
\]

\[
= \lim_{t \to 0} \gamma_t(n - 1) \cdot \frac{Q^a_t(n - 1) \left( 1 - \frac{Q^b_t(n)}{Q^a_t(n-1)} \right)}{Q^b_t(n - 1) \left( 1 - \frac{Q^a_t(n)}{Q^b_t(n-1)} \right)} = 0.
\]

Suppose there exists some \( t_1 > 0 \) such that \( \frac{\partial x_t(n)}{\partial t} |_{t=t_1} < 0 \). Then, there must exist
some \( t_2 \in (0, t_1) \) such that \( \frac{\partial x_t(n)}{\partial t} |_{t=t_2} \geq 0 \); otherwise \( x_t(n) \) decreases within \((0, t_1]\), which implies that \( x_{t_1} \leq \lim_{t \to 0} x_t(n) = 0 \). This contradicts with the fact that \( x_t(n) > 0 \) for \( t > 0 \).

\( 2 \) \( \lim_{t \to 0} \gamma_t(n - 1) > 0 \). Because of \( \lim_{t \to 0} \frac{Q_t^p(n)}{Q_t^p(n-1)} = 0 \) as shown in Theorem 2.3(ii), \( \lim L(t) = \lim_{t \to 0} x_t(n - 1) - \frac{x_t(n)}{\gamma_t(n-1)} \). Again using L'Hopital's rule, we have

\[
\lim_{t \to 0} L(t) = \lim_{t \to 0} x_t(n - 1) - \lim_{t \to 0} \frac{1}{\gamma_t(n-1)} \cdot \frac{H^a(Q_t^p(n)) \cdot Q_t^p(n-1)}{H^b(Q_t^p(n)) \cdot Q_t^p(n-1)} = 0.
\]

Suppose there exists some \( t_1 > 0 \) such that \( \frac{\partial x_t(n)}{\partial t} |_{t=t_1} < 0 \), i.e., \( L(t_1) < 0 \). Then, there must exist some \( t_2 \in (0, t_1) \) such that \( \frac{\partial x_t(n)}{\partial t} |_{t=t_2} \geq 0 \); otherwise, \( x_t(n) \) decreases within \((0, t_1]\). Combined with the results that \( x_t(n-1), \gamma_t(n-1) \) and \( \frac{Q_t^p(n)}{Q_t^p(n-1)} \) all increase in \( t \), we have that \( M(t) \) increases in \((0, t_1]\), which suggests that \( L(t_1) \geq \lim_{t \to 0} L(t) = 0 \). This contradicts with the preceding argument that \( L(t_1) < 0 \).

Therefore, if there exists some \( t_1 \) such that \( L(t_1) < 0 \), there must exist a \( t_2 \in (0, t_1) \) such that \( L(t_2) \geq 0 \). Coupling with the continuity of \( L(t) \), there exists a \( t_3 \in [t_2, t_1) \) such that \( L(t_3) = 0 \). This implies that \( x_t(n) \) strictly decreases within \((t_3, t_1]\). Combined with the results that \( x_t(n-1), \gamma_t(n-1) \) and \( \frac{Q_t^p(n)}{Q_t^p(n-1)} \) all increase in \( t \), we have that \( L(t) \) increases in \((t_3, t_1]\), which suggests that \( L(t_1) \geq L(t_3) = 0 \). This contradicts with the preceding argument that \( L(t_1) < 0 \). Therefore, we conclude that \( \frac{\partial x_t(n)}{\partial t} \geq 0 \) for all \( t > 0 \).

Given that \( x_t(n) \) increases in \( t \), for any \( \delta > 0 \), we have

\[
\frac{H^b(Q_t^p(n))}{H^b(Q_t^p(n-\delta))} = \frac{H^b(x_t(n)Q_t^p(n))}{H^b(Q_t^p(n-\delta)(n))} \geq \frac{H^b(x_t(n)Q_t^p(n))}{H^b(Q_t^p(n))} \geq \frac{H^b(x_t(n)Q_t^p(n))}{H^b(Q_t^p(n))} = \frac{H^b(Q_t^p(n))}{H^b(Q_t^p(n))},
\]

where the second inequality is a result of Assumption 1(iii). Hence \( H^b(Q_t^p(n)) \) increases in \( t \). Combining with the assumption that \( H^a(q) \) increases in \( q \), we conclude that \( H^a(Q_t^p(n)) \) increases in \( t \).

Next we prove that \( x_t(n) \) and \( \gamma_t(n) \) decrease in \( n \). Because \( x_t(n) \) increases in \( t \), we
have \( M(t) = x_t(n - 1) - x_t(n) - \left( \frac{1}{\gamma_t(n-1)} - 1 \right) \left( 1 - \frac{Q_t^a(n)}{Q_t^b(n-1)} \right) x_t(n) > 0 \) for any \( t > 0 \).

Coupling with the results that \( \gamma_t(n-1) \leq 1 \), \( x_t(n) \geq 0 \) and Theorem 2.3(i), we thus have that \( x_t(n - 1) > x_t(n) \).

Given that \( \frac{Q_t^a(n)}{Q_t^b(n)} \) decreases in \( n \), we have

\[
\frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))} = \frac{H^a\left( \frac{Q_t^a(n)}{Q_t^b(n)} Q_t^b(n) \right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left( \frac{Q_t^a(n-1)}{Q_t^b(n-1)} Q_t^b(n) \right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left( \frac{Q_t^a(n-1)}{Q_t^b(n-1)} Q_t^b(n - 1) \right)}{H^a(Q_t^b(n - 1))} = \frac{H^a(Q_t^a(n - 1))}{H^a(Q_t^b(n - 1))},
\]

where the second inequality is a result of Assumption 1(iii). Moreover, \( \frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))} \leq \frac{H^a(Q_t^a(n-1))}{H^a(Q_t^b(n-1))} \) because of the assumption that \( \frac{H^a(q)}{H^b(q)} \) increases in \( q \). Therefore, we have

\[
\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))} = \frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))} \frac{H^a(Q_t^b(n))}{H^b(Q_t^b(n))} \leq \frac{H^a(Q_t^a(n - 1))}{H^a(Q_t^b(n - 1))} \frac{H^a(Q_t^b(n - 1))}{H^b(Q_t^b(n - 1))} \frac{H^a(Q_t^a(n - 1))}{H^b(Q_t^b(n - 1))}.
\]

We thus complete the proof. ■

**Proof of Theorem 2.9** (i) Since \( J_{T,N}^b = Np \cdot Q_T(N) \) and \( J_{T,N}^a = J_T^a(N) \), it is sufficient to show that \( \frac{Q_t(n)}{J_t(n)} \) increases in \( t \).

When \( n = 0 \), the statement is obvious as \( Q_t(0) = 1 \) and \( J_t^a(0) = Np \). Now assume that \( \frac{Q_t(n-1)}{J_t(n-1)} \) weakly increases in \( t \). In that case:

When \( t < \tau^a(n) \), \( J_t^a(n) = (N - n_0) p \cdot Q_t((n - n_0)^+) \). Therefore \( \frac{Q_t(n)}{J_t(n)} = \frac{Q_t(n)}{(N - n_0) p \cdot Q_t((n - n_0)^+)} \).

According to Theorem 2.3, it increases in \( t \).

When \( t \geq \tau^a(n) \),

\[
\frac{\partial Q_t(n)}{\partial t} J_t^a(n) = \lambda_t H_t(n) \frac{J_t^a(n)}{J_t^a(n)} - \frac{Q_t(n)}{J_t^a(n)} \lambda_t H_t(n) \frac{J_t^a(n)}{J_t^a(n)} \leq \lambda_t H_t(n) \frac{Q_t(n)}{J_t^a(n)} \frac{J_t^a(n)}{J_t^a(n)} - \frac{Q_t(n)}{J_t^a(n)} \frac{J_t^a(n)}{J_t^a(n)} = \lambda_t H_t(n) \frac{J_t^a(n - 1)}{J_t^a(n)} \frac{Q_t(n - 1)}{J_t^a(n - 1)} - \frac{Q_t(n)}{J_t^a(n)} \frac{J_t^a(n)}{J_t^a(n)}.
\]
When \( t = \tau^s(n) \), because \( J_t^s(n) = (N - 1)p \cdot Q_t(n - 1) \),
\[
\frac{Q_t(n - 1)}{J_t^s(n - 1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{Q_t(n - 1)}{J_t^s(n - 1)} - \frac{Q_t(n)}{(N - 1)p \cdot Q_t(n - 1)}.
\]

Also, according to Equation (2.3), at \( t = \tau^s(n) \), \( J_t^s(n - 1) = (N - 1)p \cdot \left[ \frac{H_t(n-1)}{H_t(n)} Q_t(n - 2) - \left( \frac{H_t(n-1)}{H_t(n)} - 1 \right) Q_t(n - 1) \right] \). Hence,
\[
\frac{Q_t(n - 1)}{J_t^s(n - 1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{1}{(N - 1)p} \frac{H_t(n-1)}{H_t(n)} Q_t(n - 2) - \left( \frac{H_t(n-1)}{H_t(n)} - 1 \right) Q_t(n - 1) - \frac{Q_t(n)}{(N - 1)p \cdot Q_t(n - 1)}
\]
\[
= \frac{1}{(N - 1)p} \frac{H_t(n-1)}{H_t(n)} Q_t(n - 2) - \left( \frac{H_t(n-1)}{H_t(n)} - 1 \right) Q_t(n - 1)
\]
\[
\frac{Q_t(n - 1)}{Q_t(n)} - \frac{H_t(n-1)}{H_t(n)} \frac{Q_t(n - 2) - \left( \frac{H_t(n-1)}{H_t(n)} - 1 \right) Q_t(n - 1)}{Q_t(n - 1) - 1}.
\]

Recall that in the proof of Theorem 2.3, we have shown that for any \( t > 0 \), \( \frac{H_t(n-1)}{H_t(n)} \left( \frac{Q_t(n-1)}{Q_t(n)} - 1 \right) > 0 \). Therefore \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \big|_{t=\tau^s(n)} > 0 \).

Suppose that there exists a \( t' > \tau^s(n) \) such that \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} < 0 \), then because of continuity, there must exists a \( \tau^s(n) < t_1 < t' \) such that \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \big|_{t=t_1} = 0 \) and \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} < 0 \) when \( t \in (t_1, t') \). This also means that \( \frac{Q_t(n-1)}{J_t^s(n-1)} \) decreases in \( t \) over the interval. However, since \( \frac{Q_t(n-1)}{J_t^s(n-1)} \) increases in \( t \), \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \) must be increasing in \( t \) within \( (t_1, t') \). This indicates \( \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} \big|_{t=t'} > 0 \). This leads to contradiction. Therefore \( \frac{Q_t(n)}{J_t^s(n)} \) increases in \( t \) for any \( t > 0 \).

(ii) Note that \( J_{T,N}^b \leq J_{T,N}^s \leq Np \cdot Q_T((N - n_0)^+) \). Consequently, we have
\[
0 \leq J_{T,N}^s - J_{T,N}^b \leq Np \cdot \left( Q_T((N - n_0)^+) - Q_T(N) \right).
\]

Letting \( T \to \infty \) or \( T \to 0 \), we thus obtain the announced results. 

\[\blacksquare\]
Proof of Theorem 2.10.} We show that \(\tau^u(n)\) is given by

\[
\tau^u(n) = \sup \left\{ t : \tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n) \geq H_t(n) \frac{J^u_t(n-1)}{Np - K} \right\}, \tag{2.5}
\]

where \(J^u_t(n)\) is the expected revenue at state \((t, n)\). It is given by

- when \(t \leq \tau^u(n)\), \(J^u_t(n) = (Np - K)\tilde{Q}_t(n)\);
- when \(t > \tau^u(n)\),

\[
\frac{\partial J^u_t(n)}{\partial t} = \lambda_t H_t(n) \left[ J^u_t(n-1) - J^u_t(n) \right], \tag{2.6}
\]

with boundary conditions \(J^u_{\tau^u(n)}(n) = (Np - K)\tilde{Q}_t(n)\), and \(J^u_t(0) = Np\).

We add to the statement that \(\frac{J^u_t(n)}{\tilde{Q}_t(n)}\) increases in \(t\), and prove by induction. First, we show that the statement is correct for \(n = 1\). We can rewrite Equation (2.5) as

\[
\tau^u(1) = \sup \left\{ t : \tilde{Q}_t(1) \leq \frac{\tilde{H}_t(1)}{H_t(1)} - \frac{Np}{H_t(1)} \right\}. \tag{2.5}
\]

Notice that LHS of the inequality in the bracket strictly increases in \(t\), and RHS in the bracket decreases in \(t\). That is, for any \(t \leq \tau^u(1)\), the inequality holds; whereas the direction of the inequality is flipped for any \(t > \tau^u(1)\).

Suppose that there exists some \(t_1 > \tau^u(1)\), such that the creator’s optimal policy is to upgrade immediately, i.e., \(J^u_{t_1}(1) = (Np - K) \cdot \tilde{Q}_{t_1}(1)\). Then, we have

\[
(Np - K)\tilde{Q}_{t_1}(1) > (1 - \delta \lambda_{t_1} H_{t_1}(1))J^u_{t_1-\delta}(1) + \delta \lambda_{t_1} H_{t_1}(1)Np + o(\delta)
\]

\[
\geq (1 - \delta \lambda_{t_1} H_{t_1}(1))(Np - K)\tilde{Q}_{t_1-\delta}(1) + \delta \lambda_{t_1} H_{t_1}(1)Np + o(\delta).
\]

Plugging \(\tilde{Q}_{t_1}(1) = (1 - \delta \lambda_{t_1} \tilde{H}_{t_1}(1))\tilde{Q}_{t_1-\delta}(1) + \delta \lambda_{t_1} \tilde{H}_{t_1}(1) + o(\delta)\) into the inequality above, rearranging and taking the limit as \(\delta \to 0\), we have

\[
(Np - K)(\tilde{H}_{t_1}(1) - H_{t_1}(1))\tilde{Q}_{t_1}(1) \leq \tilde{H}_{t_1}(1)(Np - K) - H_{t_1}(1)Np.
\]

This implies that \(\tilde{Q}_{t_1}(1) \leq \frac{\tilde{H}_{t_1}(1)}{H_{t_1}(1)} - \frac{Np}{H_{t_1}(1)}\), i.e., \(t_1 \leq \tau^u(1)\), which contradicts with the assumption that \(t_1 > \tau^u(1)\). Therefore, the creator would not upgrade its features immediately for any \(t > \tau^u(1)\), i.e., \(J^u_t(1) > (Np - K) \cdot \tilde{Q}_t(1)\) for any \(t > \tau^u(1)\).

We next show that the creator’s optimal policy is to upgrade immediately when \(t < \tau^u(1)\). Suppose that there exists some \(t_2 < \tau^u(1)\), such that \(J^u_{t_2}(1) = (Np - K)\tilde{Q}_{t_2}(1)\) for any \(t \leq t_2\), and \(J^u_t(1) > (Np - K)\tilde{Q}_t(1)\) when \(t \in (t_2, t_2 + \delta]\). (Because \(J^u_{t_2}(n) = 0\) for any \(n \geq 1\), we can always find a \(t_2\) such that \(J^u_{t_2}(n) = (Np - K)\tilde{Q}_{t_2}(1)\) for any \(t \leq t_2\).)
Then, we have
\[
(Np - K)\dot{Q}_{t_2+\delta}(1) < J_{t_2+\delta}^u(1) = (1 - \delta \lambda_{t_2+\delta}H_{t_2+\delta}(1))J_{t_2}^u(1) + \delta \lambda_{t_2+\delta}H_{t_2+\delta}(1)Np + o(\delta)
\]
\[
= (1 - \delta \lambda_{t_2+\delta}H_{t_2+\delta}(1))(Np - K)\dot{Q}_{t_2}(1) + \delta \lambda_{t_2+\delta}H_{t_2+\delta}(1)Np + o(\delta).
\]

Plugging \(\dot{Q}_{t_2+\delta}(1) = (1 - \delta \lambda_{t_2+\delta}\dot{H}_{t_2+\delta}(1))\dot{Q}_{t_2}(1) + \delta \lambda_{t_2+\delta}\dot{H}_{t_2+\delta}(1) + o(\delta)\) into the inequality above, rearranging and taking the limit as \(\delta \to 0\), we have
\[
(Np - K)(\dot{H}_{t_2}(1) - H_{t_2}(1))\dot{Q}_{t_2}(1) \geq \dot{H}_{t_2}(1)(Np - K) - H_{t_2}(1)Np.
\]

This implies that \(\dot{Q}_{t_2}(1) \geq \frac{\dot{H}_{t_2}(1) - H_{t_2}(1)}{H_{t_2}(1)}\frac{Np}{Np - K}\), i.e., \(t_2 \geq \tau^u(1)\), which contradicts with the assumption that \(t_2 < \tau^u(1)\). Therefore, the creator would upgrade its features immediately for any \(t < \tau^u(1)\), i.e., \(J_{t}^u(1) = (Np - K)\dot{Q}_t(1)\) for any \(t < \tau^u(1)\).

Next we show that \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)}\) increases in \(t\). This is obvious when \(t \leq \tau^u(1)\), because \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)} = Np - K\). When \(t > \tau^u(1)\), taking the derivative of \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)}\) w.r.t. \(t\), we have
\[
\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)} = \frac{\lambda_t H_t(1)\dot{Q}_t(1) - \lambda_t J_{t}^u(1)H_t(1)[1 - \dot{Q}_t(1)]}{\dot{Q}_t(1)^2}.
\]
\[
= \frac{\lambda_t H_t(1)[Np - J_{t}^u(1)] - \lambda_t J_{t}^u(1)H_t(1)(1 - \dot{Q}_t(1))}{\dot{Q}_t(1)^2}.
\]

Suppose that there exists some \(t_3\) such that \(\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)}|_{t=t_3} \leq 0\). Then, there must be some \(t_4 \in (\tau^u(1), t_3)\) such that \(\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)}|_{t=t_4} > 0\); otherwise \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)} \leq \frac{J_{\tau^u(1)}^u(1)}{\dot{Q}_{\tau^u(1)}(1)} = Np - K\), which contradicts with the result that \(J_{t}^u(1) > (Np - K)\cdot \dot{Q}_t(1)\) for any \(t > \tau^u(1)\). Due to the continuity of \(\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)}\), there exists some \(t_5 \in (t_4, t_3)\) such that \(\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)}|_{t=t_5} = 0\), and \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)}\) decreases in \(t\) for any \(t \in [t_5, t_3]\). Thus, we have
\[
\frac{\partial}{\partial t} \frac{J_{t}^u(1)}{\dot{Q}_t(1)}|_{t=t_3} = \frac{\lambda_{t_3} H_{t_3}(1)\dot{Q}_{t_3}(1) - \lambda_{t_3} J_{t_3}^u(1)H_{t_3}(1)(1 - \dot{Q}_{t_3}(1))}{\dot{Q}_{t_3}(1)^2}.
\]
\[
> \frac{\lambda_{t_3} H_{t_3}(1)\dot{Q}_{t_3}(1) - \lambda_{t_3} J_{t_3}^u(1)H_{t_3}(1)(1 - \dot{Q}_{t_3}(1))}{\dot{Q}_{t_3}(1)^2} = 0,
\]
which leads to contradiction. Therefore, \(\frac{J_{t}^u(1)}{\dot{Q}_t(1)}\) increases in \(t\) for any \(t > \tau^u(1)\).

Now assume that the statement is true for \(n - 1\). We can rewrite the inequality in
the inequality holds; whereas the direction of the inequality is flipped for any
\( J \)
not upgrade when
\( t > \tau \)
while RHS increases in
\( u \).

According to Theorem 2.3(iii), LHS of the above inequality strictly decreases in
\( t \); while RHS increases in \( t \) due to our induction hypothesis. Therefore, for any \( t < \tau^u(n) \), the inequality holds; whereas the direction of the inequality is flipped for any \( t > \tau^u(n) \).

Suppose that there exists some \( t_1 > \tau^u(n) \), such that the creator’s optimal policy is to upgrade immediately, i.e., \( J^u_{t_1}(n) = (Np - K)\tilde{Q}_{t_1}(n) \). Then, we have
\[
(Np - K)\tilde{Q}_{t_1}(n) > (1 - \delta \lambda_t H_t(n))J^u_{t_1-\delta}(n) + \delta \lambda_t H_t(n)J^u_{t_1-\delta}(n - 1) + o(\delta)
\]
\[
\geq (1 - \delta \lambda_t H_t(n))(Np - K)\tilde{Q}_{t_1-\delta}(n) + \delta \lambda_t H_t(n)J^u_{t_1-\delta}(n - 1) + o(\delta).
\]

Plugging \( \tilde{Q}_{t_1}(n) = (1 - \delta \lambda_t H_t(n))\tilde{Q}_{t_1-\delta}(n) + \delta \lambda_t H_t(n)\tilde{Q}_{t_1-\delta}(n - 1) + o(\delta) \) into the inequality above, rearranging and taking the limit as \( \delta \to 0 \), we have
\[
\dot{H}_t(n)\tilde{Q}_{t_1}(n) - \left( \dot{H}_t(n) - H_t(n) \right)\tilde{Q}_{t_1}(n) \geq \frac{H_t(n)J^u_{t_1}(n - 1)}{Np - K}.
\]

This contradicts with our assumption that \( t_1 > \tau^u(n) \). Therefore, the creator would not upgrade when \( t > \tau^u(n) \), i.e., \( J^u_t(n) > (Np - K)\tilde{Q}_t(n) \) for any \( t > \tau^u(n) \). Consider what happens in a small time interval \( \delta \), we have
\[
J^u_t(n) = (1 - \delta \lambda_t H_t(n))J^u_{t-\delta}(n) + \delta \lambda_t H_t(n)J^u_{t-\delta}(n - 1) + o(\delta).
\]

Rearranging and taking the limit as \( \delta \to 0 \), we thus obtain Equation (2.6).

We next show that the creator’s optimal policy is to upgrade immediately when \( t < \tau^u(n) \). Suppose that there exists some \( t_2 < \tau^u(n) \), such that \( J^u_t(n) = (Np - K)\tilde{Q}_t(n) \) for all \( t \leq t_2 \), and \( J^u_t(n) > (Np - K)\tilde{Q}_t(n) \) for \( t \in (t_2, t_2 + \delta) \). Then, we have
\[
(Np - K)\tilde{Q}_{t_2+\delta}(n) < J^u_{t_2+\delta}(n) = (1 - \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n))J^u_{t_2}(n) + \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n)J^u_{t_2}(n - 1) + o(\delta)
\]
\[
= (1 - \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n))(Np - K)\tilde{Q}_{t_2}(n) + \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n)J^u_{t_2}(n - 1) + o(\delta).
\]

Plugging \( \tilde{Q}_{t_2+\delta}(n) = (1 - \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n))\tilde{Q}_{t_2}(n) + \delta \lambda_{t_2+\delta} H_{t_2+\delta}(n)\tilde{Q}_{t_2}(n - 1) + o(\delta) \) into the inequality above, rearranging and taking the limit as \( \delta \to 0 \), we have
\[
(Np - K)\left( \dot{H}_{t_2}(n)\tilde{Q}_{t_2}(n - 1) - (\dot{H}_{t_2}(n) - H_{t_2}(n))\tilde{Q}_{t_2}(n) \right) \leq H_{t_2}(n)J^u_{t_2}(n - 1).
\]
This contradicts with the assumption that \( t_2 < \tau^u(n) \). Therefore, the creator would upgrade immediately when \( t < \tau^u(n) \), i.e., \( J^u_t(n) = (Np - K)\hat{Q}_t(n) \) for any \( t < \tau^u(n) \).

Lastly, we complete the proof by showing that \( \frac{J^u_t(n)}{Q_t(n)} \) increases in \( t \). It is trivial when \( t \leq \tau^u(n) \) because \( \frac{J^u_t(n)}{Q_t(n)} = Np - K \). Consider next when \( t > \tau^u(n) \). Suppose that there exists some \( t_3 > \tau^u(n) \) such that \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_3} < 0 \). Then, there must exist some \( t_4 \in (\tau^u(n), t_3) \) such that \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_4} \geq 0 \); otherwise, \( \frac{J^u_t(n)}{Q_t(n)} < \frac{J^u_{t_3}(n)}{Q_{t_3}(n)} = Np - K \), which contradicts with the result that \( J^u_t(n) > (Np - K) \cdot \hat{Q}_t(n) \) for any \( t > \tau^u(n) \). Due to the continuity of \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \), there exists some \( t_5 \in [t_4, t_3) \), such that \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_5} = 0 \). That is,

\[
\frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_5} = \frac{\lambda_t H_t(n) J^u_t(n) - \lambda_{t_5} H_{t_5}(n) J^u_{t_5}(n)}{Q_{t_5}(n)} = \lambda_t H_t(n) J^u_t(n) \frac{J^u_{t_5}(n) - J^u_{t_5}(n)}{Q_{t_5}(n)} - \lambda_{t_5} H_{t_5}(n) J^u_{t_5}(n) \left[ \frac{Q_{t_5}(n)}{[Q_{t_5}(n)]^2} \right] = 0.
\]

Because \( \frac{Q_{t_5}(n)}{Q_{t_5}(n-1)} \) increases in \( t \), \( \frac{\hat{H}_t(n)}{\hat{H}_t(n-1)} \) decreases in \( t \) as shown in Theorem 2.3(iii), and the induction hypothesis that \( \frac{J^u_{t_5}(n-1)}{Q_{t_5}(n-1)} \) increases in \( t \), we have \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_3} \geq 0 \), which contradicts with the assumption that \( \frac{\partial J^u_t(n)}{\partial Q_t(n)} \big|_{t=t_3} < 0 \). Therefore, \( \frac{J^u_t(n)}{Q_t(n)} \) increases in \( t \) for any \( t > \tau^u(n) \), and we thus complete the proof. ■

**Proof of Corollary 2.12.** Suppose that there exists an \( n \), such that \( \tau^u(n) < \tau^u(n-1) \). For any \( t \in (\tau^u(n), \tau^u(n-1)) \), \( J^u_t(n-1) = (Np - K)\hat{Q}_t(n-1) \). Using the definition of \( \tau^u(n) \), we have

\[
(Np - K) \left[ \hat{H}_t(n)\hat{Q}_t(n-1) - (\hat{H}_t(n) - H_t(n))\hat{Q}_t(n) \right] < H_t(n) J^u_t(n-1) - \hat{Q}_t(n) < 0.
\]

This contradicts with Theorem 2.3(i) and Assumption [1](i). We thus obtain the announced results. ■

**Proof.** Proof of Corollary 2.12 (i) Since \( J^u_t(n) = Np \cdot Q_t(n) \). It is equivalent to show that \( \frac{Q_t(n)}{J^u_t(n)} \) increases in \( t \) and decreases in \( t \). We first prove that \( \frac{Q_t(n)}{J^u_t(n)} \) increases in \( t \).

When \( n = 0 \), the statement is obvious as \( Q_t(n) = 1 \) and \( J^u_t(n) = Np \). Now assume that it’s true for \( n - 1 \). Then for \( n \):

When \( t < \tau^u(n) \), \( J^u_t(n) = (Np - K)\hat{Q}_t(n) \). Hence \( \frac{Q_t(n)}{J^u_t(n)} = \frac{1}{Np - K} \frac{Q_t(n)}{Q_t(n)} \). According to Proposition 2.4 \( \frac{Q_t(n)}{J^u_t(n)} \) increases in \( t \).
When $t \geq \tau^u(n)$,

$$\frac{\partial Q_t(n)}{\partial t} J^u_t(n) = \frac{\lambda_t H_t(n)\left[Q_t(n-1) - Q_t(n)\right]}{J^u_t(n)} - \frac{Q_t(n)\lambda_t H_t(n)\left[J^u_t(n-1) - J^u_t(n)\right]}{J^u_t(n)}$$

$$= \lambda_t H_t(n)\frac{Q_t(n)}{J^u_t(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - \frac{J^u_t(n-1)}{J^u_t(n)}\right]$$

$$= \lambda_t H_t(n)\frac{J^u_t(n-1)}{J^u_t(n)}\left[\frac{Q_t(n-1)}{J^u_t(n-1)} - \frac{Q_t(n)}{J^u_t(n)}\right].$$

At $t = \tau^u(n)$, because $J^u_t(n) = (Np - K)\tilde{Q}_t(n)$,

$$\frac{Q_t(n-1)}{J^u_t(n-1)} - \frac{Q_t(n)}{J^u_t(n)} = \frac{Q_t(n-1)}{J^u_t(n-1)} - \frac{Q_t(n)}{(Np - K)\tilde{Q}_t(n)}.$$

Also according to Equation (2.5), at $t = \tau^u(n)$, $J^u_t(n) = \frac{Np - K}{H_t(n)}\left[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)\right]$. Hence,

$$\frac{Q_t(n-1)}{J^u_t(n-1)} - \frac{Q_t(n)}{J^u_t(n)} = \frac{H_t(n)\cdot Q_t(n-1)}{(Np - K)\tilde{Q}_t(n)} - \frac{Q_t(n)}{(Np - K)\tilde{Q}_t(n)}$$

$$= \frac{1}{Np - K}\left[\frac{H_t(n)\tilde{Q}_t(n-1)}{\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)} - \frac{Q_t(n)}{\tilde{Q}_t(n)}\right]$$

$$= \frac{1}{Np - K}\left[\frac{H_t(n)\cdot Q_t(n-1)}{\tilde{H}_t(n)\cdot Q_t(n-1) - (\tilde{H}_t(n) - H_t(n))\cdot Q_t(n)} - \frac{Q_t(n)}{\tilde{Q}_t(n)}\right].$$

Now recall from the proof of Proposition 2.4, we have shown that for any $t > 0$, $H_t(n)\left(\frac{Q_t(n)}{Q_t(n)} - 1\right)$ $\tilde{H}_t(n)\left(\frac{Q_t(n)}{Q_t(n)} - 1\right)$ $> 0$. Therefore, $\frac{\partial Q_t(n)}{\partial t} J^u_t(n)\big|_{t=\tau^u(n)+} > 0$. Suppose there exists some $t' > \tau^u(n)$ such that $\frac{Q_t(n)}{J^u_t(n)} - \frac{Q_t(n)}{J^u_t(n)} < 0$, then according to the continuity of the functions, there must exist some $\tau^u(n) < t' < t$, such that $\frac{\partial Q_t(n)}{\partial t} J^u_t(n)\big|_{t=t_0} = 0$ and $\frac{\partial Q_t(n)}{\partial t} J^u_t(n) < 0$ in interval $(t_0, t')$. However since $\frac{Q_t(n)}{J^u_t(n)}$ increases in $t$, $\frac{Q_t(n)}{J^u_t(n)} - \frac{Q_t(n)}{J^u_t(n)}$ must strictly increase in interval $(t_0, t')$, implying $\frac{\partial Q_t(n)}{\partial t} J^u_t(n)\big|_{t=t_0} > 0$. This leads to contradiction. Therefore, $\frac{\partial Q_t(n)}{\partial t} J^u_t(n) \geq 0$ for any $t > 0$. This would also imply $\frac{Q_t(n)}{J^u_t(n)} > \frac{Q_t(n)}{J^u_t(n)}$, thus completing the proof that $\frac{Q_t(n)}{J^u_t(n)}$ decreases in $n$.  

Proof of Theorem 2.15. Denote $G_t(n)$ the optimal expected revenue at state $(t, n)$
assuming that the creator has not ended LTO yet. The optimal expected revenue over the course of the entire pledging process is denoted by \( J_t^l(n) \). We show that \( \tau^l(n) \) is given by

\[
\tau^l(n) = \sup \left\{ t : G_t(n) \geq [(N - n)(p - k) + np] \cdot Q_t(n) \right\},
\]

(2.7)

where \( G_t(n) \) is the solution of

\[
\frac{\partial G_t(n)}{\partial t} = \lambda_t \hat{H}_t(n) \left[ J_t^l(n - 1) - G_t(n) \right],
\]

(2.8)

with boundary conditions \( G_0(n) = 0 \) for any \( n \geq 1 \), and \( G_t(0) = N(p - k) \).

Expected revenue \( J_t^l(n) \) at state \((t, n)\) is given by

\[
J_t^l(n) = \begin{cases} 
G_t(n), & \text{if } t < \tau^l(n) \\
\left[(N - n)(p - k) + np \right] \cdot Q_t(n), & \text{if } t \geq \tau^l(n)
\end{cases}
\]

Denote \( d_t(n) = \frac{G_t(n)}{Q_t(n)} \). We add to the statement that \( d_t(n) \) decreases in \( t \) and prove by induction. It’s trivial when \( n = 0 \) because \( d_t(0) = \frac{G_t(0)}{Q_t(0)} = N(p - k) \). Suppose that the statement is true for \( n - 1 \). Taking the derivative of \( d_t(n) \) w.r.t. \( t \), we have

\[
\frac{\partial d_t(n)}{\partial t} = \frac{\lambda_t \hat{H}_t(n) \left[ J_t^l(n - 1) - G_t(n) \right]}{Q_t(n)} - \frac{\lambda_t H_t(n) G_t(n) \left[ Q_t(n - 1) - Q_t(n) \right]}{[Q_t(n)]^2} \]

\[
= \frac{\lambda_t \left\{ \hat{H}_t(n) \left[ J_t^l(n - 1) Q_t(n - 1) - G_t(n) \right] - G_t(n) \left[ Q_t(n - 1) - Q_t(n) \right] \right\}}{Q_t(n)} \]

\[
= \frac{\lambda_t \hat{H}_t(n) Q_t(n - 1) \left[ J_t^l(n - 1) \right] - \left[ (1 - \left( \frac{H_t(n)}{\hat{H}_t(n)} \right) \right] \left( 1 - \frac{Q_t(n)}{Q_t(n - 1)} \right) d_t(n)}{Q_t(n) Q_t(n - 1)}.
\]

Taking the limit as \( t \to 0 \) and using L’Hopital’s rule, we have

\[
\lim_{t \to 0} d_t(n) = \lim_{t \to 0} \frac{\lambda_t \hat{H}_t(n) \left[ J_t^l(n - 1) - G_t(n) \right]}{\lambda_t H_t(n) \left[ Q_t(n - 1) - Q_t(n) \right]} = \lim_{t \to 0} \frac{\hat{H}_t(n) J_t^l(n - 1)}{H_t(n) Q_t(n - 1)}.
\]

\[
\lim_{t \to 0} \frac{H_t(n)}{\hat{H}_t(n)} \text{ exists due to Theorem 2.3(iii), where we show that } \frac{H_t(n)}{\hat{H}_t(n)} \text{ increases in } t.
\]

Next we show that, if there exists some \( t_1 \) such that \( \frac{\partial d_t(n)}{\partial t} \bigg|_{t=t_1} > 0 \), there must be some \( t_2 \in (0, t_1) \) such that \( \frac{\partial d_t(n)}{\partial t} \bigg|_{t=t_2} < 0 \). Consider the following two cases.
(1) \( \lim_{t \to 0} \frac{\dot{H}_t(n)}{H_t(n)} = \infty \). If \( \lim_{t \to 0} d_t(n) < \infty \), we have

\[
\lim_{t \to 0} d_t(n) = \lim_{t \to 0} \frac{\dot{H}_t(n)}{H_t(n)} \frac{J_t'(n-1)}{Q_t(n-1)} \geq \lim_{t \to 0} \frac{\dot{H}_t(n)}{H_t(n)} [(N - n + 1)(p - k) + (n - 1)p] = \infty,
\]

where the inequality is due to the fact that the creator’s optimal expected revenue is greater than or equal to the expected revenue if the creator chooses to end the limited-time offer immediately at time \( t \), i.e., \( J_t'(n-1) \geq [(N - n + 1)(p - k) + (n - 1)p] \cdot Q_t(n-1) \).

The result contradicts with the assumption that \( \lim_{t \to 0} d_t(n) < \infty \), and we thus conclude that \( \lim_{t \to 0} d_t(n) = \infty \). If there exists some \( t_1 \) such that \( \frac{\partial d_t(n)}{\partial t} \big|_{t=t_1} > 0 \), then there must be some \( t_2 \in (0, t_1) \) such that \( \frac{\partial d_t(n)}{\partial t} \big|_{t=t_2} < 0 \); otherwise, \( d_t(n) \) increases within \([0, t_1]\). By the induction hypothesis, we know that \( \frac{J_t'(n-1)}{Q_t(n-1)} \) decreases in \( t \) for any \( t \in [0, t_1] \) because \( J_t'(n-1) \) is equal to either \( G_t(n-1) \) or \([(N - n + 1)(p - k) + (n - 1)p] \cdot Q_t(n-1) \).

Coupling with the results that \( \frac{H_t(n)}{H_t(n)} \) and \( \frac{Q_t(n)}{Q_t(n-1)} \) both increase in \( t \), we conclude that \( B(t) \) decreases in \( t \). A direct consequence is that \( B(t_1) \leq B(0) = 0 \), which contradicts with the assumption that \( \frac{\partial d_t(n)}{\partial t} > 0 \).

Consequently, if there exists some \( t_1 \) such that \( \frac{\partial d_t(n)}{\partial t} \big|_{t=t_1} > 0 \), there must be some \( t_2 \in [0, t_1] \) such that \( \frac{\partial d_t(n)}{\partial t} \big|_{t=t_2} < 0 \). Due to the continuity of \( \frac{\partial d_t(n)}{\partial t} \), there exists some \( t_3 \in [t_2, t_1] \) such that \( B(t_3) = 0 \), and \( B(t) > 0 \) for any \( t \in (t_3, t_1) \). However, because that \( d_t(n) \), \( \frac{H_t(n)}{H_t(n)} \) and \( \frac{Q_t(n)}{Q_t(n-1)} \) increase in \( t \), and \( J_t'(n-1) \) decreases in \( t \) for any \( t \in (t_3, t_1) \), \( B(t) \) should decrease in \( t \), which contradicts with the preceding result. Therefore, \( d_t(n) \) must decrease in \( t \) for any \( t > 0 \). Moreover, because \( \frac{Q_t(n)}{Q_t(n-1)} \) strictly increases in \( t \), \( B(t) \neq 0 \) for any \( t \). Therefore for any \( t > 0 \), \( B(t) < 0 \) and \( d_t(n) \) strictly decreases in \( t \). As a result, \( G_t(n) = (N - n)(p - k) + np \cdot Q_t(n) \) for any \( t < \tau'(n) \), and the direction of the inequality is flipped for any \( t > \tau'(n) \).

Next we show that the creator’s optimal policy is to end the limited-time offer if and only if \( t > \tau'(n) \). Suppose that there exists some \( t_4 < \tau'(n) \), such that the creator’s optimal decision is to end the limited-time offer immediately, i.e., \( J_{t_4}'(n) = [(N - n)(p - p_0) - \]
\[ k + np \cdot Q_{t_4}(n) \]. Then, we have
\[
\left[ (N - n)(p - k) + np \right] \cdot Q_{t_4}(n) > \lambda_{t_4} \hat{H}_{t_4}(n) \delta J^l_{t_4}(n - 1) + (1 - \lambda_{t_4} \hat{H}_{t_4}(n) \delta) J^l_{t_4}(n) + o(\delta)
\]
\[ \geq \lambda_{t_4} \hat{H}_{t_4}(n) \delta J^l_{t_4}(n - 1) + (1 - \lambda_{t_4} \hat{H}_{t_4}(n) \delta) G_{t_4}(n) + o(\delta) = G_{t_4}(n) + o(\delta), \]

which contradicts with \( t_4 < \tau'(n) \). Therefore, the creator would not end the limited-time offer for any \( t \leq \tau'(n) \). Consider what happens in a small time interval \( \delta \), we have
\[ J^l_t(n) = (1 - \delta \lambda_H(n)) J^l_{t_4}(n) + \delta \lambda_H(n) J^l_{t_4}(n - 1) + o(\delta). \]

Rearranging, taking the limit as \( \delta \to 0 \), and replacing \( J^l_t(n) \) with \( G_t(n) \), we thus have
\[
\frac{\partial G_t(n)}{\partial t} = \lambda_H(n) \left[ J^l_t(n - 1) - G_t(n) \right].
\]

Suppose that there exists some \( t_5 \geq \tau'(n) \) such that \( J^l_t(n) = [ (N - n)(p - k) + np ] \cdot Q_t(n) \) when \( t \leq t_5 \) but \( J^l_t(n) \geq [ (N - n)(p - k) + np ] \cdot Q_t(n) \) for any \( t \in (t_5, t_5 + \delta) \). (Because \( J^l_{t_5}(n) = G_t(n) = [ (N - n)(p - k) + np ] \cdot Q_t(n) \), we can always find some \( t_5 \geq \tau'(n) \) such that it is optimal to continue the limited-time offer for \( t > t_5 \).) Thus, we have
\[
\left[ (N - n)(p - k) + np \right] \cdot Q_{t_5}(n)
\]
\[
< \lambda_{t_5} \hat{H}_{t_5}(n) \delta J^l_{t_5}(n - 1) + (1 - \lambda_{t_5} \hat{H}_{t_5}(n) \delta) J^l_{t_5}(n) + o(\delta)
\]
\[
= \lambda_{t_5} \hat{H}_{t_5}(n) \delta J^l_{t_5}(n - 1) + (1 - \lambda_{t_5} \hat{H}_{t_5}(n) \delta) \left[ [ (N - n)(p - k) + np ] \cdot Q_{t_5}(n) + o(\delta) \right].
\]

Plugging \( Q_{t_5}(n) = \lambda_{t_5} \hat{H}_{t_5}(n) \delta Q_{t_5}(n - 1) + (1 - \lambda_{t_5} \hat{H}_{t_5}(n) \delta) Q_{t_5}(n) + o(\delta) \) into the inequality above, rearranging, and taking the limit as \( \delta \to 0 \), we have
\[
\left[ (N - n)(p - k) + np \right] \cdot \left( H_{t_5}(n) Q_{t_5}(n - 1) + (\hat{H}_{t_5}(n) - H_{t_5}(n)) Q_{t_5}(n) \right) \leq \hat{H}_{t_5}(n) J^l_{t_5}(n - 1).
\]

Because that \( B(t) = \frac{J^l_{t_5}(n - 1)}{Q_{t_5}(n - 1)} - \left[ 1 - \left( 1 - \frac{H_{t_5}(n)}{H_{t_5}(n)} \right) \left( 1 - \frac{Q_{t_5}(n)}{Q_{t_5}(n - 1)} \right) \right] d_t(n) < 0 \) for any \( t \), we have
\[
\hat{H}_{t_5}(n) J^l_{t_5}(n - 1) < \left[ H_{t_5}(n) Q_{t_5}(n - 1) + (\hat{H}_{t_5}(n) - H_{t_5}(n)) Q_{t_5}(n) \right] d_t(n).
\]

Combining the preceding two inequalities, we have that \( d_t(n) = \frac{G_{t_5}(n)}{Q_{t_5}(n)} > [ (N - n)(p - k) + np ] \), which contradicts with \( t_5 > \tau'(n) \). Therefore, the creator’s optimal policy is to end the limited-time offer for any \( t > \tau'(n) \), i.e., \( J^l_t(n) = [ (N - n)(p - k) + np ] \cdot Q_t(n) \).
for any \( t > \tau^l(n) \). We thus obtain the announced results. \( \blacksquare \)

**Proposition 2.19**

\( (i) \) If \( \Lambda \hat{H}(1) \leq N \), \( \lim_{m \to \infty} \frac{J^*_m}{m} = 0 \). (If \( \Lambda \hat{H}(1) < N \), \( J^*_m = 0 \), \( \forall m \).)

\((ii)\) If \( \Lambda \hat{H}(1) \geq N \), \( \lim_{m \to \infty} \frac{J^*_m}{m} = 1 \).

**Proof of Proposition 2.19.** First, when \( \Lambda \hat{H}(1) \leq N \), using the same argument as in the proof of Theorem 2.16, we know that that success likelihood will converge to zero even if the creator commits to never end LTO until time expires. Hence \( \lim \frac{J^*_m}{m} = 0 \).

Next we consider the case when \( \Lambda H(1)ds > N \). In the deterministic problem, the optimal timing to end LTO is at time-to-go \( x^* = T \), i.e., the creator would choose not to use LTO at all. The corresponding revenue is given by \( J^*_{m,d} = mNp \), which is the maximum amount of fund the creator can possibly collect by ending LTO immediately. To this end, we need to show that the success likelihood would also converge to 1 in the stochastic model if the creator commits never to use LTO. Similar to the proof of Theorem 2.16, we could show that the success likelihood will converge to 1 uniformly at any time almost surely.

Next consider the case when \( \Lambda H(1) \leq N < \Lambda \hat{H}(1) \). We denote \( x^* \in [0,T) \) the optimal time-to-go to end LTO in the deterministic problem. The corresponding optimal revenue is given by \( J^*_{m,d} = m \left( (p-k) \int_x^T \lambda_s \hat{H}(1)ds + p(N - \int_x^T \lambda_s \hat{H}(1)ds) \right) \). Next consider the stochastic problem. Suppose the probability that the creator would end LTO before time-to-go \( x^* + h \) does not converge to 0 for any \( h > 0 \). We could always find a \( \Psi \) such that \( \int_{x^*+h}^T \lambda_s \hat{H}(1)ds < \Psi < N - \int_0^{x^*+h} \lambda_s H(1)ds \). Due to the law of large numbers, the probability that the number of pledgers for the project before time \( t = x^* + h \) is greater than or equal to \( m \Psi \) converges to 0 when \( m \to \infty \). On the other hand, the probability that more than \( m(N - \Psi) \) backers pledge between \( t = x^* + h \) and \( t = 0 \) also converges to 0 because \( \int_0^{x^*+h} \lambda_s H(1)ds + \Psi < N \). The preceding arguments imply that the creator would end LTO with a positive probability, knowing that the move would lead to a success likelihood of 0 when \( m \to \infty \), which is impossible. As a result, when \( m \to \infty \), we conclude that the creator would not end LTO before \( t = x^* + h \) with probability 1 for any \( h > 0 \). A direct consequence is that the total amount of funds to be collected is at most \( m \left( (p-k) \int_{x^*+h}^T \lambda_s \hat{H}(1)ds + p(N - \int_{x^*+h}^T \lambda_s \hat{H}(1)ds) \right) + \omega(m) \). Taking the limit as \( m \to \infty \), we have \( \lim \frac{\text{DS}_m}{m} \leq (p-k) \int_{x^*+h}^T \lambda_s \hat{H}(1)ds + p(N - \int_{x^*+h}^T \lambda_s \hat{H}(1)ds) \).

On the other hand, consider a heuristic where the creator ends LTO at time-to-go \( x^* - h \) for any \( h > 0 \). Because \( \int_{x^*+h}^T \lambda_s \hat{H}(1)ds + \int_0^{x^*+h} \lambda_s H(1)ds > N \), we can follow a similar approach to that in the case when \( \int_0^T \lambda_s H(1)ds \geq N \), and show that the success likelihood would converge to 1 in the stochastic model when \( m \to \infty \). Consequently, the
creator’s revenue is given by
\[ m \left[ (p - k) \int_{x^* - h}^{T} \lambda_s \hat{H}(1) ds + p(N - \int_{x^* - h}^{T} \lambda_s \hat{H}(1) ds) \right] + o(m). \]
Taking the limit as \( m \to \infty \), we have
\[ \lim_{m \to \infty} \frac{j_m^h}{\sqrt{m}} \geq \lim_{m \to \infty} \frac{j_m^h}{m} \geq (p - k) \int_{x^* - h}^{T} \lambda_s \hat{H}(1) ds + p(N - \int_{x^* - h}^{T} \lambda_s \hat{H}(1) ds). \]

Combining the results above and taking the limit as \( h \to 0 \), we have
\[ \lim_{m \to \infty} \frac{j_m^h}{m} = (p - k) \int_{x^*}^{T} \lambda_s \hat{H}(1) ds + p(N - \int_{x^*}^{T} \lambda_s \hat{H}(1) ds). \]
We thus obtain the announced results. ■

**Proof of Theorem 2.15.** We study a family of static heuristics where the creator would end LTO at a deterministic time-to-go \( x^* - h(m) \). As we have shown in the proof of Proposition 2.19, \( h(m) \geq 0 \); otherwise the project’s success likelihood would converge to zero. We also assume that \( \lim_{m \to \infty} h(m) = 0 \) and \( \lim_{m \to \infty} \sqrt{m} h(m) = \infty \), as we can see in the proof later that if this is not true, the static heuristic will not be asymptotically optimal. The expected revenue with such heuristics is denoted by \( J_m^{l,h} \). To show that
\[ \lim_{m \to \infty} \sqrt{m} \left( 1 - \frac{j_m^h}{j^h_m} \right) = \infty, \] notice that the pledging likelihood \( H(\cdot) \leq H(1) \) for any \( t \) and \( n \), and thus it suffices to investigate the special case where \( H(\cdot) \equiv H(1) \), as in general \( J_m^{l,h} \) can be much smaller.

Denote \( M_1 \) as the number of pledgers between time-to-go \( T \) and \( x^* - h(m) \). Due to the central limit theorem, \( \xi_1^m \equiv \frac{1}{\sqrt{m}} \left[ M_1 - m \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds \right] \to \mathcal{N}(0, \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds) \), as \( m \to \infty \). Similarly, denote \( M_2 \) as the number of pledgers between time-to-go \( x^* + h(m) \) and 0. Then \( \xi_2^m \equiv \frac{1}{\sqrt{m}} \left[ M_2 - m \int_{0}^{x^* - h(m)} \lambda_s H(1) ds \right] \to \mathcal{N}(0, \int_{0}^{x^* - h(m)} \lambda_s H(1) ds) \), as \( m \to \infty \). Thus, the expected revenue \( J_m^{l,h} \) can be written as
\[
J_m^{l,h} = \mathbb{E} \left( (p - k) \min\{M_1, mN\} + p \cdot \max\{mN - M_1, 0\} | M_1 + M_2 \geq mN \right) \mathbb{P} \left( M_1 + M_2 \geq mN \right)
= \mathbb{E} \left( mN p - k \min\{M_1, mN\} | M_1 + M_2 \geq mN \right) \mathbb{P} \left( M_1 + M_2 \geq mN \right)
= \left[ mN p - k \mathbb{E} \left( \min\{M_1, mN\} | M_1 + M_2 \geq mN \right) \right] \mathbb{P} \left( M_1 + M_2 \geq mN \right)
\leq \left[ mN p - k \mathbb{E} \min\{M_1, mN\} \right] \mathbb{P} \left( M_1 + M_2 \geq mN \right)
= \left[ mN p - k \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds + k \mathbb{E} \left( \min\{M_1, mN\} - m \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds \right) \right] \mathbb{P} \left( M_1 + M_2 \geq mN \right)
= \left[ mN p - k \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds + k \sqrt{m} \mathbb{E} \left( \min\left\{ \xi_1^m, \sqrt{m} \left( N - \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds \right) \right\} \right) \right] \mathbb{P} \left( M_1 + M_2 \geq mN \right).
\]

Next let us consider each term in the last equation. Recall that \( J_m^{l,d} = m \left[ (p - k) \int_{x^*}^{T} \lambda_s \hat{H}(1) ds + p(N - \int_{x^*}^{T} \lambda_s \hat{H}(1) ds) \right] \). There exists a positive constant \( a \) such that \( m(N p - k \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds) = m(N p - k \int_{x^*}^{T} \lambda_s \hat{H}(1) ds - \int_{x^* - h(m)}^{T} \lambda_s \hat{H}(1) ds) \leq J_m^{l,d} - a \cdot mh(m) \). Because \( 0 < \lambda_t \leq \Lambda \) for
any $t$ and $0 < \hat{H}(1) \leq 1$, the existence of $a$ is guaranteed. Regarding the second term, because $N = \int_0^x \lambda_s H(1) ds + \int_x^T \lambda_s \hat{H}(1) ds$, we know that $\sqrt{m} \left( N - \int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds \right) \to \infty$ when $m \to \infty$. Therefore, we have

$$\min \{ \xi_m^1, \sqrt{m} \left( N - \int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds \right) \} \to N(0, \int_x^T \lambda_s \hat{H}(1) ds), \text{ as } m \to \infty.$$ 

A direct consequence is that $k \sqrt{m} \mathbb{E} \left( \min \{ \xi_m^1, \sqrt{m} \left( N - \int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds \right) \} \right) = o(\sqrt{m})$.

Then we consider the term $P \left( M_1 + M_2 \geq mN \right)$. Because $\int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds + \int_0^{x^*-h(m)} \lambda_s H(1) ds = N + \int_{x^*-h(m)}^T \lambda_s (\hat{H}(1) - H(1)) ds$, there exists a positive constant $b$, such that:

$$\lim_{m \to \infty} P(M_1 + M_2 \geq mN) = \lim_{m \to \infty} P \left( \xi_m^1 + \xi_m^2 \geq -\sqrt{m} \int_{x^*-h(m)}^T \lambda_s (\hat{H}(1) - H(1)) ds \right) \geq \lim_{m \to \infty} P \left( \xi_m^1 + \xi_m^2 \geq -b \cdot \sqrt{mh(m)} \right) = 1.$$

Because $0 < \lambda_t \leq \bar{\lambda}$ for any $t$ and $0 < \hat{H}(1) - H(1) \leq 1$, the existence of $b$ is guaranteed. The last equality is due to our assumption that $\lim_{m \to \infty} \sqrt{mh(m)} = \infty$.

Combining the results above, we have

$$\lim_{m \to \infty} \sqrt{m} \left( 1 - \frac{J_{m}^{l,h}}{J_{m}^{l,d}} \right) \geq \lim_{m \to \infty} \sqrt{m} \left( 1 - \frac{\left[ J_{m}^{l,d} - a \cdot mh(m) + o(\sqrt{m}) \right] P(M_1 + M_2 \geq mN)}{J_{m}^{D}} \right)$$

$$= \lim_{m \to \infty} \sqrt{m} \frac{a \cdot mh(m) + o(\sqrt{m})}{J_{m}^{D}} P(M_1 + M_2 \geq mN) = \infty.$$

On the other hand, the expected revenue can be rewritten as

$$\frac{J_{m}^{l,h}}{m} = \frac{1}{m} \left[ mNP - k \mathbb{E} \left( \min \{ M_1, mN \} \mid M_1 + M_2 \geq mN \right) \right] P(M_1 + M_2 \geq mN)$$

$$\geq mNP \cdot P(M_1 + M_2 \geq mN) - \frac{k}{m} \mathbb{E} \left( M_1 \mid M_1 + M_2 \geq mN \right)$$

$$= mNP \cdot P(M_1 + M_2 \geq mN) - k \int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds + o(1).$$

Thus, we have

$$\lim_{m \to \infty} \frac{J_{m}^{l,h}}{J_{m}^{l,d}} \geq \lim_{m \to \infty} \frac{mNP \cdot P(M_1 + M_2 \geq mN) - k \int_{x^*-h(m)}^T \lambda_s \hat{H}(1) ds + o(1)}{mNP - k \int_{x^*}^T \lambda_s \hat{H}(1) ds} = 1.$$
where the inequality is due to $\lim_{m \to \infty} P(M_1 + M_2 \geq mN) = 1$ and $J_{m}^{l,d} = m\left(Np - k \int_{x}^{T} \lambda_s \hat{H}(1)ds\right)$. We have also shown that $\lim_{m \to \infty} \sqrt{m} \left(1 - \frac{\mu_{m}^{l,h}}{\nu_{m}}\right) = +\infty$, which implies $\lim_{m \to \infty} \frac{\mu_{m}^{l,h}}{\nu_{m}} \leq 1$. Combining the preceding two claims, we thus conclude that $\lim_{m \to \infty} \frac{\mu_{m}^{l,h}}{\nu_{m}} = 1$. \qed
Chapter 3

Revenue Management with All-or-Nothing Constraint

3.1 Introduction

Managing business performances under deadline is a fundamental problem in almost all industry. A salesperson at a car dealership is evaluated monthly and her monthly compensation heavily depends on her sales number. She must decide how much effort she exerts to maximize her compensation. In politics, legislatures are to be voted on election date, and politicians need to campaign for the proposals to gather support for the legislatures before the vote.

In addition to the deadline, in many business problems, there is also an all-or-nothing constraint. The decision maker would only be rewarded only if the performance reaches a predetermined milestone. The compensation contracts of the salespeople often includes lump-sum bonuses that are awarded to the salesperson only if her sales number exceeds a quota. The salesperson’s decision on how much efforts she exerts thus depends on both the deadline and how far her numbers are to the quota. An episode of the popular podcast This American Life (2013) shows vividly how the sales activities of the car dealership and the salespeople employed by the dealership all evolve around meeting the monthly sales quota of 129 cars set by the manufacturer. The manufacturer pays out bonuses between $65,000 to $85,000 if they meet the sales quota, but otherwise they get nothing.

In US politics, many states allow initiated state statutes, which allow any citizen or organization to propose legislatures to be put to vote on election day. The initiative campaign will be successful only if it gathers enough signatures before the deadline. For example, according to Aull (2015), on average it takes to over two million dollars to
gather enough signatures and get the initiative on the ballot in California. The signature gathering process is typically handled by petition drive management firms. They must decide how intensive their canvassing campaign needs to be based on time and number of signatures needed. In the cases where they fail to gather enough signature, the consequences are abysmal as all the money spent on the campaign would be wasted (see Egan (2016)).

Crowdfunding, a form of alternative financing that raises money from a large pool of people, is becoming ever more popular in recent years. A crowdfunding project is typically hosted on a platform such as Kickstarter and Indiegogo. The crowdfunding project creator sets up a project that needs funding from public. She then sets up the goal and the limited time, typically between one and sixty days, to achieve that goal. Once the parameters are set up and the project goes live, backers arrive sequentially and pledge their money on the project. One of the unique features about crowdfunding is that unless the goal is reached within the predetermined deadline, the project cannot get any of the funds. According to Du et al. (2017), less than 30% of the Kickstarter projects reaches their predetermined goal. Advertising to promote the project is particularly important in crowdfunding. The project creators are typically early-stage innovators and are relatively unknown. There are also thousands of projects live on the platform at any give time. In addition to that, the project needs to reach the goal within a relatively short deadline. Advertising is a great way for the creators to attract additional backers to increase success rate. However the spending on advertising will be wasted if the project still fails to reach its goal in the end. The project creator faces a dilemma on how much she needs to spend on advertising during the crowdfunding campaign.

In this paper, we study the general problem of how a decision maker should manage the business performance under a deadline with an all-or-nothing constraint. Specifically we consider a model where demand arrives according to a nonhomogeneous Poisson process, and the seller can control the sales intensity under a cost. The seller will be rewarded at the end of the time period only if the total sales is greater than a predetermined threshold. We consider a general form of reward that includes both a lump-sum payout and over-threshold commissions. We formulate this problem as a dynamic programming problem and show that contrary to the traditional revenue management problem, the optimal demand rate is not monotone with respect to time-to-go and the distance to the threshold. We demonstrate that that there are two forces affecting the seller’s decision: On one hand she have incentive to increase the rate to meet the threshold and get the reward; On the other hand, the cost of inducing the high demand rate and the possibility of failing to reach the threshold discourages the seller from increasing the sale intensity.
Those two factors join forces and result in a turning point in time: The seller will increase the sales intensity as it gets closer to the deadline, but only to a point. After the turning point the optimal rate will decrease as time-to-go decreases.

We then formulate the deterministic fluid approximation of the problem and investigate the relationship between the stochastic problem and the deterministic problem. Traditional revenue management literature has stated that the deterministic problem provides an upper bound for the original stochastic problem. However, we construct an example to show that this statement is no longer true under the all-or-nothing constraint. Nonetheless, we construct a series of problems that scale up the time window, the threshold and the rewards. We prove that when the scale of the problem is large enough, the optimal profit of the deterministic problem does serve as an upper bound of that of the stochastic problem. Moreover, we observe that the scaled-up problem is analogous to extending the evaluation window and rewarding the seller only once at the time period. We show that when the time window is long enough, the seller can no longer be opportunistic and “game” the system, and her expected sales must be higher than the threshold.

Because of the complex structure of optimal demand rate, we investigate heuristics that are simple to compute. Motivated by the connection between the deterministic problem and the stochastic problem, we derive static heuristics from the solutions of the deterministic problem. We show that when the revenue maximizing rate is larger than the threshold rate, static heuristics perform extremely well, as the revenue loss drops to zero as the scale of the problem increases. However, when revenue maximizing rate is smaller than the threshold rate, the performance of the static heuristics is compromised. We show that because of the all-or-nothing constraint, the seller must increase her rate to higher than the threshold rate, making the performance gap larger than the square root of the scale factor, which is the performance gap of the static heuristics in the traditional RM problems.

As the static heuristics doesn’t perform well when the revenue maximizing rate is smaller than the threshold rate, to improve the performance, we investigate the periodic updating heuristics where the seller updates the demand rate with a predetermined frequency. In dynamic pricing, Jasin (2014) has shown that the resolving heuristic, which frequently resolves the deterministic problem with updated threshold and time period, yields a logarithmic performance gap. We adopt the same heuristic and show that the standard resolving heuristic is not asymptotically optimal due to the all-or-nothing constraint. Instead, we propose a modified two-stage resolving heuristic. In the first stage the heuristic re-optimizes the profit responsively, and in the second stage it switches to a higher rate to make sure the threshold will be reached. We prove that the modified
resolving heuristic is asymptotically optimal, and has a logarithmic performance gap, the same gap as the standard resolving heuristic in the dynamic pricing problem. This means that through periodically adjusting her sales intensity, the seller is able to recover most of the revenue losses despite the all-or-nothing constraint.

3.2 Literature Review

This paper contributes to the literature in revenue management. Revenue management and dynamic pricing have been studied extensively (for comprehensive reviews see McGill and Van Ryzin (1999) and Talluri and Van Ryzin (2006)), but traditional RM problems solve the revenue maximization problem with the inventory constraint where the total sales is bounded from above. In our work we consider the all-or-nothing constraint where the total sales is bounded from below. Unless the total sales number reaches predetermined target, the seller will get no payout at all. Besbes and Maglaras (2012) considered dynamic pricing problems where the seller is penalized for failing to reaching the revenue and sales targets during the selling season. Besbes et al. (2017) studies the dynamic pricing problem under financial debt. In both papers the seller’s penalties are continuous w.r.t the sales and revenue. In contrast in our paper, with the all-or-nothing constraint, the seller is paid nothing if her sales falls short of the threshold. We demonstrate that the discontinuity around the threshold in the seller’s payout function forces the seller to implement strategies that ensures the sales to go over the threshold. It creates both motivation and deterrent for the seller, leading to nontrivial distinctions in the seller’s strategies comparing to traditional RM problems.

Our work is also related to the works that study the deterministic approximation in RM and the heuristics that based on it. Gallego and Van Ryzin (1994, 1997) show that the optimal profit of the deterministic problem is an upper bound to that of the stochastic problem. Also when the problem is scaled up, the static heuristic that adopts the solution in the deterministic problem is asymptotically optimal and has a performance gap of the square root of the scale parameter. In contrast, we demonstrate that under the all-or-nothing constraint, the deterministic problem in general is no longer an upper bound. We modify the static heuristics so it can still be asymptotically optimal, however we show that the performances are compromised. We also the investigate dynamic “resolving” heuristics that updates the sales intensities by resolving the deterministic problem during the time horizon. “Resolving” heuristics have been studied by Cooper (2002), Maglaras and Meissner (2006), Jasin and Kumar (2012) and Jasin (2014) in different contexts of revenue management. Yet we demonstrate that due to the all-or-nothing constraint, a
standard resolving heuristic is not asymptotically optimal. As a result we devise a two-stage modified resolving heuristic that reduces cost at the first stage, and maximizes the probability of hitting the threshold at the second stage.

All-or-nothing constraints has been discussed in the context of crowdfunding. Zhang et al. (2017) and Alaei et al. (2016) study the dynamics of the crowdfunding projects to find the optimal design of project parameters, including the goal, the duration and the pledge level. They assume predetermined customer arrival intensities whereas in paper we investigate how the seller should control her sales intensities during the selling season. Du et al. (2017) study contingent stimulus policies in crowdfunding where project creators dynamically offer stimuli including feature upgrade and limited time offers to induce the customers’ pledging. The distinction is that the crowdfunding projects are typically at the design stage, so the costs of the stimulus policies are only incurred if the project reaches the goal. This feature results in a cutoff timing structure for the optimal stimulus offering policies. In contrast, the all-or-nothing constraint in this paper requires the seller to pay immediately for the increase in sales intensity. This creates a dilemma for the seller as the spending is wasted if she couldn’t reach the threshold by deadline.

Our work is also related to the widely studied sales-force compensation problem (see Coughlan (1993) for a comprehensive survey). Basu et al. (1985) apply the agency theory framework in Hölmstrom (1979) and show that the optimal compensation structure is nonlinear function of the sales. Holmstrom and Milgrom (1987) study the the problem where the principal sets up the compensation scheme at the beginning of the time period and the agent adjusts her effort throughout the period. They showed that under constant absolute risk-aversion, the optimal compensation is a linear function of the total sales over the entire selling season. Optimal compensation structures in the majority of works in sales-force compensation in are continuous with respect to sales, yet Joseph and Kalwani (1998) suggest that in practice incentive contracts with discrete jumps in the sales numbers are prevalent. Raju and Srinivasan (1996) show numerically that even though the quota-bonus system may not be optimal, the performance is close to that of the optimal compensation policy. Oyer (2000) shows that when the agent’s participation constraint is not binding, the quota-based bonus structure can be the optimal compensating mechanism. Empirically, Steenburgh (2008) and Chung et al. (2013) show that the quota-based compensation plans incentivise the sales-force to work harder. In this paper, rather than arguing whether quota-based compensations are a good plan, we offer guidance on how to maximize their profits for sellers who already accept an all-or-nothing contract.

In operations management literature, Chen (2000) and Sohoni et al. (2010) show that
under a quota system the salesperson may delay her effort decision until the later in the selling horizon and creates a “hockey-stick” phenomenon in sales. Sales fluctuations has an adverse effect on the manufacturer’s production and inventory planning. To mitigate the hockey-stick phenomenon, Chen (2000) proposes a moving time window evaluation schedule. Sohoni et al. (2010) show that the reduction of demand variability with better information dampens the hockey stick phenomenon.

3.3 The Model

3.3.1 Model Description

We consider a seller selling a single product for a fixed price. The seller’s performance is evaluated at the end of a finite selling season \( T \). The seller has agreed upon an all-or-nothing contract predetermined before the selling season starts. We define the all-or-nothing contract as follows:

**Definition 2 (All-or-Nothing Contract)** Let \( x \) be the total sales during the selling period. There exists a threshold \( N \) such that the payout to the seller \( R(x) \) is

\[
R(x) = \begin{cases} 
  b + p(x - N) & x \geq N \\
  0 & x < N,
\end{cases}
\]

(3.1)

where \( b, p \geq 0 \).

In Definition 2 the seller is only rewarded if her sales reaches the threshold \( N \) at the end of the selling season. The payout consists of two parts: a lump sum payment of \( b \) and a bonus commission of \( p \) for each unit sold beyond the threshold.

We note that under our definition, many commonly used reward plans are special cases of the equation 3.1. For example when \( N = 0 \) the contract is reduced to the regular commission plan where the seller is paid for every unit she has sold. Sohoni et al. (2010) consider two contracts: the \( \Delta \)-contract and the \( D \)-contract. The \( \Delta \)-contract pays the seller a marginal commission for units sold over threshold. This corresponds to the case in Definition 2 where \( b = 0 \) and \( p, N > 0 \). The \( D \)-contract pays the seller a fixed bonus for reaching the threshold but doesn’t pay any over-achievement commission. This corresponds to the case in Definition 2 where \( b = 0 \) and \( p, N > 0 \).

We assume that the customers arrive and make their purchases according to a non-homogeneous Poisson process \( D_t \). The seller can control the intensity \( \lambda_t \) of the sales process. In the context of sales-force management, the salesperson decides how much
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effort she exerts and the stochastic sales is responds to that (Basu et al., 1985). Since there is a one-to-one correspondence between effort and sales rate, we use the sales intensity $\lambda_t$ as our decision variables as is conventional in the revenue management literature. See Gallego and Van Ryzin (1994), Maglaras and Meissner (2006) and Jasin (2014) for examples. We assume that $\lambda_t \in [\underline{\lambda}, \bar{\lambda}]$.

There is a cost associated with the sales intensities that the seller chooses. In the context of car sales, part of the cost comes from the discounts offered to the customers. According to the podcast This American Life (2013), the car dealership offers deep discounts, sometimes at a loss, to induce a higher sales rate. The seller can also pay online advertisers and promote her business to increase the exposure. We denote the cost rate for the seller as $c(\lambda) \geq 0$ for intensity $\lambda$.

**Assumption 2** (Properties of the Cost Rate Function $c(\cdot)$)

1. $c(\underline{\lambda}) = 0$.
2. $c(\lambda)$ is increasing and convex in $\lambda$.

Assumption 2(i) says that there exists a natural cost-free sales rate $\underline{\lambda}$. Assumption 2(ii) means that the cost is higher if a higher intensity is to be induced. Moreover the convexity of $c(\cdot)$ indicates that the spending has a diminishing return on the demand. This assumption is consistent with Lal and Staelin (1986), Oyer (2000) and Sohoni et al. (2010) in which the sales response function, the inverse of the cost rate function, is concave.

The seller is risk neutral and she selects a sales intensity strategy $u$ over the selling season to maximize her expected profit $\Pi_u(\lambda)$ at the end of the selling horizon. Hence we can formulate the problem as

$$\Pi_{opt} = \max_u \pi_u(\lambda) = \mathbb{E}_u \left[ R \left( \int_0^T dD_s \right) - \int_0^T c(\lambda_s) ds \right].$$  \hspace{1cm} (3.2)

We end this section by showing an example of applying our modeling framework to the advertising problem in crowdfunding.

**Example 3.1** (Advertising in Crowdfunding) For a crowdfunding project, the goal is $X$ and the time duration is $T$. For simplicity we assume that there’s only one price tier $p$. This is equivalent to needing $N = \lceil \frac{X}{p} \rceil$ backers before time expires. Backers pledge on the project according to a Poisson process. The pledge intensity $\lambda(a)$ is an increasing and concave function of the creator’s advertising spending rate $a$. The project creator must
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We denote \( t \) as time-to-go and \( n \geq 0 \) as the number of sales needed to get to the threshold. It is easy to see that the decision on the optimal sales intensity only depends on \( t \) and \( n \). In the next proposition we show that problem [3.2] can be formulated as a Hamilton-Jacobi-Bellman (HJB) equation.

**Proposition 3.1** Denote \( J^*_t(n) \) as the optimal expected pay-off at state \((t, n)\). \( J^*_t(n) \) is the solution of:

\[
\frac{\partial J^*_t(n)}{\partial t} = \max_{\lambda} \left\{ \lambda [J^*_t(n-1) - J^*_t(n)] - c(\lambda) \right\},
\]

with boundary conditions \( J^*_0(0) = b \) for \( n = 0 \). \( J^*_0(n) = 0 \) for any \( n \geq 1 \). For notation convenience, we let \( J^*_t(-1) = J^*_t(0) + p \).

The optimal \( \lambda(t, n) \) is the unique solution of:

\[
c'(\lambda) = \max\{J^*_t(n-1) - J^*_t(n), c'(\Lambda)\}.
\]

In the next proposition, we show some structural properties of \( \lambda(t, n) \) and \( J^*_t(n) \).

**Theorem 3.2** For each \( n \geq 1 \), there exists a time \( 0 \leq \tau(n) \leq \infty \) such that:

(i) When \( t \leq \tau(n) \), \( \lambda(t, n) \) increases in \( t \). \( \lambda(t, n) \leq \lambda(t, n-1) \). \( J^*_t(n-1) - J^*_t(n) \) increases in \( t \).

(ii) When \( t > \tau(n) \), \( \lambda(t, n) \) decreases in \( t \). \( \lambda(t, n) \geq \lambda(t, n-1) \). \( J^*_t(n-1) - J^*_t(n) \) decreases in \( t \).

Moreover, the time threshold \( \tau(n) \) increases in \( n \).

Proposition [3.2] shows that the there exists a turning point \( \tau(n) \). When \( t \) is less than \( \tau(n) \), the probability of reaching the threshold is low. The cost cannot justify the increase in sales intensity. The seller thereby reduces the intensity as time gets closer to deadline or sales gets further from the threshold. When \( t \) is greater than \( \tau(n) \), the probability of reaching the threshold is relatively high. The benefits of increasing the intensities to
reach the threshold outweigh the cost, so the seller increases her sales intensity as time gets closer to deadline or sales gets further from the threshold.

In traditional revenue management problems, the optimal demand rate decreases with time-to-go and increases with the remaining inventory (Gallego and Van Ryzin (1994)). In contrast, the relationships are no longer monotone with all-or-nothing constraint. On the one hand, when as the time-to-go decreases, the lump sum payout $b$ incentivises the seller to increase the sales intensity to reach the threshold. On the other hand, this also means that the chance of not reaching the threshold is higher, and this discourages the seller from inducing the high intensity. The two forces combine and creates a turning point. Theorem 3.2 is also consistent with the empirical findings in Steenburgh (2008), where they find that under the lump-sum bonus contract, the salespeople stretches themselves to reach the quota, but may give up if they feel that they are too far away.

### 3.4 Deterministic Problem

We study the deterministic version of problem 3.2, where demand arrives in a “fluid fashion”. As there is no demand uncertainty in the deterministic problem, the decision variable is simply $\lambda_t$’s. We formally state the problem as follows:

$$\Pi_D = \max \pi_D(\lambda) = b + p \left( \int_0^T \lambda_t dt - N \right) - \int_0^T c(\lambda_t) dt$$

$$\text{s.t. } \int_0^T \lambda_t dt \geq N. \tag{3.5}$$

Note that in problem we implicitly assume that $\Pi_D \geq 0$, as otherwise the seller would simply choose $\lambda$ for the entire time horizon and this problem becomes trivial.

**Proposition 3.3** The solution to the deterministic problem is $\lambda_t = \lambda_D = \max \left\{ \frac{N}{T}, \lambda^* \right\}$, where $\lambda^* = \arg \max \left\{ \lambda \left| \lambda p - c(\lambda) : \underline{\lambda} \leq \lambda \leq \bar{\lambda} \right. \right\}$.

Intuitively, the seller chooses the sales rate $N/T$ that just reaches the threshold by deadline, unless the revenue maximizing rate $\lambda^*$ is higher. In that case, since the per unit reward $p$ for the sales that exceeds the threshold is high enough, the seller exerts at an effort level higher than $N/T$. Proposition 3.3 that if $p = 0$, $\lambda^* = \underline{\lambda}$ and $\lambda_D = N/T$. Intuitively if there is no over-threshold commission, the seller chooses the intensity that just reaches the threshold.

In traditional revenue management problems, Gallego and Van Ryzin (1994, 1997) has shown that the optimal revenue in the deterministic problem is an upper bound for
that in the stochastic problem. However with the all-or-nothing constraint, this is in
general not true. Depending on the parameters it is possible to have $\Pi_{opt} > \Pi_D$. We
illustrate that in a simple example.

**Example 3.2** When $T = 1$, $N = 1$, $p = 5$ and $b = 5$. $c(\lambda) = (3\lambda - 1)^2$. Then
$\lambda^D = N/T = 1$. $\pi^D = 5 - (3 \times 1 - 1)^2 = 1$. However if we set the $\lambda = 1/3$, the total
cost is 0.

The number of sales before time expires now follows a Poisson distribution with rate
$1/3$. Since $N=1$, the expected reward

$$
E_u R \left( \int_0^T dD_s \right) = E_u \left[ 5 + 5 \left( \int_0^T dD_s - 1 \right) \right] \mathbb{P}_u \left[ \int_0^T dD_s \geq 1 \right]
$$

$$
= 5E_u \left[ \int_0^T dD_s \left| \int_0^T dD_s \geq 1 \right. \right] \mathbb{P}_u \left[ \int_0^T dD_s \geq 1 \right]
$$

$$
= 5E_u \left[ \int_0^T dD_s \right] = 5 \times \frac{1}{3} = 1.67.
$$

This means that the expected $\Pi_{opt} \geq 1.67 > \Pi_D = 1$.

Example 3.2 illustrates that seller may have incentives to “game” the system in the
sales-force compensation problem. For the firm, the intention of setting up a quota $N$ is
to force the agent to keep the sales rate at $N/T$. Yet since the firm can’t directly observe
the efforts, the demand uncertainties may make the agents opportunistic and give them
incentives to exert lower efforts.

However we can show that when $N$ and $T$ are “large enough”, $\Pi_{opt}^\theta < \Pi_D^\theta$. To do
that, similar to Besbes and Maglaras (2012) and Jasin (2014), we introduce the scaling
factor $\theta$, and investigate a series of problems where $T\theta = \theta T$, $N(\theta) = \theta N$ and $b(\theta) = \theta b$. Intuitively, this construction can be interpreted as a less frequent performance evaluation
scheme. Instead of evaluating the sales number and rewarding the seller every $T$ time
period, essentially making the seller solve the original problem in 3.2 repeatedly for $\theta$
times, in the $\theta$’th problem, the sales number is evaluated only once at the end of the $\theta T$
time.

For the $\theta$’th problem, we denote $\Pi_{opt}^{\theta}$ as the optimal expected profit of the $\theta$’th
stochastic problem, and $\Pi_D^{\theta}$ as the optimal profit of the deterministic problem. It is
easy to see that without demand uncertainty, the optimal solution in the deterministic
problem remains the same and $\Pi_D^{\theta} = \theta \left[ b + p(\lambda_D T - N) - c(\lambda_D) T \right]$. However $\Pi_{opt}^{\theta}$ must
be recalculated from the dynamic programming problem in Proposition 3.1. In the next
proposition we show that when $\theta$ is large enough, $\Pi_D^{\theta}$ still serves as an upper bound of
$\Pi_{opt}^{\theta}$. 

Proposition 3.4 There exists a $\Theta > 0$, such that for any $\theta > \Theta$,

(i) For any policy $u$, if $\mathbb{E}_u \int_0^\theta \lambda_s ds < \theta N$, $\pi_u^{(\theta)}(\lambda) \leq 0$.

(ii) $\Pi_{opt}^{(\theta)} < \Pi_D^{(\theta)}$.

Chung et al. (2013) show empirically that shorter evaluation periods are preferred as they serve as pacers to keep the sales-force to be on track so they can hit the long term sales goal, whereas when the evaluation period is long the salespeople don’t have much guidance and might lose motivation. Chen (2000) also suggests that breaking the annual quota based bonus system to shorter time windows smooths the demand process and mitigates the negative impact of the “hockey-stick” sales pattern on the supply chain. We show in Proposition 3.4 the upside of having a long evaluation time window. As the size of the problem increases and the evaluation period gets longer, the probability that the seller would reach the threshold with a low sales intensity drops to zero. This means that any attempt by the seller to “game” the system and keep the sales intensity low is futile and results in a negative expected profit. Therefore, for a long evaluation window, the seller must comply with the threshold and keep the expected sales higher than the threshold.

3.5 Heuristics

Proposition 3.1 shows that the optimal solution for the stochastic problem is both time and state dependent. This is hard to implement in real life as the seller must constantly adjust the intensity throughout the selling period. Moreover as there’s in general no close form solution for the dynamic programming problem in Proposition 3.1, the calculation of the optimal $\lambda(t, n)$’s needs a lot of computational resources, especially for large problems. For statute initiation, California requires at least 365,880 signatures in 180 days. Calculating the optimal intensities and implementing them for a problem with such a big scale is impractical. For those reasons, we would like to investigate simple-to-implement heuristics and their performances.

3.5.1 Static Heuristics

First we study the static heuristic where the demand rate $\lambda_{SH}$ is fixed throughout the selling period. It is the simplest heuristic that requires no supervision from the seller during the selling season. The solution of the deterministic problem in Proposition 3.3 is appealing as it does not depend on the demand trajectory and can be easily implemented.
Hence we adopt the solution to the stochastic problem and investigate its performance in the stochastic setting.

We denote the optimal expected profit of the static heuristic as \( \Pi_{\text{SH}}^{(\theta)} \) and assess its performance by calculating \( \Pi_D^{(\theta)} - \Pi_{\text{SH}}^{(\theta)} \). Proposition 3.4 ensures that this difference provides an upper bound of the performance gap \( \Pi^{(\theta)}_{\text{opt}} - \Pi_{\text{SH}}^{(\theta)} \) when \( \theta \) sufficiently large.

We first analyze the case when \( \lambda^* > N/T \). In this case the unit price \( p \) is high enough that in the deterministic problem it is optimal for the seller to exert efforts higher than the threshold rate. We set \( \lambda_{\text{SH}}^{(\theta)} = \lambda^* \). We can show that

**Proposition 3.5** When \( \lambda^* > N/T \), \( \lim_{\theta \to \infty} (\Pi_D^{(\theta)} - \Pi_{\text{SH}}^{(\theta)}) = 0 \).

Proposition 3.5 is a remarkable result as \( \Pi_D^{(\theta)} \) increases linearly in \( \theta \), yet a simple fixed rate policy pretty much eliminates the gap to zero. It implies that when the revenue maximizing rate \( \lambda^* \) is higher than the threshold rate \( N/T \), the seller can simply ignore the deadline and use \( \lambda^* \) throughout the selling period and be confident that there is little loss in performance.

Next we analyze the static heuristics when \( \lambda^* \leq N/T \). In this case the solution to the deterministic problem is \( \lambda_D = N/T \) or the threshold rate. It is tempting to also use \( \lambda_D \) as the fix rate directly to the stochastic problem, however we show in the following proposition that using \( \lambda_D \) is not an asymptotically optimal solution to the problem. In fact in Proposition 3.6 we state a stronger statement.

**Proposition 3.6** When \( \lambda^* \leq N/T \), let \( \lambda_{\text{SH}}^{(\theta)} = \lambda_D + f(\theta) \). If \( \lim_{\theta \to \infty} \sqrt{\theta} f(\theta) < \infty \),

\[
\lim_{\theta \to \infty} \frac{\Pi_{\text{SH}}^{(\theta)}}{\Pi_D^{(\theta)}} < 1.
\]

In traditional RM problems, the static heuristics are asymptotically optimal because the seller’s reward is continuous around the threshold. With the all-or-nothing constraint however, the seller collects her rewards only when her sales reaches the threshold. This means that the high probability of reaching the threshold is essential for any heuristic to be asymptotically optimal. The formulation of the deterministic problem 3.5 does not take into consideration the demand uncertainties. Adopting the solution of problem 3.5 directly thereby isn’t asymptotically optimal.

From Proposition 3.6, in order to find static heuristics that are asymptotically optimal, we need the rate \( \lambda_{\text{SH}}^{(\theta)} = \lambda_D + f(\theta) \) where \( \lim_{\theta \to \infty} \sqrt{\theta} f(\theta) = \infty \). In the next proposition, we show that such heuristics are indeed asymptotically optimal, however, their performances are compromised.
Proposition 3.7 When $\lambda^* \leq N/T$, for any deterministic heuristic, if $\lambda^{SH} = \lambda_D + f(\theta)$ where $\lim_{\theta \to \infty} \sqrt{\theta} f(\theta) = \infty$, $\lim_{\theta \to \infty} \frac{\Pi^{D}(\theta)}{\Pi^{SH}(\theta)} = 1$. However, $\lim_{\theta \to \infty} \frac{1}{\sqrt{\theta}} \left( \Pi^{D}(\theta) - \Pi^{SH}(\theta) \right) = \infty$.

Gallego and Van Ryzin (1994) has shown that the static heuristics, while not the most efficient, have a performance gap of at most $\sqrt{\theta}$. However in our context where the selling is bound by an all-or-nothing constraint, the seller must increase the sales intensity by $f(\theta)$ so that the probability of reaching the threshold will converge to 1. This further decreases the efficiency and results in a performance gap that is worse than $\sqrt{\theta}$.

3.5.2 Dynamic Heuristics

In Proposition 3.7 we have shown that when $\lambda^* \leq N/T$, static heuristics don’t perform well as the intensity is set at the beginning of the time period, and cannot adjust to any demand fluctuations. As a result the seller must increase her effort to make sure the total sales would go over the threshold. In this section, we focus on the case where $\lambda^* \leq N/T$ and search for heuristics that would improve the performance gap of the static heuristics. Since the deterministic problem is easy to solve, it is natural to study the type of dynamic heuristics that resolve the deterministic problem periodically at a distinct time points $\mathcal{T} = \{1, 2, \ldots, T\}$. We outline the heuristic in Algorithm 1. We denote $\hat{\lambda}_t$ where $t \in \mathcal{T}$ as the updated sales intensity at the the resolve time points.

**Algorithm 1 Standard Periodic Resolving Heuristic**

1: At time-to-go $T$, set $\hat{\lambda}_T = \lambda_D$.
2: At time-to-go $t > 1$, find the updated distance to threshold $\hat{n}_t$.
3: if $\hat{n}_t \leq 0$ then
4: Set $\hat{\lambda}_t = \lambda^*$.
5: else
6: Compute $\hat{\lambda}_t$ as the solution to problem 3.5 with threshold $N = \hat{n}_t$ and $T = t$.
7: end if

The periodic resolving heuristics have been applied to different problems in the traditional revenue management settings. In particular, Jasin (2014) shows that the resolving heuristic has a logarithmic revenue loss in the dynamic pricing problem. The resolving heuristics are appealing as it offers a nice trade-off between the static heuristic and the optimal solutions. Comparing to the static heuristic, the resolving heuristic is responsive to demand fluctuations, yet unlike the optimal solution that needs to solve HJB equations through backwards induction, it only solves a simple deterministic problem $T$ times. Unfortunately, the standard resolving heuristic outlined in Algorithm 1 is not asymptotically optimal under the the all-or-nothing constraint.
Proposition 3.8 Let $\Pi^{(\theta)}_{RH}$ be the expected profit of the standard resolving heuristic.

$$\lim_{\theta \to \infty} \frac{\Pi^{(\theta)}_{RH}}{\Pi^{(\theta)}_D} < 1.$$ 

The periodic resolves adjust the sales intensities to past demand realizations. It is therefore able to reduce excess costs when the current sales number is higher than expected. However, at each time period, the seller still solves the deterministic problem that doesn’t take into account the future demand uncertainties. As a result, the probability that the sales number will reach the threshold at the end of the time period does not converge to 1 as $\theta \to \infty$.

With the presence of the all-or-nothing constraint, a good heuristic should be responsive to past demand realizations, and also guard against future demand uncertainties by making sure the probability of reaching the threshold will converge to 1. We design the two-stage modified resolving heuristic (MRH) in Algorithm 2.

**Algorithm 2: Two-Stage Modified Resolving Heuristic**

1. At time-to-go $T$, set $\lambda^M_{MRH} = \lambda^D$.
2. At time-to-go $t > 1$, find the updated threshold $\hat{n}_t$.
3. if $\hat{n}_t \leq 0$ then
4. Set $\lambda^M_{MRH} = \lambda^*$.
5. else
6. Compute $\hat{\lambda}_t$ as the solution to problem 3.5 with threshold $N = \hat{n}_t$ and $T = t$.
7. if $t < M \log \theta$ or $\hat{\lambda}_t \geq \frac{1}{2} (\lambda_D + \bar{\lambda})$ then
8. Set $\lambda^M_{MRH} = \bar{\lambda}$
9. else
10. Set $\lambda^M_{MRH} = \hat{\lambda}_t$
11. end if
12. end if

At the first stage the heuristic does resolving to respond to past demand realizations. This reduces the costs while still keeps the sales number on track to reach the threshold. Later in the selling period when time-to-go is less than $M \log \theta$ or when the resolving solution is larger than $\frac{1}{2} (\lambda_D + \bar{\lambda})$, the seller switches to the second stage in which she increases her effort to $\bar{\lambda}$. This second stage ensures that the total sales will reach the threshold regardless of the demand uncertainties.

The surge in the later stage of the selling period in the modified resolving heuristic resembles to the “hockey-stick” phenomenon discussed in Chen (2000) and Sohoni et al. (2010). In Chen (2000) and Sohoni et al. (2010), the seller lets the stochastic demand realize first before she decides her efforts. By delaying her decision the seller won’t exert
excess efforts that could have been avoided. However we have shown in Proposition 3.4 that for a large scale problems, any strategy that tries to “game” the system is futile, so there’s no point postponing the efforts as the expected sales must be greater than the threshold anyways. In Algorithm 2, the expected sales equals the threshold at any resolving time in the first stage. The increase in sales intensity in the second stage serves as an insurance policy to make sure the seller will reach the threshold.

**Theorem 3.9** The modified resolving heuristic is asymptotically optimal. Moreover,

\[ \Pi_D^{(\theta)} - \Pi_{MRH}^{(\theta)} = O(\log \theta). \]

Theorem 3.9 demonstrates that by adding a second stage that maximizes the probability of reaching the threshold and switching at the right time, we can achieve the same performance gap of \( \log \theta \) as the resolving heuristic in the traditional dynamic pricing problem demonstrated in Jasin (2014). Comparing with Proposition 3.7 in which we show that the revenue loss of the static heuristics is more than \( \sqrt{\theta} \), Theorem 3.9 highlights the importance of dynamic adjustments. It ensures that the negative impact of the all-or-nothing constraint can be eliminated if the seller can readjust her sales intensities periodically. Du et al. (2017) argues the importance of using the stimulus policies contingently. In Theorem 3.9 we quantify the benefits by showing a logarithmic revenue loss.

### 3.6 Numerical Examples

In this section, we present some numerical experiments to demonstrate the structure of the optimal effort rate discussed in Section 3.3 and the performances of the heuristics discussed in Section 3.5.

First we consider the setting where the threshold \( N = 20 \), \( b = 40 \) and \( p = 2 \). The deadline \( T = 5 \). The cost rate function \( c(\lambda) = (\lambda - 2)^2 \). We can calculate the optimal sales intensity \( \lambda(t, n) \) and expected profit \( J^*_t(n) \) with Proposition 3.1. From Figure 3.1(a) we observe that as time-to-go \( t \to 0 \), \( \lambda(t, n) \) is close to the cost-free rate \( \lambda = 2 \). Since the probability of reaching the threshold is small when it’s close to deadline, the seller’s optimal strategy is to avoid incurring any extra cost. When \( t \) grows larger, the seller now has a reasonable chance of getting the rewards so she increases her sales intensity as \( t \) increases. However when \( t \) is large enough, the seller can comfortably hit the threshold without stretching herself. So her sales intensity decreases after a turning point. From Figure 3.1(b), we observe that the expected profit \( J^*_t(n) \) is close to zero when \( t \) is small,
as there’s little chance of the seller to reach the threshold. \( J^*_t(n) \) is initially convex in \( t \), as the seller picks up the sales intensity, but becomes concave after the turning point as the seller tunes down the intensity.

\[
c(\lambda) = (\lambda - 2)^2. \quad b = 40. \quad p = 2. \quad N = 20.
\]

Next we evaluate the performances of the static heuristics discussed in Section 3.5.1 through simulations. We keep the bonus commission \( p = 2 \) per unit, and the cost rate function \( c(\lambda) = (\lambda - 2)^2 \). We construct a series of problems, such that in the \( \theta \)’th problem, \( N^{(\theta)} = 20\theta \), \( T^{(\theta)} = 5\theta \) and \( b^{(\theta)} = 40\theta \). This means that the threshold rate \( N^{(\theta)}/T^{(\theta)} = 4 \). To show how the performance as we consider two cases separately where the profit maximizing rate \( \lambda^* > N^{(\theta)}/T^{(\theta)} \), and where \( \lambda^* \leq N^{(\theta)}/T^{(\theta)} \). For each problem \( \theta \), we repeat the simulation experiments for 50,000 times and record the average of the seller’s profits as an approximation of the expected profit \( \Pi^{(\theta)}_{SH} \). We then compare it with the profit of the deterministic problem \( \Pi^{(\theta)}_D \), which we have shown in Proposition 3.4 is an upper bound for the optimal profit of the stochastic problem when \( \theta \) is large enough.

When \( p = 6, \lambda^* = \arg \max \{\lambda : \lambda p - c(\lambda)\} = \arg \max \{\lambda : 6\lambda - (\lambda - 2)^2\} = 5 > N^{(\theta)}/T^{(\theta)} = 4 \). In this case the optimal profit of the deterministic problem \( \Pi^{(\theta)}_D = 25\theta \). As proposed in Proposition 3.5, we set the sales intensity \( \lambda^{(\theta)}_{SH} = \lambda^* = 5 \) in the static heuristics. In Figure 3.2(a), we observe that when \( \theta \) increases, the difference between the optimal profit of the deterministic problem and the expected profit using static heuristic decreases to 0. This confirms the statement in Proposition 3.5 that there is hardly any performance loss for static heuristics when \( \lambda^* > N/T \).

Next we let \( p = 2 \). In this case \( \lambda^* = \arg \max \{\lambda : \lambda p - c(\lambda)\} = \arg \max \{\lambda : 2\lambda - (\lambda - 2)^2\} = 5 > N^{(\theta)}/T^{(\theta)} = 4 \). In this case the optimal profit of the deterministic problem \( \Pi^{(\theta)}_D = 25\theta \). As proposed in Proposition 3.5, we set the sales intensity \( \lambda^{(\theta)}_{SH} = \lambda^* = 5 \) in the static heuristics. In Figure 3.2(a), we observe that when \( \theta \) increases, the difference between the optimal profit of the deterministic problem and the expected profit using static heuristic decreases to 0. This confirms the statement in Proposition 3.5 that there is hardly any performance loss for static heuristics when \( \lambda^* > N/T \).
(2) \{(\theta / T(\theta)) = 4. The optimal profit of the deterministic problem \( \Pi_D^{(\theta)} = 20\theta \).

We let the sales intensity \( \lambda_{SH}(\theta) = \lambda_D + f(\theta) \). We consider four different \( f(\theta) \)'s, where \( f(\theta) = 0 \) and \( \theta^{-\beta}, \beta \in \{0.1, 0.4, 0.7\} \). When \( \beta \) is small, the sales intensity is high to make sure the seller can hit the threshold, yet the high intensity is also very costly. When \( \beta \) is large, the cost for the seller is lower, but hitting the threshold cannot be guaranteed. As shown in Figure 3.2(b), none of the four static heuristics performs particularly well, as the performance gaps swiftly increase in \( \theta \). This is consistent with our findings in Proposition 3.6 and Proposition 3.7.

![Figure 3.2: Performances of the Static Heuristics: \( \Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} \)](image)

Finally we evaluate the performances of the dynamic heuristics discussed in Section 3.5.2 and compare them with those of the static heuristics. The results are provided in Table 3.1. We confirm again that the static heuristic where \( \lambda_{SH}^{(\theta)} = \lambda_D \) and the standard resolving heuristic are not asymptotically optimal. We adopt the modified resolving heuristic outlined in Algorithm 2 and set the constant \( M = 10 \). Even though the performance gap in Proposition 3.7 and Theorem 3.9 are stated in the asymptotic regime, the modified resolving heuristics consistently produce higher average profits and smaller standard deviations than the static heuristics in moderately sized problems as well. It is further evidenced in Figure 3.3 as we show that the revenue loss percentage converges to zero promptly as the scale factor \( \theta \) increases.
Table 3.1: Expected Profit of Static Heuristics and Dynamic Heuristics

<table>
<thead>
<tr>
<th>θ</th>
<th>Static Heuristics</th>
<th>Dynamic Heuristics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SH a</td>
<td>MSH b</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
</tr>
<tr>
<td>100</td>
<td>58</td>
<td>2039</td>
</tr>
<tr>
<td>200</td>
<td>144</td>
<td>4048</td>
</tr>
<tr>
<td>300</td>
<td>0</td>
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<tr>
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<td>70</td>
<td>8075</td>
</tr>
<tr>
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<td>-181</td>
<td>10083</td>
</tr>
<tr>
<td>600</td>
<td>-709</td>
<td>12069</td>
</tr>
<tr>
<td>700</td>
<td>63</td>
<td>14099</td>
</tr>
<tr>
<td>800</td>
<td>-181</td>
<td>16114</td>
</tr>
<tr>
<td>900</td>
<td>-329</td>
<td>18110</td>
</tr>
<tr>
<td>1000</td>
<td>70</td>
<td>20120</td>
</tr>
</tbody>
</table>

a $\lambda_{SH}^{(0)} = \lambda_D$.
b $\lambda_{MSH}^{(0)} = \lambda_D + \theta^{-0.4}$.
c Standard resolving heuristic outlined in Algorithm 1.
d Modified resolving heuristic outlined in Algorithm 2.

Figure 3.3: Revenue Loss %
3.7 Conclusions

In this paper, we study a general problem of managing business performances under an all-or-nothing constraint. We consider a seller who would receive her rewards only when her total sales exceeds a threshold by the end of the selling period. The all-or-nothing contract is a double-edged sword. The reward to reach the threshold encourages the seller to increase her sales intensity, yet the seller is also deterred as her spending will be wasted if she fails to reach the target. The optimal strategy is thereby non-monotone in time and the distance to the threshold. We show that under an all-or-nothing constraint with a long enough evaluation period, the expected sales must be greater than the threshold for the seller to make a profit. We then provide some practical heuristics inspired by the deterministic version of the problem. Specifically we show that the static heuristics are asymptotically optimal, yet the revenue loss under the all-or-nothing constraint is larger than the loss in traditional RM problems, as the seller must increase her sales intensity to ensure she reaches the threshold. We propose a two-stage modified resolving heuristic. The seller resolves the deterministic periodically in the first stage, and switch to high sales intensity in second stage. The modified resolving heuristic adjusts the sales intensity based on past demand variations, while also ensures the total sales will reach the threshold. The modified resolving heuristic we proposed has a logarithmic performance gap, essentially removing all the adverse effects of the all-or-nothing constraint. This in turn highlights the importance of dynamic adjustments to sales intensity during the selling season.

In our work, we assume that the seller is risk neutral and her objective is to maximize her expected profit. Future work could look into studying the strategies of risk averse sellers, as in sales-force compensation literature, the salesperson are often assumed to be risk averse. In addition, we focus on the analysis of the seller’s strategies facing an all-or-nothing contract, yet it also sheds light to the optimal design of the contract from the firm’s perspective. We have already alluded that having a longer evaluation period prevents the seller from exerting lower efforts than the firm desires. We have bounded expected profit of the all-or-nothing contract for the seller. It is worth further investigation the theoretical bound of the firm’s profit loss by offering an all-or-nothing contracts to the seller.
3.8 Proofs

Proof of Proposition 3.1. When \( n = 0 \), the seller already reaches the threshold. It is obvious that at time \( t \) the optimal \( \lambda(t, 0) \) is the one that maximizes the profit rate \( \lambda p - c(\lambda) \). Hence \( \lambda(t, 0) \) is the unique solution of \( c'(\lambda) = p \). It is also not hard to verify that \( \frac{\partial J^*_t(0)}{\partial t} = \max \{ \lambda [J^*_t(-1) - J^*_t(0)] - c(\lambda) \} \).

When \( n \geq 1 \), for any \( t > 0 \), consider a some time period \( \delta \). Then,

\[
J^*_{t+\delta}(n) = \max_{\lambda} (1 - \lambda \delta) \cdot J^*_t(n) + \lambda \delta \cdot J^*_t(n-1) - c(\lambda) \delta + o(\delta).
\]

Rearrange the terms and let \( \delta \to 0 \), we get equation 3.3. Because \( c(\lambda) \) is convex in \( \lambda \), \( \lambda(t, n) \) is the unique solution of equation 3.4. \[\blacksquare\]

Proof of Theorem 3.2. Denote \( \Delta_t(n) = J^*_t(n - 1) - J^*_t(n) \). First we show that \( \lambda(t, n) \) increases in \( t \) if and only if \( \Delta_t(n) \) increases in \( t \). To see that, we recall that \( c'(\lambda(t, n)) = \Delta_t(n) \) and \( \frac{\partial J^*_t(n)}{\partial t} = \lambda(t, n) \cdot \Delta_t(n) - c(\lambda(t, n)) \). Therefore,

\[
\frac{\partial \Delta_t(n)}{\partial t} = \left[ \lambda(t, n-1) c'(\lambda(t, n-1)) - c(\lambda(t, n-1)) \right] - \left[ \lambda(t, n) c'(\lambda(t, n)) - c(\lambda(t, n)) \right]
\]

Since \( c(\lambda) \) is convex in \( \lambda \), \( \lambda' \cdot c'(\lambda) \) increases in \( \lambda \). Therefore \( c_t(n) \leq c_t(n-1) \) if and only if \( \frac{\partial \Delta_t(n)}{\partial t} \leq 0 \).

Next we show that

\[
\lambda(t, n) [\Delta_t(n-1) - \Delta_t(n)] \leq \frac{\partial \Delta_t(n)}{\partial t} \leq \lambda(t, n-1) \left[ \Delta_t(n-1) - \Delta_t(n) \right]. \quad (3.6)
\]

To see that, for any time \( t \), we consider a small time interval \( \delta \).

\[
J^*_{t+\delta}(n) = \max_{\lambda} (1 - \lambda \delta) \cdot J^*_t(n) + \lambda \delta \cdot J^*_t(n-1) - c(\lambda) \delta + o(\delta)
\]

\[
\geq (1 - \lambda(t, n-1)) \delta \cdot J^*_t(n) + \lambda(t, n-1) \delta \cdot J^*_t(n-1) - c(\lambda(t, n-1)) \delta + o(\delta).
\]

Rearrange the terms and let \( \delta \to 0 \), \( \frac{\partial J^*_t(n)}{\partial t} \geq \lambda(t, n-1) \Delta_t(n) - c(\lambda(t, n-1)) \). Therefore,

\[
\frac{\partial \Delta_t(n)}{\partial t} = \frac{\partial J^*_t(n-1)}{\partial t} - \frac{\partial J^*_t(n)}{\partial t} \\
\leq [\lambda(t, n-1) \Delta_t(n-1) - c(\lambda(t, n-1))] - [\lambda(t, n-1) \Delta_t(n) - c_t(n-1)] \\
= \lambda(t, n-1) [\Delta_t(n-1) - \Delta_t(n)].
\]

Similarly, we can also show that \( \frac{\partial \Delta_t(n)}{\partial t} \leq \lambda(t, n) [\Delta_t(n-1) - \Delta_t(n)] \). Now we are ready to prove this proposition. We do this by induction.
When \( n = 1 \), suppose the statement in the proposition is not true, then there must exist \( t_3 > t_2 > t_1 \) such that \( \frac{\partial \Delta_t(1)}{\partial t} |_{t=t_2} = 0 \). \( \frac{\partial \Delta_t(1)}{\partial t} < 0 \) between \((t_1, t_2)\) and \( \frac{\partial \Delta_t(1)}{\partial t} > 0 \) between \((t_2, t_3)\). Since \( \Delta_t(1) - c'(\lambda(t,1)) = p - c'(\lambda(t,0)) = 0 \), and \( \lambda(t_2, 1) = \lambda(t_2, 0) \), we get \( p = \Delta_t(1) \). Since \( \Delta_t(1) \) strictly increases between \((t_2, t_3)\), \( p - \Delta_t(1) < 0 \) when \( t \in (t_2, t_3) \). However according to the inequality 3.6 this in turn means that \( \frac{\partial \Delta_t(1)}{\partial t} < 0 \) between \((t_2, t_3)\). This leads to contradiction.

Now suppose the statement is true for \( n - 1 \), then for \( n \), we first show that when \( t \leq \tau(n - 1) \), \( \frac{\partial \Delta_t(n)}{\partial t} > 0 \). Because \( \Delta_t(n) > 0 \) when \( t > 0 \), and \( \Delta_0(n) = 0 \), \( \Delta_t(n) \) increases in \( t \) when \( t \) is small. Suppose there exists a \( t_1 < t_2 < t_3 < \tau(n - 1) \) such that \( \frac{\partial \Delta_t(n)}{\partial t} |_{t=t_2} = 0 \), \( \frac{\partial \Delta_t(n)}{\partial t} > 0 \) when \( t \in (t_1, t_2) \), and \( \frac{\partial \Delta_t(n)}{\partial t} < 0 \) when \( t \in (t_2, t_3) \). This means that \( \Delta_{t_2}(n - 1) = \Delta_{t_2}(n) \). Then from inequality 3.6 we know that \( \frac{\partial \Delta_t(n)}{\partial t} \geq \lambda(t, n) \left[ \Delta_t(n - 1) - \Delta_t(n) \right] \). However because \( \Delta_t(n - 1) \) increases in \( t \) and \( \Delta_t(n) \) decreases in \( t \), this must mean that \( \Delta_t(n - 1) - \Delta_t(n) > 0 \) when \( t \in (t_2, t_3) \). This is not possible. Therefore \( \Delta_t(n) \) increases in \( t \) when \( t \leq \tau(n - 1) \).

Next we show that there exists at most one \( t \geq \tau(n - 1) \) such that \( \frac{\partial \Delta_t(n)}{\partial t} = 0 \). Suppose this is not true, then there must exist \( t_3 > t_2 > t_1 \geq \tau(n - 1) \) such that \( \frac{\partial \Delta_t(n)}{\partial t} |_{t=t_2} = 0 \). \( \frac{\partial \Delta_t(n)}{\partial t} < 0 \) between \((t_1, t_2)\) and \( \frac{\partial \Delta_t(n)}{\partial t} > 0 \) between \((t_2, t_3)\). However, according to inequality 3.6 because when \( t \in (t_2, t_3) \) \( \Delta_t(n - 1) \) decreases in \( t \) and \( \Delta_t(n) \) increases in \( t \), \( \frac{\partial \Delta_t(n)}{\partial t} \) must be less than zero. This leads to contradiction. This completes the proof. □

**Proof of Proposition 3.3.** First we solve the following maximization problem:

\[
\tilde{\Pi}_D = \max_{\lambda} \quad \tilde{\pi}_D(\lambda) = b + p (\Lambda T - N) - c(\Lambda) T \\
\text{s.t.} \quad \Lambda T \geq N.
\]

It is not hard to see that the optimal solution is \( \Lambda^* = \lambda_D \) and \( \tilde{\Pi}_D = \pi_D(\lambda_D) \). Using Jensen’s inequality, for any \( \lambda \),

\[
\pi_D(\lambda) = b + p \left( \int_0^T \lambda_t dt - N \right) - \int_0^T c(\lambda_t) dt \\
\leq b + p \left( \int_0^T \lambda_t dt - N \right) - T \cdot c \left( \frac{1}{T} \int_0^T \lambda_t dt \right) \\
= \tilde{\pi}_D \left( \frac{1}{T} \int_0^T \lambda_t dt \right).
\]
Therefore,

\[ \Pi_D = \max \left\{ \pi_D(\lambda) : \int_0^T \lambda_t dt \geq N \right\} \leq \max \left\{ \tilde{\pi}_D \left( \frac{1}{T} \int_0^T \lambda_t dt \right) : \int_0^T \lambda_t dt \geq N \right\} = \tilde{\Pi}_D. \]

Since \( \pi_D(\lambda_D) = \tilde{\Pi}_D \), \( \lambda_D \) must be the optimal solution of problem 3.5. \( \blacksquare \)

**Proof of Proposition 3.4** Without loss of generality, assume that \( T = 1 \). The optimal solution of the deterministic problem is then \( \lambda_D = \max \{ \lambda^*, N \} \). By definition, \( p\lambda^* - c(\lambda^*) \geq pN - c(N) \). This means \( \Pi_D^{(\theta)} \geq \theta [b - c(N)] \).

For any policy \( u \), if \( \mathbb{E}_u \int_0 \theta dD_s < \theta N \), using Chebyshev’s inequality, we have

\[
\mathbb{P}_u \left( \int_0 \theta dD_s \geq \theta N \right) \leq \frac{\text{Var}_u \left( \int_0 \theta dD_s - \mathbb{E}_u \int_0 \theta dD_s \right)^2}{\left[ \theta N - \mathbb{E}_u \int_0 \theta dD_s \right]^2}. 
\]

Because \( \int_0 \theta dD_s - \mathbb{E}_u \int_0 \theta dD_s \) is a martingale, \( \left( \int_0 \theta dD_s - \mathbb{E}_u \int_0 \theta dD_s \right)^2 - \int_0 \theta dD_s \) is also a martingale. Hence \( \mathbb{E}_u \left( \int_0 \theta dN_s - \mathbb{E}_u \int_0 \theta dN_s \right)^2 = \mathbb{E}_u \int_0 \theta dD_s = \mathbb{E}_u \int_0 \theta \lambda_s ds = O(\theta) \). This means that \( \mathbb{P}_u \left( \int_0 \theta dD_s \geq \theta N \right) \to 0 \) as \( \theta \to \infty \). Hence

\[
\Pi_u^{(\theta)} = \theta (b - pN) + p \cdot \mathbb{E}_u \left( \int_0 \theta dD_s \bigg| \int_0 \theta dD_s \geq \theta N \right) \mathbb{P}_u \left( \int_0 \theta dD_s \geq \theta N \right) - \mathbb{E}_u \int_0 \theta c(\lambda_s) ds
\]

\[
= -\mathbb{E}_u \int_0 \theta c(\lambda_s) ds + o(\theta).
\]

Therefore when \( \theta \) is large enough, \( \Pi_u^{(\theta)} < 0 \). We can focus on the policies where \( \mathbb{E}_u \int_0 \theta dD_s \geq \theta N \). Since \( c(\lambda) \) is convex, using Jensen’s inequality, \( \mathbb{E}_u \int_0 \theta c(\lambda_s) ds \geq \theta \cdot c \left( \frac{1}{\theta} \mathbb{E}_u \int_0 \theta \lambda_s ds \right) \). Therefore,

\[
\Pi_u^{(\theta)} = \mathbb{E}_u \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \bigg| \int_0 \theta dD_s \geq \theta N \right) \mathbb{P}_u \left( \int_0 \theta dD_s \geq \theta N \right) - \mathbb{E}_u \int_0 \theta c(\lambda_s) ds
\]

\[
\leq \mathbb{E}_u \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \bigg| \int_0 \theta dD_s \geq \theta N \right) \mathbb{P}_u \left( \int_0 \theta dD_s \geq \theta N \right) - \theta \cdot c \left( \frac{1}{\theta} \mathbb{E}_u \int_0 \theta \lambda_s ds \right)
\]

\[
< \theta (b - pN) + \theta \left[ p \cdot \frac{1}{\theta} \mathbb{E}_u \int_0 \theta dD_s - c \left( \frac{1}{\theta} \mathbb{E}_u \int_0 \theta \lambda_s ds \right) \right].
\]

Since \( \frac{1}{\theta} \mathbb{E}_u \int_0 \theta dD_s \geq N \) and \( p - c'(\lambda) \leq 0 \) for any \( \lambda \geq N \), \( p \cdot \frac{1}{\theta} \mathbb{E}_u \int_0 \theta dD_s - c \left( \frac{1}{\theta} \mathbb{E}_u \int_0 \theta \lambda_s ds \right) \leq pN - c(N) \). Therefore, \( \Pi_u^{(\theta)} < \theta [b - c(N)] \leq \Pi_D^{(\theta)} \). \( \blacksquare \)

To prove Proposition 3.5, we first state a lemma that bounds the tail of a Poisson distribution.

**Lemma 3.10** Let \( X \) be a random variable with Poisson distribution with rate \( \lambda \). For
any $0 < x < \lambda$, 
\[ P(X \leq \lambda - x) \leq \exp \left( -\frac{x^2}{2\lambda} \right). \]


Proof of Proposition 3.3. Without loss of generality, assume that $T = 1$. When $\lambda^* > N/T$, $\Pi_D^{(\theta)} = \theta [b + p(\lambda^* - N) - c(\lambda^*)]$. For the stochastic problem, since the demand rate is fixed throughout the time period, the demand rate is a Poisson process with rate $\lambda^*$. The total sales $\int_0 \theta dD_s$ has a Poisson distribution with rate $\lambda^* \theta$. Using lemma 3.10,

\[ P\left( \int_0 \theta dD_s \geq \theta N \right) = 1 - P\left( \int_0 \theta dD_s < \lambda^* \theta - \theta(\lambda^* - N) \right) \geq 1 - \exp \left[ -\frac{(\lambda^* - N)^2}{2\lambda^*} \cdot \theta \right]. \]

Therefore,

\[
\Pi_{SH}^{(\theta)} = \mathbb{E} \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \right) \left( \int_0 \theta dD_s \geq \theta N \right) \mathbb{P} \left( \int_0 \theta dD_s \geq \theta N \right) - \int_0 \theta c(\lambda_s) ds \\
\geq \mathbb{E} \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \right) \cdot \left( 1 - \exp \left[ -\frac{(\lambda^* - N)^2}{2\lambda^*} \cdot \theta \right] \right) - c(\lambda^*) \theta \\
= b \theta + p(\lambda^* - N) \theta - c(\lambda^*) \theta - \mathbb{E} \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \right) \cdot \exp \left[ -\frac{(\lambda^* - N)^2}{2\lambda^*} \cdot \theta \right] \\
= J^D \theta - \mathbb{E} \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \right) \cdot \exp \left[ -\frac{(\lambda^* - N)^2}{2\lambda^*} \cdot \theta \right].
\]

Since $\mathbb{E} \left( \theta b + p \left( \int_0 \theta dD_s - \theta N \right) \right) = O(\theta)$, $\lim_{\theta \to \infty} \left( \Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} \right) = 0$. □

Proof of Proposition 3.6. Without loss of generality, assume that $T = 1$. The total sales $\int_0 \theta dD_s$ has a Poisson distribution with mean $\theta N + \theta f(\theta)$. First it is easy to show that when $\lim_{\theta \to \infty} |f(\theta)| > 0$, the static heuristic is not asymptotically optimal. So we focus on the heuristics that have $\lim_{\theta \to \infty} f(\theta) = 0$.

Let $Y\theta = \int_0 \frac{\theta dD_s - (\theta N + \theta f(\theta))}{\theta N + \theta f(\theta)}$. $\int_0 \theta dD_s \geq \theta N$ is equivalent to $Y\theta \geq -\frac{\theta f(\theta)}{\sqrt{\theta N + \theta f(\theta)}}$. If $\lim_{\theta \to \infty} \sqrt{\theta} f(\theta) < \infty$, there exists a $-\infty < b < +\infty$, such that $\lim_{\theta \to \infty} \frac{\theta f(\theta)}{\sqrt{\theta N + \theta f(\theta)}} = b$. Using central limit theorem,

\[ P \left( Y\theta \geq -\frac{\theta f(\theta)}{\sqrt{\theta N + \theta f(\theta)}} \right) = \Phi(b) + o(1), \]

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Also from the Strong Law of
Large Numbers, $\frac{\int_0^\theta dD_s}{\theta^N + \theta f(\theta)} \to 1$ almost surely as $\theta \to \infty$. Hence $\mathbb{P}\left(\frac{\theta}{2} N < \int_0^\theta dD_s < \theta N\right) = 1 - \Phi(b) + o(1)$, This means that

$$
\mathbb{E}\left(\int_0^\theta dD_s \left| \int_0^\theta dD_s < \theta N\right) \cdot \mathbb{P}\left(\int_0^\theta dD_s < \theta N\right)
= \mathbb{E}\left(\int_0^\theta dD_s \left| \frac{\theta}{2} N < \int_0^\theta dD_s < \theta N\right) \cdot \mathbb{P}\left(\frac{\theta}{2} N < \int_0^\theta dD_s < \theta N\right)
= \frac{\theta}{2} N \cdot \mathbb{P}\left(\frac{\theta}{2} N < \int_0^\theta dD_s < \theta N\right) = O(\theta).
$$

Therefore,

$$
\Pi^{(\theta)}_{SH} = \mathbb{E}\left(\theta b + p \left(\int_0^\theta dD_s - \theta N\right) \left| \int_0^\theta dD_s \geq \theta N\right) \mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) - \int_0^\theta \theta c(\lambda_s) ds
= \theta b - \theta p N + p \left[\mathbb{E}\left(\int_0^\theta dD_s \right) - \mathbb{E}\left(\int_0^\theta dD_s \left| \int_0^\theta dD_s < \theta N\right) \cdot \mathbb{P}\left(\int_0^\theta dD_s < \theta N\right)\right] \right.
\left. - \theta c(\lambda^D + f(\theta)) \right.
\leq \theta b + \theta f(\theta) - \theta c(\lambda^D) - O(\theta)
= \Pi^{(\theta)}_D - O(\theta).
$$

Because $\Pi^{(\theta)}_D = O(\theta)$, $\lim_{\theta \to \infty} \frac{\Pi^{(\theta)}_{SH}}{\Pi^{(\theta)}_D} < 1$. ■

**Proof of Proposition 3.7** Without loss of generality, assume that $T = 1$. When $\lambda^* < N/T$, $\Pi^{(\theta)}_D = \theta \left[b - c(N)\right]$.

Using Chebyshev’s inequality, we have

$$
\mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right)
= \mathbb{P}\left(\int_0^\theta dD_s - (\theta N + \theta f(\theta)) \geq -\theta f(\theta)\right)
\geq 1 - \mathbb{P}\left(\left|\int_0^\theta dD_s - (\theta N + \theta f(\theta))\right| \geq \theta f(\theta)\right)
\geq 1 - \frac{\theta N + \theta f(\theta)}{(\theta f(\theta))^2}.
$$

Since $\lim_{\theta \to \infty} \sqrt{\theta} f(\theta) = \infty$, $\lim_{\theta \to \infty} \frac{1}{\theta} (\theta f(\theta))^2 = \infty$. This means that when $\theta \to \infty$, $\mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) \to 1$. As a result,

$$
\frac{1}{\theta} \Pi^{(\theta)}_{SH} = b - p N + \frac{1}{\theta} \mathbb{E}\left(\int_0^\theta dD_s \left| \int_0^\theta dD_s \geq \theta N\right) \mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) - \frac{1}{\theta} \int_0^\theta \theta c(\lambda_s) ds
\geq b - p N + \frac{1}{\theta} \mathbb{E}\left(\int_0^\theta dD_s \right) \mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) - c(\lambda_D + f(\theta))
= b - p N + p N \cdot \mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) - c(\lambda_D + f(\theta))
= b - p N + p N \cdot \mathbb{P}\left(\int_0^\theta dD_s \geq \theta N\right) - c(\lambda_D + f(\theta)).
$$
\[
\rightarrow b - c(\lambda_D) = \frac{1}{\theta} \Pi_D^{(\theta)}.
\]

Therefore, the static heuristic is asymptotically optimal. Also,
\[
\frac{1}{\theta} \Pi_{SH}^{(\theta)} \leq b - pN + \frac{p}{\theta} \mathbb{E} \left( \int_0^1 \theta dD_s \right) - c(\lambda_D + f(\theta)) = b + pf(\theta) - c(\lambda_D + f(\theta)).
\]

With Taylor’s expansion, since \( f(\theta) \rightarrow 0 \), we know that \( c(\lambda_D + f(\theta)) = c(\lambda_D) + c'(\lambda_D) f(\theta) + o(\theta) \). So,
\[
\frac{1}{\theta} \Pi_{SH}^{(\theta)} \leq b - c(\lambda_D) + f(\theta) [p - c'(\lambda_D)] + o(f(\theta)) = \frac{1}{\theta} \Pi_D^{(\theta)} + f(\theta) (p - c'(\lambda_D)) + o(f(\theta)).
\]

Because \( \lambda^* < \lambda_D, p - c'(\lambda_D) < 0 \). Therefore, \( \frac{1}{\theta} \left( \Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} \right) = O(f(\theta)) \). And since \( \lim_{\theta \to \infty} \sqrt{\theta} f(\theta) = \infty \), \( \frac{1}{\sqrt{\theta}} \left( \Pi_D^{(\theta)} - \Pi_{SH}^{(\theta)} \right) = \infty \). ■

**Lemma 3.11** Denote \( \hat{D}_t \) as the realized demand between time-to-go \( t \) and \( t-1 \). \( \delta_s = \hat{D}_s - \mathbb{E} \hat{D}_s \). If \( \hat{\lambda}_s \geq \lambda^* \) for any \( s \geq t+1 \),
\[
\hat{\lambda}_t = \lambda_D - \sum_{s=t+1}^T \frac{\delta_s}{s-1}.
\]  \hspace{1cm} (3.7)

**Proof of Lemma 3.11** We do this by induction.

At \( t = T \), \( \hat{\lambda}_T = \lambda_D = N/T \). \( \hat{n}_T = N \). Because \( \hat{D}_T \) is Poisson distributed with mean \( \hat{\lambda}_1, \mathbb{E} \hat{D}_T = N \). The threshold is updated as
\[
\hat{n}_{T-1} = N - \hat{D}_T = N - \delta_T - \mathbb{E} \hat{D}_T = \frac{T - 1}{T} N - \delta_T.
\]

Since \( \hat{\lambda}_{T-1} \geq \lambda^* \), \( \hat{\lambda}_{T-1} = \frac{\hat{n}_{T-1}}{T-1} \). Therefore,
\[
\hat{\lambda}_{T-1} = \frac{\hat{n}_{T-1}}{T-1} = \frac{\frac{T-1}{T} N - \delta_T}{T-1} = \frac{\lambda_D - \delta_T}{T-1}.
\]

Now suppose equation (3.7) holds for \( t \). Then \( \hat{\lambda}_t = \lambda_D - \sum_{s=t+1}^T \frac{\delta_s}{s-1} \). Since \( \hat{D}_t \) is Poisson distributed with mean \( \hat{\lambda}_t, \mathbb{E} \hat{D}_t = \hat{\lambda}_t \). Because \( \hat{\lambda}_t \geq \lambda^* \), \( \hat{n}_t = \hat{\lambda}_t t \). The updated threshold
\[
\hat{n}_{t-1} = \hat{n}_t - \hat{D}_t = \hat{n}_t - \delta_t - \mathbb{E} \hat{D}_t = \hat{\lambda}_t t - \delta_t - \hat{\lambda}_t = \hat{\lambda}_t (t-1) - \delta_t.
\]
Therefore,
\[
\hat{\lambda}_{t-1} = \frac{n_{t-1}}{t-1} = \frac{\hat{\lambda}_t(t-1) - \delta_t}{t-1} = \hat{\lambda}_t - \frac{\delta_t}{t-1} = \lambda_D - \sum_{s=t}^{T} \frac{\delta_s}{s-1}.
\]

\[\blacksquare\]

**Lemma 3.12** For any \(0 < x < \bar{\lambda} - \lambda_D\), let \(\tau(x)\) be the first time-to-go that \(|\hat{\lambda}_t - \lambda_D| \geq x\). Then there exists a \(\Psi(x) > 0\) independent of \(t\), such that for any \(1 \leq t \leq T - 2\),
\[
\mathbb{P}[\tau(x) > t] < \frac{\Psi(x)}{t}.
\]

**Proof of Lemma 3.12** \(\sum_{s=t}^{T} \frac{\delta_s}{s-1}\) is a backwards martingale w.r.t filtration \(\mathcal{F}_t\), where \(\mathcal{F}_t\) is the observed history up to the beginning of time-to-go \(t\). Using Doob’s maximal inequality:
\[
\mathbb{P}(\tau(x) > t) = \mathbb{P}\left(\max_{t+1 \leq l \leq T} \left| \sum_{s=l+1}^{T} \frac{\delta_s}{s-1} \right| \geq x \right) \leq \frac{1}{x^2} \mathbb{E}\left( \sum_{s=t+2}^{T} \frac{\delta_s}{s-1} \right)^2.
\]

For any \(s < t\), \(\mathbb{E}\delta_s \delta_t = \mathbb{E}[\delta_t \mathbb{E}(\delta_s | \delta_t)] = 0\). Therefore,
\[
\mathbb{E}\left( \sum_{s=t+2}^{T} \frac{\delta_s}{s-1} \right)^2 = \sum_{s=t+2}^{T} \frac{\mathbb{E}\delta_s^2}{(s-1)^2} = \sum_{s=t+2}^{T} \frac{\text{var}(\hat{\lambda}_s)}{(s-1)^2} < \sum_{s=t+2}^{T} \frac{\bar{\lambda}}{(s-1)^2} < \sum_{s=t+2}^{T} \frac{\bar{\lambda}}{(s-1)(s-2)} < \frac{\bar{\lambda}}{t}.
\]

Let \(\Psi(x) = \frac{\bar{\lambda}}{x^2}\), \(\mathbb{P}(\tau(x) \geq t) < \frac{\Psi(x)}{t}\). \(\blacksquare\)

**Proof of Proposition 3.8** Without loss of generality, assume that \(T = 1\). Because
\[
\Pi_{RH}^{(\theta)} = \theta b - p \mathbb{E}\left( \sum_{t=1}^{\theta N} \theta \hat{D}_t \bigg| \sum_{t=1}^{\theta N} \theta \hat{D}_t < \theta N \right) \mathbb{P}\left( \sum_{t=1}^{\theta N} \theta \hat{D}_t < \theta N \right) - \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t),
\]
we just need to show that \(\lim_{\theta \to \infty} \mathbb{P}\left( \sum_{t=1}^{\theta N} \theta \hat{D}_t < \theta N \right) < 1\).

To see that, \(x = \frac{\lambda_D - \lambda^*}{2}\) and \(t = 2\Psi(x)\). We apply lemma 3.12 and get \(\mathbb{P}[\tau(x) > 2\Psi(x)] < \frac{1}{2}\). This means that the probability \(\hat{\lambda}_t > \lambda^*\) when \(t = 2\Psi(x)\) is greater than 1/2. This implies that the updated threshold at 2\(\Psi(x)\) is greater than zero. Since 2\(\Psi(x)\) is finite and doesn’t depend on \(\theta\), the probability that the threshold will be reached when time expires must be strictly less than 1 and doesn’t converge to 1. \(\blacksquare\)

**Proof of Theorem 3.9** Without loss of generality, assume that \(T = 1\). Let \(x = \frac{\lambda_D - \lambda^*}{2}\).
From Lemma 3.12, we can see that $\mathbb{P}(\tau^{(\theta)}(x) > t) < \frac{\Psi(x)}{t}$.

$$\mathbb{E}\tau^{(\theta)}(x) = \sum_{t=1}^{T-1} \mathbb{P}(\tau(x) \geq t) < 1 + \sum_{t=1}^{T-2} \mathbb{P}(\tau(x) > t) < 1 + \Psi(x) \sum_{t=1}^{T-2} \frac{1}{t} = O(\log \theta).$$

Therefore, there exists a $M$ that is independent of $\theta$, such that $\mathbb{E}\tau\theta(x) \leq M \log \theta$. Denote $\hat{\tau}\theta = \max\{\tau\theta(x), M \log \theta\}$.

$$\mathbb{E}\hat{\tau}\theta = \mathbb{E}\max\{\hat{\tau}\theta, M \log \theta\} \leq \mathbb{E}(\hat{\tau}\theta + M \log \theta) \leq 2M \log \theta.$$

This means that $M \log \theta \leq \mathbb{E}\hat{\tau}\theta \leq 2M \log \theta$.

$$\Pi^{(\theta)}_{MRH} = \theta(b - pN) + \mathbb{E}\left(\sum_{t=1}^{\theta} \hat{\theta} \hat{D}_t \left| \sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right.\right) \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) - \mathbb{E}\sum_{t=1}^{\theta} \mathbb{E}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) - \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) - O(\log \theta)$$

$$= \theta(b - pN) + \mathbb{E}\left(p \sum_{t=1}^{\theta} \hat{D}_t - c(\hat{\lambda}_t) \right) \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) - \left(1 - \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right)\right) \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) - O(\log \theta).$$

Because $\mathbb{E}\hat{\tau}\theta \leq 2M \log \theta$ and $c(\hat{\lambda}_t) < c(\bar{\lambda})$, $\left(1 - \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right)\right) \mathbb{E}\sum_{t=1}^{\theta} c(\hat{\lambda}_t) = O(\log \theta)$. Therefore,

$$\Pi^{(\theta)}_{MRH} \geq \theta(b - pN) + \mathbb{E}\left(p \sum_{t=1}^{\theta} \hat{D}_t - c(\hat{\lambda}_t) \right) \mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) - O(\log \theta).$$

Next we bound the two terms $\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right)$ and $\mathbb{E}\left(p \sum_{t=1}^{\theta} \hat{D}_t - c(\hat{\lambda}_t) \right)$.

First for $\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right)$, because $\sum_{s=t}^{\theta} \frac{\delta_s}{s-1} \geq -\frac{\bar{\lambda} - \lambda N}{2}$ for any $t \geq \hat{\tau}\theta + 1$, $\theta N - \sum_{t=\hat{\tau}\theta+1}^{\theta} \hat{D}_t \leq \frac{\bar{\lambda} + \lambda N}{2} \hat{\tau}\theta$.

$$\mathbb{P}\left(\sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) = \mathbb{P}\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t \geq \theta N - \sum_{t=\hat{\tau}\theta+1}^{\theta} \hat{D}_t \right)$$

$$\geq \mathbb{P}\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t \geq \frac{\bar{\lambda} + \lambda N}{2} \hat{\tau}\theta \right)$$
Conditional on $\hat{\tau}\theta$, $\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t$ has a Poisson distribution with mean $\bar{\lambda}\tau\theta$. From lemma 3.10, we have

$$
P\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t \geq \theta N\right) \geq \mathbb{E} \left[1 - \mathbb{P}\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t < \bar{\lambda}\hat{\tau}\theta - \frac{\bar{\lambda} - \lambda D}{2}\hat{\tau}\theta\right)\right]
$$

$$
\geq 1 - \mathbb{E} \left[\exp\left(-\frac{(\bar{\lambda} - \lambda D)^2}{2\lambda \hat{\tau}\theta}\right)\right]
$$

$$
= 1 - \mathbb{E} \left[\exp\left(-\frac{(\bar{\lambda} - \lambda D)^2}{8\lambda \hat{\tau}\theta}\right)\right].
$$

Because $\hat{\tau}\theta \geq M \log \theta$,

$$
P\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t \geq \theta N\right) \geq 1 - \mathbb{E} \left[\exp\left(-\frac{(\bar{\lambda} - \lambda D)^2}{8\lambda \hat{\tau}\theta}\right)\right].
$$

We can make $M \geq \frac{8\lambda}{(\bar{\lambda} - \lambda D)^2}$, then there exists a $\Gamma$ such that $P\left(\sum_{t=1}^{\hat{\tau}\theta} \hat{D}_t \geq \theta N\right) \geq 1 - \frac{\Gamma}{\theta}$.

Next we look at $\mathbb{E} \left(p \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \hat{D}_t - c(\hat{\lambda}_t)\right)$. Denote $\epsilon_t = \sum_{s=t}^{\theta} \delta_s / s - 1$. Recall that $\delta_t = \hat{D}_t - \mathbb{E}\hat{D}_t$, using Taylor’s expansion,

$$
p \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \hat{D}_t - c(\hat{\lambda}_t) = \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \left[p(\lambda_D - \epsilon_{t+1} + \delta_t) - c(\lambda_D) + c'(\lambda_D)\epsilon_{t+1} - \frac{1}{2}c''(z_t)\epsilon_{t+1}^2\right]
$$

$$
= \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \left[p\lambda_D - c(\lambda_D)\right] - \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \left(p - c'(\lambda_D)\right)\epsilon_{t+1} - \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \frac{1}{2}c''(z_t)\epsilon_{t+1}^2 + p \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \delta_t.
$$

Note $\sum_{s=t}^{\theta} \delta_s$ and $\sum_{s=t}^{\theta} \epsilon_s$ are backwards martingales. Because $E\hat{\tau}\theta > 0$, using the optional stopping time theorem, $\mathbb{E} \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \delta_t = \mathbb{E} \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \epsilon_{t+1} = 0$. Also,

$$
\mathbb{E} \sum_{t=\hat{\tau}\theta}^{\hat{\tau}\theta+\theta} c''(z_t)\epsilon_t^2 = \mathbb{E} \sum_{t=\hat{\tau}\theta+1}^{\hat{\tau}\theta+\theta} \sum_{s=1}^{\theta} \sum_{v=1}^{t} \frac{\delta_s \delta_v}{(\theta - s)(\theta - v)}
$$

$$
= \mathbb{E} \sum_{t=1}^{\hat{\tau}\theta+\theta} \sum_{s=1}^{t} \frac{c''(z_t)\delta_s^2}{(\theta - s)^2}.
$$
\[ \leq \mathbb{E} \sum_{t=1}^{\theta} \sum_{s=1}^{t} c''(z_t) \frac{\delta^2_s}{(\theta - s)^2} = O(\log \theta). \]

Therefore,

\[ \mathbb{E} \left[ p \sum_{t=\hat{\tau}+1}^{\theta} \hat{D}_t - c(\hat{\lambda}_t) \right] = \mathbb{E} \left[ \sum_{t=\hat{\tau}+1}^{\theta} [p\lambda_D - c(\lambda_D)] \right] - O(\log \theta) = [p\lambda_D - c(\lambda_D)] (\theta - \mathbb{E} \hat{\tau} \theta) - O(\log \theta). \]

Because \( \mathbb{E} \hat{\tau} \theta \leq 2M \log \theta \) and \( p\lambda_D - c(\lambda_D) = pN - c(N) \),

\[ \mathbb{E} \left[ p \sum_{t=\hat{\tau}+1}^{\theta} \hat{D}_t - c(\hat{\lambda}_t) \right] = [pN - c(N)] \theta - O(\log \theta). \]

Finally we assemble the pieces together,

\[ \Pi_{MRH}^{(\theta)} \geq \theta(b - pN) + \mathbb{E} \left( p \sum_{t=1}^{\hat{\tau}} \hat{D}_t - c(\hat{\lambda}_t) \right) \mathbb{P} \left( \sum_{t=1}^{\theta} \hat{D}_t \geq \theta N \right) - O(\log \theta) \]

\[ \geq \theta(b - pN) + ([pN - c(N)] \theta - O(\log \theta)) \cdot \left( 1 - \frac{\Gamma}{\theta} \right) - O(\log \theta) \]

\[ = \theta b - c(N) \theta - O(\log \theta) \]

\[ = \Pi_D^{(\theta)} - O(\log \theta). \]
Chapter 4

Price and Wage Competition of On-Demand Matching Platforms

4.1 Introduction

Peer-to-peer ridesharing services have revolutionized urban transportation. The first ride-sharing platform Lyft was launched in the summer of 2012. Shortly afterwards Uber follows suit with its own UberX. Since then more firms have joined the market, including Via, Juno and Didi Chuxing. While there are some differences between platforms, the services they offer are largely similar. As a customer enters her destination on the ridesharing platform she is quoted a price. She can either accept the price or decline and choose other modes of transportation. Once she accepts the price the customer will be matched to a driver for the service. The driver is in turn paid with the price less a certain commission. In other words, the platforms act as intermediaries that coordinates demand and supply by setting prices for customers and wages for suppliers.

Because of the similarities between platforms, there’s not much brand loyalty. Many riders and drivers are active users in multiple platforms. According to the survey conducted by the Ridesharing Guy\textsuperscript{1} in 2018 over 78% of the drivers use at least two platforms. On the demand side, there is also 10% of the riders who regularly use services from multiple platforms\textsuperscript{2}. Due to the nature of ridesharing, in order for the platforms to expand, they need to actively recruit both customers and drivers, since keeping a balanced number of customers and drivers leads to a more efficient matching. This has made


the ridesharing platforms, especially Uber and Lyft, arch enemies since their inceptions. The platforms compete fiercely both for customers and for drivers in all the markets. This means constant price and wage promotions to encourage utilization. For example in January 2015, in an effort to attract more customers, Uber\footnote{https://newsroom.uber.com/us-california/5-uberpool-rides/ (accessed May 15, 2018)} and Lyft\footnote{https://blog.lyft.com/lyftline5/ (accessed August 20, 2018)} both offered $5 flat rate rides, a heavily discounted price, to woo more riders in San Francisco. To recruit drivers, both Uber and Lyft offer sign-up incentives as much as $500, and reduced commissions\footnote{https://www.forbes.com/sites/ellenhuet/2014/05/30/how-uber-and-lyft-are-trying-to-kill-each-other (accessed August 20, 2018)}.

Yet despite the competitions, the market has grown tremendously over the years for all the platforms. Thanks to the numerous promotions, ridesharing is becoming a popular transportation mode. The dominance of Uber and Lyft didn’t deter new players, such as Via and Juno, from joining the game. As seen in Figure 4.1, the number of trips in New York city has been increasing steadily for all of the platforms. This creates a head-scratching problem for the platforms about their price and wage strategies and whether they can remain profitable when they have to compete for unloyal customers and drivers. In order to gain market share, the platforms must offer competitive prices and wages. However, the competition also make the services more popular. This gives the platforms some leeway to increase their margin. In this paper we investigate competition of on-demand matching platforms and its impact on prices, wages, customers’ welfare and the platforms’ profitability when users are unloyal to the platforms.

We assume that there is a finite demand and supply population. We model the customers and suppliers sign-up decisions with nested logit models, in which they make

![Figure 4.1: Market Size of Ride-Hailing Platforms in New York City](image)
decisions sequentially, first deciding whether they would use the service, and then choose the platform. We analyze three representative scenarios: the *monopoly* scenario where each platform has its own customer and supplier base; the *price competition* scenario where the customers have access to multiple platforms whereas the suppliers are loyal to their platforms; and the *price and wage competition* scenario where both the customers and the suppliers have access to multiple platforms. We prove that there exists a unique equilibrium under all three scenario. We also show that the equilibrium quantities, prices, wages and the platforms’ profits all increase with the size of demand and supply pool.

We then compare the equilibrium solutions of the three scenarios. As customers gain access to multiple platforms, the utilization of the services for all platforms increase. Comparing the price competition scenario with the monopoly scenario, the platforms pay a higher wage to the drivers to accommodate the higher usage. However, the prices that platforms charge to their customers do not necessarily decrease because the customers are willing to accept a higher price due to the increase in the overall attraction of the services. We then show that when the supply side is stringent enough, the price in the case of price competition is guaranteed to be higher that of the monopoly. Similarly, as the suppliers have access to multiple platforms, the pool of supply for the platforms is larger, and the platforms could charge a lower price to their customers. However, the wage may not increase. We show that when the demand side is stringent enough, the wage in the case of wage competition is guaranteed to be lower that when there’s no competition on the supply side. Finally through numerical examples, we demonstrate that competition may in fact increase the platforms’ profit, creating a win-win situation for all parties.

### 4.2 Related Literature

There is a growing list of papers that study the ride-hailing platforms. In particular, there are a number of recent works that investigate platforms’ pricing decisions. Bimpikis et al. (2016) study the impact of the spatial demand pattern impacts the platform’s pricing and profitability. They show that a balanced network benefits both the platform and the users. Cachon et al. (2017) find that in spite of the public outcry, surge pricing by the platforms can in fact benefit both customers and drivers. Hu and Zhou (2017) study a monopolistic platform’s price and wage decision under market uncertainty. They show that despite the uncertainty, commission contracts, in particular fix-rate contracts, performs very well. Cohen and Zhang (2017) study the competition between platforms by establishing a joint service. They show that with a well-designed profit sharing contract, it
is possible for the platforms, riders and drivers to all benefit from coopetition mechanism. Nikzad (2018) and Benjaafar et al. (2018) show that in on-demand service matching, it may be beneficial for the platform to hire reserve of drivers to keep the waiting cost low. In this case, the wages might increase with labor pool size when the pool is thin. Nikzad (2018) also show that platforms competition in the same market increases the workers’ welfare, but the customers’ welfare might decrease when the labor market is thin. In our paper, rather than focusing on the cannibalism of platforms in a single segment of the market, we study the impact on prices, wages and welfare when the individual markets of previously monopolistic platforms gain access to competing platforms.

Methodology wise, we model the sign up decision of customers and suppliers with discrete choice models (for comprehensive review see Train (2009)). One of the most widely used discrete choice models is the multinomial logit (MNL) model, which was first proposed in McFadden (1974). One key assumption of MNL is the Independence from Irrelevant Alternatives (IIA) property, first proposed in Luce (1959), that states the probability of choosing one alternative over the other is independent of other alternatives. The IIA assumption can be restrictive as it would be violated if some alternatives are more similar to each other than to the rest of the choice set. The famous “red-bus/blue-bus” paradox, which resonates our setting, illustrates the potential problems of IIA properties. While there are some differences between ridesharing platforms, they are clearly more similar to each other than to other transportation modes. To resolve this, in this paper we adopt the nested logit (NL) model to study the users’ sign-up decisions. In NL, the users decide first whether to use the service, and then choose between the platforms. In the NL model, the IIA property still holds within the nest but not across the nest. Because NL has a close form solution and is relatively easy to estimate, it has been applied to study problems in different areas including marketing and transportation (see Ben-Akiva and Lerman (1985), Train et al. (1987), McFadden (1986) and Bhat (1995) for examples).

Discrete choice models have been adopted to study various operations management problems. Gallego et al. (2006) show that there exists a unique Nash equilibrium under Bertrand competition when the cost function is monotonically increasing and convex in sales. Li and Huh (2011) extend the uniqueness result in Gallego et al. (2006) to nested logit demand functions. Gallego and Wang (2014) study the price optimization problem the price sensitivities are different between products. Cachon and Kök (2007) study assortment planning problem where the customers’ demand model follows the nested logit framework. They characterize the equilibrium under duopoly competition where retailers decides the prices and the variety levels within the categories.
4.3 Model

We assume that there are $K \geq 1$ platforms operating in the two-sided marketplace. Platform $1 \leq k \leq K$ charges a price of $p_k$ to its customers and a wage of $w_k$ to its service providers.

On the demand side, there are $L$ segments in the customer population, each has $N_i, i = 1, 2, \ldots, L$ number of users. Each customer segment $i$ has access to a set of platforms $\mathcal{S}_i$. In addition, the customers also have the option of not using the service. We model the customer’s choice with a nested logit model. A customer in segment $i$ is first presented with prices from each platform she has access to. Let $a - bp_k$ be the attractiveness of platform $k$, where $a$ represents the value of the service, and $b > 0$ is the price sensitivity. She then decides whether she would like to use the service, and platform to use. Conditional on her using the service, a customer in segment $i$ chooses platform $k$ with probability:

$$q_{ik}(p) = \frac{e^{a-bp_k}}{\sum_{k \in \mathcal{S}_i} e^{a-bp_k}}$$  \hspace{1cm} (4.1)

The probability that the customer uses the service is:

$$\rho_i(p) = \frac{\left[ \sum_{k \in \mathcal{S}_i} e^{a-bp_k} \right]^{\gamma}}{1 + \left[ \sum_{k \in \mathcal{S}_i} e^{a-bp_k} \right]^{\gamma}}, \hspace{1cm} (4.2)$$

where $0 < \gamma \leq 1$ is the parameter that measures the degree of dissimilarity between the different platforms. A higher $\gamma$ indicates a lesser similarity between platforms. When $\gamma = 1$, the model is reduced to the multinomial logit model.

Let $D_{ik}(p)$ be the number of customers in group $i$ that choose platform $k$. $D_{ik}$ can be expressed as

$$D_{ik}(p) = N_i \cdot \rho_i(p) \cdot q_{ik}(p) = N_i \times \frac{\left[ \sum_{k \in \mathcal{S}_i} e^{a-bp_k} \right]^{\gamma}}{1 + \left[ \sum_{k \in \mathcal{S}_i} e^{a-bp_k} \right]^{\gamma}} \times \frac{e^{a-bp_k}}{\sum_{k \in \mathcal{S}_i} e^{a-bp_k}}. \hspace{1cm} (4.3)$$

Similarly on the supply side, we segment the supplier pool into $H$ segments, each has $M_i$ number of suppliers. Each supplier segment $j$ has access to a set of platforms $\mathcal{R}_i$ and she can choose not to participate. She is presented wages $w_k$ from the platforms she has access to. The attraction of platform $k$ is $-c + dw_k$, where $c$ is the cost of
providing the service and \( d > 0 \) is wage sensitivity. The supplier decides first whether she will participate, and then choose the specific platform. The total supply for each driver group \( j \) towards platform \( k \) is

\[
S_{jk}(w) = M_j \cdot \frac{\sum_{r \in \mathcal{R}_j} e^{-c+dw_r}}{1 + \sum_{r \in \mathcal{R}_j} e^{-c+dw_r}} \cdot \frac{e^{-c+dw_k}}{\sum_{r \in \mathcal{R}_j} e^{-c+dw_r}}. \tag{4.4}
\]

For platform \( k \), its total demand \( D_k(p) = \sum_{i=1}^{L} D_{ik}(p) \) and its total supply \( S_k(p) = \sum_{i=1}^{H} S_{ik}(w) \). We could then express platform \( k \)'s profit function as

\[
\pi_k(p, w) = \min\{D_k(p), S_k(w)\} \times (p_k - w_k). \tag{4.5}
\]

The platform’s problem is to set price \( p_k \) and wage \( w_k \) so as to maximize its profit. We use the following Lemma to reduce the problem into a single variable maximization.

**Lemma 4.1** It’s optimal for platform \( k \) to set \( p_k, w_k \) jointly such that the demand and supply exactly matches.

Lemma 4.1 is intuitive, as if the total demand is larger than total supply for the platform, it can always increase the price to make additional profit. Similarly if the total supply is larger, the platform can always decrease the wages and make a higher profit. As a result of Lemma 4.1, there is a one-to-one correspondence between the matching quantities \( Q \) and \( p, w \).

Now we investigate three representative scenarios. For simplicity, we assume that \( N_i = N, M_i = M \) and \( L = H = K \).

**Scenario 1: Monopoly**

In this scenario, we assume that each platform has its own customer and driver base, i.e. \( \mathcal{S}_i = \{i\} \) for each customer group \( i \), and \( \mathcal{R}_j = \{j\} \) for each driver group \( j \). The demand for platform \( i \) is

\[
D_i(p_i) = N \times \frac{(e^{a-bp_i})^\gamma}{1 + (e^{a-bp_i})^\gamma}. \tag{4.6}
\]

The supply of platform \( i \) is

\[
S_i(w_i) = M \times \frac{(e^{-c+dw_i})^\eta}{1 + (e^{-c+dw_i})^\eta}. \tag{4.7}
\]

We derive the equilibrium solution under monopoly in Proposition 4.2.
Proposition 4.2 (Monopoly) There exists a unique equilibrium under monopoly such that for each platform \( i \), the equilibrium service quantity \( Q_i = Q^{(1)} \), where \( Q^{(1)} \) is the unique solution to

\[
H^{(1)}(Q) = \frac{1}{b\gamma} \ln \frac{Q}{N - Q} + \frac{1}{d\eta} \ln \frac{Q}{M - Q} + \frac{N}{b\gamma(N - Q)} + \frac{M}{d\eta(M - Q)} = \frac{a}{b} - \frac{c}{d}.
\]  

(4.8)

The equilibrium price \( p^{(1)} \) and wage \( w^{(1)} \) are given by:

\[
p^{(1)} = \frac{1}{b} \left( a - \frac{1}{\gamma} \ln \frac{Q^{(1)}}{N - Q^{(1)}} \right),
\]

\[
w^{(1)} = \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q^{(1)}}{M - Q^{(1)}} \right).
\]

Scenario 2: Oligopoly Price Competition

In this scenario, customers have access to all the platforms, whereas each platform has its own driver base, i.e. \( \mathcal{I}_i = \{1, 2, \ldots, K\} \) for each customer group \( i \), and \( \mathcal{R}_j = \{j\} \) for each driver group \( j \).

The demand for platform \( i \) is

\[
D_i(p_i, p_{-i}) = KN \cdot \frac{\left[ \sum_k e^{a-bp_k} \right]^{\gamma-1} e^{a-bp_i}}{1 + \left[ \sum_k e^{a-bp_k} \right]^{\gamma}}.
\]  

(4.9)

The supply for platform \( i \) is

\[
S_i(w_i) = M \times \frac{(e^{-c+dw_i})^{\eta}}{1 + (e^{-c+dw_i})^{\eta}}.
\]  

(4.10)

We derive the equilibrium solution under price competition in Proposition 4.3.

Proposition 4.3 (Oligopoly Price Competition) There exists a unique equilibrium under price competition such that for each platform \( i \), the equilibrium service quantity \( Q_i = Q^{(2)} \), where \( Q^{(2)} \) is the unique solution to

\[
H^{(2)}(Q) = \frac{1}{b\gamma} \ln \frac{Q}{N - Q} + \frac{1}{d\eta} \ln \frac{Q}{M - Q} + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q]} + \frac{M}{d\eta(M - Q)} - \frac{1}{b} \ln K
\]

\[= \frac{a}{b} - \frac{c}{d}.
\]  

(4.11)
The equilibrium price $p^{(2)}$ and wage $w^{(2)}$ are given by:

$$ p^{(2)} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{Q^{(2)}}{N - Q^{(2)}} \right), \quad (4.12) $$

$$ w^{(2)} = \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q^{(2)}}{M - Q^{(2)}} \right). \quad (4.13) $$

**Scenario 3: Oligopoly Price and Wage Competition**

In this scenario, customers now have access to all the platforms, whereas each platform has its own driver base, i.e. $\mathcal{S}_i = \{1, 2, \ldots, K\}$ for each customer group $i$, and $\mathcal{R}_j = \{1, 2, \ldots, K\}$ for each driver group $j$. The total demand for platform $i$ is

$$ D_i(p_i, p_{-i}) = KN \cdot \frac{\left[ \sum_k e^{a-bp_k} \right]^{\gamma-1} e^{a-bp_i}}{1 + \left[ \sum_k e^{a-bp_k} \right]^\gamma}. \quad (4.14) $$

The total supply for platform $i$ is

$$ S_i(w_i, w_{-i}) = KM \cdot \frac{\left[ \sum_k e^{-c+dw_k} \right]^\eta}{1 + \left[ \sum_k e^{-c+dw_k} \right]} \cdot \frac{e^{-c+dw_i}}{\sum_k e^{-c+dw_k}}. \quad (4.15) $$

We derive the equilibrium solution under price and wage competition in Proposition 4.4.

**Proposition 4.4 (Oligopoly Price and Wage Competition)** There exists a unique equilibrium under price and wage competition such that for each platform $i$, the equilibrium service quantity $Q_i = Q^{(3)}$, where $Q^{(3)}$ is the unique solution to

$$ H^{(3)}(Q) = \frac{1}{b\gamma} \ln \frac{Q}{N - Q} + \frac{1}{d\eta} \ln \frac{Q}{M - Q} + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q]} + \frac{KM}{d[(K + \eta - 1)M - \eta Q]} - \left( \frac{1}{b} + \frac{1}{d} \right) \ln K = \frac{a}{b} - \frac{c}{d}. \quad (4.16) $$

The equilibrium price $p^{(3)}$ and wage $w^{(3)}$ are given by:

$$ p^{(3)} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{Q^{(3)}}{N - Q^{(3)}} \right), $$

$$ w^{(3)} = \frac{1}{d} \left( c - \ln K + \frac{1}{\eta} \ln \frac{Q^{(3)}}{M - Q^{(3)}} \right). $$
Cachon and Kök (2007), Li and Huh (2011) and Gallego and Wang (2014) establish the uniqueness of the equilibrium under oligopoly price competition with multiple products. Note that in their setups the customers first choose the firm and then decide on the product. In our model, the users choose the mode of transportation first and if they have chosen ride-hailing, decide on the specific platform. Nonetheless, in Proposition 4.2, 4.3 and 4.4, we show that when the platforms are symmetric, the equilibrium is unique under price and wage competition.

Next we study how $N$ and $M$, the size of demand and supply pool, affect the equilibriums.

**Proposition 4.5 (Comparative Statics)** For each of the three scenarios $s \in \{1, 2, 3\}$,

(i) The equilibrium quantity $Q^{(s)}$ increase in $N$ and $M$.

(ii) Price $p^{(s)}$, wage $w^{(s)}$ increase in $N$ and decrease in $M$.

(iii) Profit for each platform $\Pi^{(s)}$ increase in $N$ and $M$.

In Proposition 4.5, we show that for all three scenarios, the equilibrium quantity increases in the size of demand and supply pool. When demand pool gets larger, the demand curve shifts outwards, resulting in a higher equilibrium quantity and a higher a market clearing price. And because the equilibrium quantity is higher, the wage must also increase to accommodate the increase in equilibrium quantity. Yet overall, the platform’s profit increases with the size of the demand pool.

Similarly, a larger supply pool increases the equilibrium quantity. Both the prices and the wages decrease as a result. Nonetheless, the platform’s profit increases in the size of the supply pool.

**4.4 Comparisons**

In this section, we compare the equilibrium solutions of the aforementioned three scenarios.

**4.4.1 Equilibrium quantities $Q$**

First we show in Proposition 4.6 the relationship of the equilibrium quantities.

**Proposition 4.6** For the equilibrium quantities under the three scenarios:

$$Q^{(3)} > Q^{(2)} > Q^{(1)}.$$
Proposition 4.6 shows that the equilibrium quantity is highest under price and wage competition and is lowest under monopoly. As the customers and suppliers gain access to multiple platforms, the pools of supply and demand are larger for each platform. This allows for a more efficient match between supply and demand, resulting in higher matching quantities. Moreover, the platforms overall become more attractive to users, making the utilization even higher.

### 4.4.2 Equilibrium price $p$ and wage $w$

Next we compare the equilibrium prices $p^s$ and wages $w^{(s)}$ under the three scenarios.

**Proposition 4.7**

(i) Equilibrium wage under oligopoly price competition $w^{(2)}$ is higher than the wage under monopoly $w^{(1)}$.

(ii) Equilibrium price under oligopoly price and wage competition $p^{(3)}$ is lower than the price under oligopoly price competition $p^{(2)}$.

Comparing scenario 2 to scenario 1, the competitiveness on the supply side remains the same, yet the platforms are now exposed with a larger demand pool. According to Proposition 4.6, the equilibrium quantities increase as a result of platform competition. To accommodate to the increase in demand, the platforms must also increase their wages to attract additional suppliers. Similarly, comparing scenario 3 to scenario 2, a larger supply pool results in a higher equilibrium quantity, which must be induced by lower prices as the competitiveness on the demand side remains unchanged.

However, comparing prices in scenario 2 to prices in scenario 1, there are a number of forces acting against each other and it is not easy to generalize how prices change. First as the demand side gets more competitive, the equilibrium quantities are higher. This pushes the prices up. However, this means that the platforms need to pay higher wages to attract more suppliers. Also as the customers get exposed to multiple platforms, the overall attractiveness of the service also increase. Customers are more willing to use one of the platforms. These two forces in turn drive the prices up. We show in Proposition 4.8 that when the supply side is stringent enough, prices indeed may increase under price competition compared to the monopoly case.

**Proposition 4.8** There exists a threshold $\tau$, such that when $\frac{M}{N} \leq \tau$, equilibrium price under price competition $p^{(2)}$ is higher than the price under monopoly $p^{(1)}$. 
When the platforms compete on the demand side, each of them pay a higher wage to accommodate the higher demand. As the service overall becomes more attractive, the customers are willing to share the “burden” with the platforms by conceding on the prices. When $\frac{M}{N}$ is small enough, the supplies are scarce compared to the size of the demand pool. The wages must be raised dramatically to recruit enough suppliers. The downward force caused by the increase in competitiveness is outweighed by the upward force caused by the increase in wages and the attractiveness of services, resulting in an increase in prices.

Next, comparing scenario 3 to scenario 2, we show in Proposition 4.9 that when the demand side is stringent enough, the wages may decrease when the platforms competes on the supply side.

**Proposition 4.9** There exists a threshold $\psi$, such that when $\frac{M}{N} \geq \psi$, equilibrium wage under price and wage competition $w^{(3)}$ is lower than the wage under price competition $w^{(2)}$.

As $\frac{M}{N}$ is small, there is a small customer base. This means that platforms must cut prices aggressively to stay competitive. In this case the suppliers don’t have much bargaining power as the supply pool is large. As the services are more attractive to the suppliers when they gain access to multiple platforms, they are willing to take a wage cut. As a result when $\frac{M}{N}$ is small enough, the platforms decrease their wages to offset the decrease in revenue.

### 4.4.3 User’s Surplus

The consumer surplus in nested logit models can be expressed using the equilibrium quantities (see Train (2009) and Anderson and De Palma (1992)). For given prices $\mathbf{p}$, the consumer surplus of the demand side is

$$CS_D = \sum_{i=1}^{L} \frac{N_i}{b_i} \ln \left[ \left( \sum_{k \in S_i} e^{a_k - b_p k} \right)^{\gamma} + 1 \right].$$

Similarly for given wages $\mathbf{w}$, the consumer surplus of the supply side is

$$CS_S = \sum_{j=1}^{H} \frac{M_j}{b_j} \ln \left[ \left( \sum_{r \in S_j} e^{-c_r + d_{w_r}} \right)^{\eta} + 1 \right].$$

In Proposition 4.10 we show that regardless of how prices and wages change, platform competition always benefits the users.
Proposition 4.10 For the consumer surplus on both demand and supply side:
\[ CS_D^{(1)} < CS_D^{(2)} < CS_D^{(3)} \quad \text{and} \quad CS_S^{(1)} < CS_S^{(2)} < CS_S^{(3)}. \]

As users get access to multiple platforms, the market can match supply and demand more efficiently. Even though the prices may increase and the wages may decrease, the users are compensated with a better selection of platforms and a higher service level. As a result, their welfares are also higher.

4.5 Numerical Examples

In this section, we demonstrate the impact of platform competition through some numerical examples. We assume that there are \( K = 3 \) platforms. We let the value of the service \( a = 10 \), cost \( c = 5 \), the price and wage sensitivity \( b = d = 1 \) and the dissimilarity indices \( \gamma = \eta = 0.5 \). We fix the number of customers in each segment \( N = 10000 \), and evaluate the equilibrium solutions with different supply pool size \( M \).

From Figure 4.2, we observe that the equilibrium quantities increase in \( M \) for all three scenarios, and competition results in larger matching quantities as the users gain access to multiple platforms.

![Figure 4.2: Equilibrium Quantities \( Q(s) \)](image)

In Figure 4.3(a) we illustrate how prices evolve with supply size \( M \). We see that when \( M \) is small, price competition actually increases the prices, because the platforms need to heavily compensate the suppliers. Figure 4.3(a), we can see that when \( M \) is large the
equilibrium wage under scenario 3 is lower than that in scenario 2. This happens because the platforms need to greatly lower their prices to compete on the demand side.

![Figure 4.3: Price $p(s)$ and Wage $w(s)$](image)

We have identified in Section 4.4 a number of forces that affect the platforms profit: the competitions force the platforms to cut their margins, which drives the profits down; on the other hand, the competitions together make the service more popular, so the customers and suppliers are more tolerant to price increases and wage decreases. This drives the profit up. Those two forces drive the profit up. In Figure 4.4 we illustrate the relationship between the size of the supply pool and the platforms’ profits under the three scenarios. We can see that the introduction of competitions may surprisingly result in a higher profit for the platforms when the demand and supply pool are about the same size. When neither customers nor suppliers are scarce, the platforms don’t engage in cut-throat competitions. This is when the competition actually benefits the platforms as it makes the services overall more attractive to the users.

### 4.6 Conclusions

In this paper, we investigate the on-demand matching platforms’ pricing and wage decisions under competition when the users are unloyal. Platforms act as intermediaries that set prices for customers and wages for suppliers. Customers and suppliers sign up to the services according to nested logit models. We investigate three representative scenarios. In scenario 1, each platform has its own loyal customer and supplier base. In scenario 2, customers have access to all of the platforms whereas the suppliers are still loyal to
their platform. In scenario 3, both the customers and the suppliers are free to choose any platform. We prove that there exists a unique Nash equilibrium under all three scenarios when the parameters are symmetric. We show that the equilibrium quantities and platforms’ profits increase in the size of demand and supply pool. Prices and wages increase in the size of the demand pool and decreases in the size of the supply pool. We then compare the equilibrium solutions, and show that contrary to traditional one-sided market, introducing price competition may result in a price increase when the supplier resources are scarce. Similarly we show that introducing wage competition may result in a wage decrease when the customers are outnumbered by the suppliers. Nevertheless, we demonstrate that access to multiple platforms always improves users’ welfare.

There are a few limitations to our work. The scenarios we consider focus on the the impact of users having access to multiple platforms. Unlike Nikzad (2018), we do not study the impact of cannibalism. Going forward, we would like to consider cases where the number of platforms increases within a single market segment. This will also shed light on the platforms’ market entry decisions. So far in the scenarios we consider, we treat the set of available platforms $\mathcal{S}_i$ and $\mathcal{R}_j$ as exogenous. This is common when customers voluntarily install multiple apps to their cellphone. Yet the availability is also a decision the platforms can consciously make. Future works could look into how the platforms should choose the market segments they operate in and the conditions that encourage the platforms to enter an already crowded market.
4.7 Appendix

4.7.1 Extension: Platform’s Market Entry Decisions

We treat the accessibility of platforms as endogenous and study the platforms’ market entry strategies. In ridesharing, service areas of platforms are not necessarily the same, and the platforms need to decide whether to expend to “enemy territories” and compete “head-on” with other platforms. In the numerical examples in Section 4.5, we show in Figure 4.4 that platforms may in fact prefer competition in all the market rather than the monopoly it enjoys in its own segment.

We consider two platforms, 1 and 2, and two market segments $\alpha$ and $\beta$, each having a demand pool size of $N$ and supply pool size of $M$. The two platforms first decide which segment to operate in and then decide their prices and wages. Platform $i$’s strategy set $S_i = \{\alpha, \beta, B\}$, where $\alpha$ represents operating only in segment $\alpha$, $\beta$ represents operating only in segment $\beta$ and $B$ represents operating in both segments. We show in Proposition 4.11 that both platforms enjoying monopoly in a single segment is not an equilibrium.

**Proposition 4.11** Any state where both platforms operate in only one segment cannot be a Nash equilibrium.

*Proof of Proposition 4.11.* Without loss of generality, we show that neither $(\alpha, \alpha)$ nor $(\alpha, \beta)$ can be a Nash equilibrium.

(i) When the state is $(\alpha, \alpha)$, the price and wage decision of platform 2 does not change regardless of whether the platform 1 expends to segment $\beta$. Therefore by playing $B$, the profit of platform 1 strictly increases.

(ii) When the state is $(\alpha, \beta)$, platform 1 can expend to segment $\beta$ and keep the same price and wage. Its profit must be greater than that when it operates only in $\alpha$.

Therefore, both platforms operating in only one segment cannot be a Nash equilibrium. $lacksquare$

In order to derive the equilibrium solution to platforms’ market entry strategies, platform competition in a single market segment needs to be analyzed. We have planned to work on this in our future research.

4.7.2 Proofs

*Proof of Lemma 4.1.* Since $D_k(p)$ decreases in $p_k$, if the $D_k(p) > S_k(w)$, there exists a $\delta > 0$ such that platform $k$ can always increase $p_k$ by $\delta$, and result in an increase in $p_{i_k}$. 
Similarly when \( D_k(p) < S_k(w) \) the platform can decrease the wages and the profit will increase. \( \blacksquare \)

**Proof of Proposition 4.2.** Since \( D_i = S_i = Q_i \) according to lemma 4.1 we can write the inverse demand and supply function as

\[
p_i(Q_i) = \frac{1}{b} \left( a - \frac{1}{\gamma} \ln \frac{Q_i}{N - Q_i} \right) \quad \text{and} \quad w_i(Q_i) = \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q_i}{M - Q_i} \right). \tag{4.17}
\]

We take first derivative w.r.t. \( Q_i \) in equation 4.17 and get

\[
\frac{\partial p_i}{\partial Q_i} = -\frac{1}{b\gamma Q_i(N - Q_i)} \quad \text{and} \quad \frac{\partial w_i}{\partial Q_i} = \frac{1}{d\eta Q_i(M - Q_i)}.
\]

The second derivatives of \( p_i \) and \( w_i \) are given by:

\[
\frac{\partial^2 p_i}{\partial Q_i^2} = \frac{1}{b\gamma Q_i^2(N - Q_i)^2} \quad \text{and} \quad \frac{\partial^2 w_i}{\partial Q_i^2} = -\frac{1}{d\eta Q_i^2(M - Q_i)^2}.
\]

This means that

\[
Q_i \frac{\partial^2 p_i}{\partial Q_i^2} + 2 \frac{\partial p_i}{\partial Q_i} = \frac{1}{b\gamma Q_i^2(N - Q_i)^2} - \frac{1}{b\gamma Q_i(N - Q_i)} \frac{2N}{Q_i} = -\frac{1}{b\gamma Q_i(N - Q_i)} \frac{N^2}{Q_i(N - Q_i)} < 0,
\]

\[
Q_i \frac{\partial^2 w_i}{\partial Q_i^2} + 2 \frac{\partial w_i}{\partial Q_i} = -\frac{1}{d\eta Q_i^2(M - Q_i)^2} + \frac{1}{d\eta Q_i(M - Q_i)} \frac{2M}{Q_i} = \frac{1}{d\eta Q_i(M - Q_i)} \frac{M^2}{Q_i(M - Q_i)} > 0.
\]

The platform select a \( Q_i \) that maximizes its profit \( \pi(Q_i) = Q_i(p_i - w_i) \). The first order condition is thus:

\[
0 = \frac{\partial \pi_i}{\partial Q_i} = p_i - w_i + Q_i \left( \frac{\partial p_i}{\partial Q_i} - \frac{\partial w_i}{\partial Q_i} \right)
\]

\[
= \frac{1}{b} \left( a - \frac{1}{\gamma} \ln \frac{Q_i}{N - Q_i} \right) - \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q_i}{M - Q_i} \right) + Q_i \left[ -\frac{1}{b\gamma Q_i(N - Q_i)} - \frac{1}{d\eta Q_i(M - Q_i)} \right] \left( a - \frac{1}{\gamma} \ln \frac{Q_i}{N - Q_i} + c + \frac{1}{\eta} \ln \frac{Q_i}{M - Q_i} \right)
\]

\[
= \frac{a}{b} - \frac{c}{d} - \frac{1}{b\gamma} \ln \frac{Q_i}{N - Q_i} - \frac{1}{d\eta} \ln \frac{Q_i}{M - Q_i} - \frac{1}{b\gamma} \frac{N}{Q_i} - \frac{1}{d\eta} \frac{M}{Q_i}.
\]

Rearrange the terms and we get equation 4.8. Also \( \frac{\partial^2 \pi_i}{\partial Q_i^2} = Q_i \frac{\partial^2 p_i}{\partial Q_i^2} + 2 \frac{\partial p_i}{\partial Q_i} \left( Q_i \frac{\partial^2 w_i}{\partial Q_i^2} + 2 \frac{\partial w_i}{\partial Q_i} \right) < 0 \). Therefore \( Q^{(1)} \) is the unique solution that maximizes \( \pi \). \( \blacksquare \)

**Proof of Proposition 4.3.** First show that \( Q_i = Q^{(2)} \) is indeed an equilibrium. Rewrite equation 4.14 as

\[
Q_i \left[ 1 + \left( \sum_k e^{a-bp_k} \right)^\gamma \right] = KN \left[ \sum_k e^{a-bp_k} \right]^{\gamma-1} e^{a-bp_i}.
\]
Take derivative w.r.t. \( Q_i \) on both sides, and we get:

\[
\frac{\partial p_i}{\partial Q_i} = \frac{1 + \left( \sum_k e^{a-bp_k} \right) \gamma}{b e^{a-bp_i} \cdot \left( \sum_k e^{a-bp_k} \right)^{\gamma-1} \cdot (\gamma Q_i - KN - (\gamma - 1) KN \sum_k e^{a-bp_k})} = -\frac{KN}{bQ_i \cdot (-\gamma Q_i + KN + (\gamma - 1) KN \sum_k e^{a-bp_k})}.
\]

Rearrange the terms, and we have

First off, it is easy to see that equation [4.11] has a unique solution. Next we verify that there exists a unique symmetric equilibrium. If \((\bar{Q}, \bar{p}, \bar{w})\) is a symmetric equilibrium, it is sufficient to show that when \((Q_k, p_k, w_k) = (\bar{Q}, \bar{p}, \bar{w})\) for any \( k \neq i \), \( \frac{\partial \pi_i}{\partial Q_i} \bigg|_{Q_i = \bar{Q}} = 0 \) and \( \frac{\partial^2 \pi_i}{\partial Q_i^2} < 0 \). First we show the condition that satisfies \( \frac{\partial \pi_i}{\partial Q_i} \bigg|_{Q_i = \bar{Q}} = 0 \).

Since \( p_i = \bar{p} \) for any \( i \), \( \sum_k e^{a-bp_k} = \frac{1}{K} \). Therefore,

\[
Q_i \frac{\partial p_i}{\partial Q_i} \bigg|_{Q_i = \bar{Q}} = -\frac{KN}{b \left( -\gamma \bar{Q} + KN + (\gamma - 1) KN \sum_k \frac{Q_k}{Q_k} \right)} = -\frac{KN}{b \left( (K + \gamma - 1)N - \gamma \bar{Q} \right)}.
\]

Since \( \bar{Q} = N \cdot \left( \frac{Ke^{a-bp}}{1 + (Ke^{a-bp})} \right)^\gamma \), \( p^{(2)} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{\bar{Q}}{N - \bar{Q}} \right) \). Also according to lemma 4.1, \( \bar{Q} = M \frac{[e^{a+cd} - d]}{[e^{a+cd} - d]^2} \), meaning that \( \bar{w} = \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{\bar{Q}}{M - \bar{Q}} \right) \). Hence

\[
0 = \frac{\partial \pi_i}{\partial Q_i} \bigg|_{Q_i = \bar{Q}} = \bar{p} - \bar{w} + \bar{Q} \left( \frac{\partial p_i}{\partial Q_i} - \frac{\partial w_i}{\partial Q_i} \right) \bigg|_{Q_i = \bar{Q}} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{\bar{Q}}{N - \bar{Q}} \right) - \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{\bar{Q}}{M - \bar{Q}} \right) - \frac{KN}{b \left( (K + \gamma - 1)N - \gamma \bar{Q} \right)} + \frac{M}{d \eta (M - \bar{Q})}.
\]
This is identical to equation 4.11. It means that \((Q^{(2)}, p^{(2)}, w^{(2)})\) is the only possible symmetric equilibrium.

Next we show that \(\frac{\partial^2 \pi}{\partial Q_i^2} < 0\). Because \(\frac{\partial^2 \pi}{\partial Q_i^2} = 2\left(\frac{\partial p_i}{\partial Q_i} - \frac{\partial w_i}{\partial Q_i}\right) + Q_i \left(\frac{\partial^2 p_i}{\partial Q_i^2} - \frac{\partial^2 w_i}{\partial Q_i^2}\right)\), and we have shown in the proof of Proposition 4.2 that \(2\frac{\partial w_i}{\partial Q_i} + Q_i \frac{\partial^2 w_i}{\partial Q_i^2} > 0\), it is sufficient to show that \(2\frac{\partial p_i}{\partial Q_i} + Q_i \frac{\partial^2 p_i}{\partial Q_i^2} < 0\). We take second derivative of \(p_i\) and get

\[
\frac{\partial^2 p_i}{\partial Q_i^2} = \frac{KN}{bQ_i^2 \left(-\gamma Q_i + KN + (\gamma - 1)KN \sum_{k \neq i} e^{a-bp_k} \right)^2} \left(-\gamma Q_i + KN + (\gamma - 1)KN \sum_{k \neq i} e^{a-bp_k} \right) + Q_i \left(-\gamma - b(\gamma - 1)KN e^{a-bp_i} \sum_{k \neq i} e^{a-bp_k} \frac{\partial p_i}{\partial Q_i} \right)^2 \frac{\partial^2 p_i}{\partial Q_i^2}.
\]

Because \(\gamma \leq 1\) and \(\frac{\partial p_i}{\partial Q_i} < 0\),

\[
-\gamma - b(\gamma - 1)KN e^{a-bp_i} \sum_{k \neq i} e^{a-bp_k} \frac{\partial p_i}{\partial Q_i} < 0.
\]

Therefore,

\[
\frac{\partial^2 p_i}{\partial Q_i^2} < \frac{KN}{bQ_i^2 \left(-\gamma Q_i + KN + (\gamma - 1)KN \sum_{k \neq i} Q_k \right)^2} = -\frac{1}{Q_i} \frac{\partial p_i}{\partial Q_i} < -\frac{2}{Q_i} \frac{\partial p_i}{\partial Q_i}.
\]

This means that \(Q_i \frac{\partial^2 p_i}{\partial Q_i^2} + 2 \frac{\partial p_i}{\partial Q_i} < 0\). Therefore \((Q^{(2)}, p^{(2)}, w^{(2)})\) is the unique symmetric equilibrium.

Finally we show that only symmetric equilibrium exists. Suppose that there exists an asymmetric equilibrium \((\hat{Q}_k, \hat{p}_k, \hat{w}_k)\) where all \(\hat{Q}_k\)’s are not the same. Without loss of generality, we can assume that \(\hat{Q}_1 = \max \hat{Q}_k\) and \(Q_2 = \min \hat{Q}_k\). Then \(\hat{Q}_1 > \hat{Q}_2\) This means that \(\hat{p}_1 = \min \hat{p}_k < \max \hat{p}_k = \hat{p}_2\) and \(\hat{w}_1 = \max \hat{w}_k > \min \hat{w}_k = \hat{w}_2\). Because \(\gamma \leq 1\),

\[
Q_1 \left. \frac{\partial p_1}{\partial Q_1} \right|_{Q_1 = \hat{Q}_1} = -\frac{KN}{b \left(-\gamma \hat{Q}_1 + KN + (\gamma - 1)KN \sum_{k \neq i} e^{a-bp_k} \right)^2}.
\]
\[
< - \frac{KN}{b \left( -\gamma \hat{Q}_2 + KN + (\gamma - 1)KN \sum_k e^{a-bk} \right)} = \left. Q_2 \frac{\partial p_2}{\partial Q_2} \right|_{Q_2=\hat{Q}_2}.
\]

Also since \( Q_i \frac{\partial w_i}{\partial Q_i} = \frac{M}{d\eta(M-Q_i)}, \)

\[
Q_1 \left. \frac{\partial w_1}{\partial Q_1} \right|_{Q_1=\hat{Q}_1} > Q_2 \left. \frac{\partial w_2}{\partial Q_2} \right|_{Q_2=\hat{Q}_2}.
\]

Therefore,

\[
0 = \hat{p}_1 - \hat{w}_1 + Q_1 \left. \frac{\partial p_1}{\partial Q_1} \right|_{Q_1=\hat{Q}_1} - Q_1 \left. \frac{\partial w_1}{\partial Q_1} \right|_{Q_1=\hat{Q}_1} < \hat{p}_2 - \hat{w}_2 + Q_2 \left. \frac{\partial p_2}{\partial Q_2} \right|_{Q_2=\hat{Q}_2} - Q_2 \left. \frac{\partial w_2}{\partial Q_2} \right|_{Q_2=\hat{Q}_2} = 0.
\]

This leads to contradiction. Therefore, only symmetric equilibrium exists. This completes the proof.

The proof of Proposition 4.4 is similar to that of Proposition 4.3.

**Proof of Proposition 4.5.**

(i) It is not hard to see that LHS of equation 4.6, 4.11 and 4.16 all decrease in \( N \) and \( M \), and increase in \( Q \). This means that when \( N \) and \( M \) increase, \( Q^{(s)} \) as solutions to those equations, must increase.

(ii) First since \( p^{(s)} \) decreases in \( Q^{(s)} \), and \( Q^{(s)} \) increases in \( M \) according to (i), \( p^{(s)} \) decreases in \( M \). Similarly since \( w^{(s)} \) increases in \( Q^{(s)} \), and \( Q^{(s)} \) increases in \( N \) according to (i), \( w^{(s)} \) increases in \( N \).

Next we show that \( p^{(s)} \) increases in \( N \) and \( w^{(s)} \) decreases in \( M \). This is equivalent to showing that \( Q^{(s)}_N \) decreases in \( N \) and \( Q^{(s)}_M \) increases in \( M \).

Rewrite equation 4.6 as

\[
p^{(1)} = \frac{1}{b\gamma \left( 1 - \frac{Q^{(1)}_N}{N} \right)} = w^{(1)} + \frac{1}{d\eta \left( 1 - \frac{Q^{(1)}_M}{M} \right)}.
\]

LHS of the above equation decreases in \( \frac{Q^{(1)}_N}{N} \) and RHS of the above equation increases in \( \frac{Q^{(1)}_M}{M} \). Because \( \frac{Q^{(1)}_N}{M} \) increases in \( N \), and \( w^{(1)} \) increases in \( \frac{Q^{(1)}_M}{M} \), RHS increases in \( N \). LHS of the equation must also increase in \( N \). Therefore \( \frac{Q^{(s)}_N}{N} \) decreases in \( N \) leading to \( p^{(1)} \) increasing in \( N \).
Similarly because \( p^{(1)} \) decreases in \( M \), \( Q^{(1)} \) increases in \( M \). LHS decreases in \( M \). Hence \( w^{(1)} \) must also decrease in \( M \).

In a similar fashion, we could also show that \( p^{(s)} \) increases in \( N \) and \( w^{(s)} \) decreases in \( M \) for \( s \in \{2, 3\} \).

(iii) By Envelope Theorem, \( \frac{\partial \Pi^{(s)}}{\partial N} = \frac{\partial \pi_i}{\partial N} \bigg|_{Q_i = Q^{(s)}} \). Note that since \( \pi_i = (p_i(Q_i) - w_i(Q_i))Q_i \), \( \frac{\partial \pi_i}{\partial N} = Q_i \frac{\partial p_i}{\partial N} > 0 \). Therefore, \( \frac{\partial \Pi^{(s)}}{\partial N} > 0 \). Similarly we can show that \( \frac{\partial \Pi^{(s)}}{\partial M} > 0 \).

\[
\text{Proof of Proposition 4.6.} \quad Q^{(s)} \text{ is the solution to } H^{(s)}(Q) = \frac{a}{b} - \frac{z}{d}. \text{ For any given } Q, \text{ because } \frac{K N}{b(K+\gamma-1)N-\gamma Q} \text{ and } \frac{K M}{d(K+\gamma-1)M-\eta Q} \text{ decreases in } K, \quad H^{(1)}(Q) > H^{(2)}(Q) > H^{(3)}(Q). \text{ Since } H^{(s)}(Q) \text{ also increases in } Q. \quad Q^{(3)} > Q^{(2)} > Q^{(1)}. \]

\[
\text{Proof of Proposition 4.7.} \quad w^{(1)} = \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q^{(1)}}{M - Q^{(1)}} \right) < \frac{1}{d} \left( c + \frac{1}{\eta} \ln \frac{Q^{(2)}}{M - Q^{(2)}} \right) = w^{(2)},
\]
as \( Q^{(2)} > Q^{(1)} \) according to Proposition 4.6. Similarly, we have

\[
p^{(2)} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{Q^{(2)}}{N - Q^{(2)}} \right) > \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{Q^{(3)}}{N - Q^{(3)}} \right) = p^{(3)},
\]
as \( Q^{(3)} > Q^{(2)} \) according to Proposition 4.6.

\[
\text{Proof of Theorem 4.8.} \quad \text{We show that when } \ln \frac{M}{N} \leq \min \left\{ 0, (\frac{d\eta}{b\gamma} + 1) \ln \left( \frac{b\gamma^2}{d\eta} \ln K - 1 \right) - \left( \frac{a}{b} - \frac{z}{d} \right) d\eta \right\}, \quad p^{(2)} > p^{(1)}.
\]

Denote \( z^{(1)} = \ln \frac{Q^{(1)}}{N-Q^{(1)}} \) and \( z^{(2)} = \ln \frac{Q^{(2)}}{K+(N-Q^{(2)})} \). Then \( Q^{(1)} = \frac{Ne^{z^{(1)}}}{1+e^{z^{(1)}}} \) and \( Q^{(2)} = \frac{K+Ne^{z^{(2)}}}{1+K+Ne^{z^{(2)}}} \). Since \( p^{(1)} = \frac{1}{b} \left( a - \frac{1}{\gamma} \ln \frac{Q^{(1)}}{N-Q^{(1)}} \right) \) and \( p^{(2)} = \frac{1}{b} \left( a + \ln K - \frac{1}{\gamma} \ln \frac{Q^{(2)}}{N-Q^{(2)}} \right) \), we just need to show that \( z^{(2)} < z^{(1)} \). We can rewrite equation 4.6 as:

\[
\frac{a}{b} - \frac{c}{d} = \frac{1}{b\gamma} z^{(1)} + \frac{1}{d\eta} \ln \frac{Ne^{z^{(1)}}}{M - (N-M)e^{z^{(1)}}} + \frac{1}{b\gamma} e^{z^{(1)}} + \frac{1}{d\eta} \frac{M}{(M-Q^{(1)})} \>
\]
\[
> \frac{1}{b\gamma} z^{(1)} + \frac{1}{d\eta} \ln \frac{Ne^{z^{(1)}}}{M - (N-M)e^{z^{(1)}}} + \frac{1}{b\gamma} + \frac{1}{d\eta}.
\]
Because $M \leq N$ and $\ln \frac{M}{N} \leq (\frac{a}{b} + 1) \ln \left( \frac{b\gamma^2}{d\eta} \ln K - 1 \right) + \gamma \ln K - \left( \frac{a}{b} - \frac{c}{d} \right) \ln \eta$,

$$\frac{a}{b} - \frac{c}{d} > \left( \frac{1}{b\gamma} + \frac{1}{d\eta} \right) z^{(1)} - \frac{1}{d\eta} \ln \frac{M}{N}$$

$$\geq \left( \frac{1}{b\gamma} + \frac{1}{d\eta} \right) z^{(1)} - \frac{1}{d\eta} \left[ \left( \frac{d\eta}{b\gamma} + 1 \right) \ln \left( \frac{b\gamma^2}{d\eta} \ln K - 1 \right) - \left( \frac{a}{b} - \frac{c}{d} \right) \ln \eta \right]$$

$$= \left( \frac{1}{b\gamma} + \frac{1}{d\eta} \right) z^{(1)} - \left( \frac{1}{b\gamma} + \frac{1}{d\eta} \right) \ln \left( \frac{b\gamma^2}{d\eta} \ln K - 1 \right) + \frac{a}{b} - \frac{c}{d}.$$

This means that $e^{z^{(1)}} < \frac{b\gamma^2}{d\eta} \ln K - 1$. On the other hand,

$$\frac{a}{b} - \frac{c}{d} = \frac{1}{b\gamma} z^{(2)} + \frac{1}{d\eta} \ln \left[ \frac{K\gamma Ne^{z^{(2)}}}{M - K\gamma(N - M)e^{z^{(2)}}} \right] + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q^{(2)}]} + \frac{M}{d\eta(M - Q^{(2)})}$$

$$> \frac{1}{b\gamma} z^{(2)} + \frac{1}{d\eta} \ln \left[ \frac{K\gamma Ne^{z^{(2)}}}{M - K\gamma(N - M)e^{z^{(2)}}} \right] + \frac{M}{d\eta(M - Q^{(2)})}$$

$$> \frac{1}{b\gamma} z^{(2)} + \frac{1}{d\eta} \ln \left[ \frac{Ne^{z^{(2)}}}{M - (N - M)e^{z^{(2)}}} \right] + \frac{\gamma}{d\eta} \ln K + \frac{M}{d\eta(M - Q^{(2)})}.$$
Denote \( y^{(2)} = \ln\left( \frac{Q^{(2)}}{M - Q^{(2)}} \right) \) and \( y^{(3)} = \ln\left( \frac{Q^{(3)}}{K^\eta(M - Q^{(3)})} \right) \). Then \( Q^{(2)} = \frac{Me^\gamma}{1 + e^\gamma} \) and \( Q^{(3)} = \frac{K^\eta Me^\gamma}{1 + K^\eta e^\gamma} \). Since \( w^{(2)} = -\frac{c}{a} + \frac{1}{dn} y^{(2)} \) and \( w^{(3)} = -\frac{c}{d} + \frac{1}{dn} y^{(3)} \), we just need to show that \( y^{(2)} > y^{(3)} \).

We can rewrite equation [4.16] as:

\[
\frac{a}{b} - \frac{c}{d} = \frac{1}{b\gamma} \ln \left( \frac{K^{\eta - \gamma} Me^\gamma}{N - K^\eta(M - N)e^\gamma} \right) + \frac{1}{dn} y^{(3)} + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q^{(3)}]} + \frac{KM}{d[(K + \eta - 1)M - \eta Q^{(3)}]}
\]

Because \( M \geq N \) and \( K > 1 \),

\[
\frac{a}{b} - \frac{c}{d} > \frac{1}{b\gamma} \ln \left( \frac{Me^\gamma}{N} \right) + \frac{1}{dn} y^{(3)} + \frac{\eta - \gamma}{b\gamma} \ln K + \frac{K}{b(K + \gamma - 1)} + \frac{K}{d(K + \eta - 1)}
\]

\[
> \left( \frac{1}{b\gamma} + \frac{1}{dn} \right) y^{(3)} + \frac{1}{b\gamma} \ln \frac{M}{N} + \frac{\eta - \gamma}{b\gamma} \ln K.
\]

Since \( \ln \frac{M}{N} \geq b\gamma \left( \frac{\eta}{b} - \frac{c}{d} \right) + (\gamma - \eta) \ln K - \left( 1 + \frac{b\gamma}{dn} \right) \ln \left[ \frac{dn}{b\gamma} \left( \eta \ln K - \frac{\gamma K}{K - 1} \right) - 1 \right] \),

\[
\frac{a}{b} - \frac{c}{d} > \left( \frac{1}{b\gamma} + \frac{1}{dn} \right) y^{(3)} + \left( \frac{a}{b} - \frac{c}{d} \right) + \frac{\eta - \gamma}{b\gamma} \ln K - \left( \frac{1}{b\gamma} + \frac{1}{dn} \right) \ln \left[ \frac{dn}{b\gamma} \left( \eta \ln K - \frac{\gamma K}{K - 1} \right) - 1 \right] + \frac{\eta - \gamma}{b\gamma} \ln K.
\]

Therefore, \( y^{(3)} < \ln \left[ \frac{dn}{b\gamma} \left( \eta \ln K - \frac{\gamma K}{K - 1} \right) - 1 \right] \).

We can also rewrite equation [4.11] as:

\[
\frac{1}{b\gamma} \ln \left( \frac{Me^\gamma}{N - (M - N)e^\gamma} \right) - \frac{1}{b} \ln K + \frac{1}{dn} y^{(2)} + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q^{(2)}]} + \frac{1}{dn} e^{y^{(2)}} + \frac{1}{dn} = \frac{a}{b} - \frac{c}{d}
\]

Because \( Q^{(2)} < N \), \( \frac{1}{b\gamma} \ln \left( \frac{Me^\gamma}{N - (M - N)e^\gamma} \right) - \frac{1}{b} \ln K + \frac{1}{dn} y^{(2)} + \frac{KN}{b(K - 1)} + \frac{1}{dn} e^{y^{(2)}} + \frac{1}{dn} > \frac{a}{b} - \frac{c}{d} \).

On the other hand,

\[
\frac{a}{b} - \frac{c}{d} = \frac{1}{b\gamma} \ln \left( \frac{K^{\eta - \gamma} Me^\gamma}{N - K^\eta(M - N)e^\gamma} \right) + \frac{1}{dn} y^{(3)} + \frac{KN}{b[(K + \gamma - 1)N - \gamma Q^{(3)}]} + \frac{KM}{d[(K + \eta - 1)M - \eta Q^{(3)}]}
\]

\[
> \frac{1}{b\gamma} \ln \left( \frac{K^{\eta - \gamma} Me^\gamma}{N - K^\eta(M - N)e^\gamma} \right) + \frac{1}{dn} y^{(3)}.
\]
This means that

\[
\frac{1}{b \gamma} \ln \left[ \frac{Me^{y(3)}}{N - (M - N)e^{y(3)}} \right] - \frac{1}{b} \ln K + \frac{1}{d \eta} y^{(3)} + \frac{K}{b(K - 1)} + \frac{1}{d \eta} e^{y(3)} + \frac{1}{d \eta}.
\]

Since \( y^{(3)} < \ln \left[ \frac{d \eta}{b} \left( \frac{n}{\gamma} \ln K - \frac{K}{K - 1} \right) - 1 \right] \), \( e^{y(3)} < \frac{d \eta}{b} \left( \frac{n}{\gamma} \ln K - \frac{K}{K - 1} \right) - 1 \).

\[
\frac{1}{b \gamma} \ln \left[ \frac{Me^{y(2)}}{N - (M - N)e^{y(2)}} \right] - \frac{1}{b} \ln K + \frac{1}{d \eta} y^{(2)} + \frac{K}{b(K - 1)} + \frac{1}{d \eta} e^{y(2)} + \frac{1}{d \eta}.
\]

Denote \( f(y) = \frac{1}{b \gamma} \ln \left[ \frac{Me^{y}}{N - (M - N)e^{y}} \right] - \frac{1}{b} \ln K + \frac{1}{d \eta} y + \frac{K}{b(K - 1)} + \frac{1}{d \eta} e^{y} + \frac{1}{d \eta} \). \( f(y) \) increases in \( y \). We have shown that \( f(y^{(2)}) > \frac{a}{b} - \frac{c}{d} > f(y^{(3)}) \). Therefore \( y^{(2)} > y^{(3)} \). This completes the proof.

**Proof of Proposition 4.10** For scenario 1, the consumer surplus on the demand side and the supply side are

\[
CS_D^{(1)} = \frac{KN}{b \gamma} \ln \left[ \left( e^{a - by^{(1)}} \right)^{\gamma} + 1 \right] \quad \text{and} \quad CS_S^{(1)} = \frac{KM}{d \eta} \ln \left[ \left( e^{c + dw^{(1)}} \right)^{\eta} + 1 \right].
\]

Using the expression of \( p^{(1)} \) in Proposition 4.2, we have

\[
CS_D^{(1)} = \frac{KN}{b \gamma} \ln \left( \frac{N}{N - Q^{(1)}} \right) \quad \text{and} \quad CS_S^{(1)} = \frac{KM}{d \eta} \ln \left( \frac{M}{M - Q^{(1)}} \right).
\]

Similarly, for scenario 2, we have

\[
CS_D^{(2)} = \frac{KN}{b \gamma} \ln \left( \left( Ke^{a - by^{(2)}} \right)^{\gamma} + 1 \right) = \frac{KN}{\gamma b} \ln \left( \frac{N}{N - Q^{(2)}} \right), \quad \text{and}
\]

\[
CS_S^{(2)} = \frac{KM}{d \eta} \ln \left( \left( e^{c + dw^{(2)}} \right)^{\eta} + 1 \right) = \frac{KM}{d \eta} \ln \left( \frac{M}{M - Q^{(2)}} \right).
\]

Last, for scenario 3, we have

\[
CS_S^{(3)} = \frac{KN}{b \gamma} \ln \left( \left( Ke^{a - by^{(3)}} \right)^{\gamma} + 1 \right) = \frac{KN}{\gamma b} \ln \left( \frac{N}{N - Q^{(3)}} \right), \quad \text{and}
\]
\( C S_{S}^{(3)} = \frac{KM}{\eta d} \ln \left( (K e^{-c+dw^{(3)}})^{\eta} + 1 \right) = \frac{KM}{d\eta} \ln \left( \frac{M}{M - Q^{(3)}} \right) . \)

From these equations, we observe that the function form is the same, except that the equilibrium quantities are different. Since \( Q^{(3)} > Q^{(2)} > Q^{(1)} \) according to Proposition 4.6, we get the rank of the surplus. ■
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