DYNAMIC PROGRAMMING
AND
GEOGRAPHICAL SYSTEMS

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Preface

This is Report No. 13 in the series on the Environment Study prepared in the Department of Geography and the Centre for Urban and Community Studies under a grant from Bell Canada, the first to be released under Component Study No. 8. It is a thorough review of the literature on the analysis of geographical systems within dynamic programming frameworks.

The report considers dynamic planning and control processes and their geographical implications. Thus it complements the more descriptive forecasting approaches proposed by Professor Curry in Research Report No. 12. It represents a portion of the technical and theoretical framework within which the dynamics of geographical processes of Eastern Canada are to be studied. With normative models such as dynamic programming, objectives must be specified, but the sensitivity of the resulting patterns to alternative goal structures may be tested. It is hoped that these and other dynamic frameworks will be applied to rural and urban as well as transportation processes.

Initially, the dynamic programming approach is outlined, followed by a detailed discussion of significant geographical applications of the techniques and finally an evaluation of practical advantages and limitations.
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1. Introduction

Although dynamic programming can no longer be characterized as a "new" approach to systems optimization, it is not widely known even to mathematically oriented geographers. One reason for this undoubtedly lies in the fact that geographers have traditionally avoided normative frameworks preferring instead to describe selected aspects of past, current, and, even on occasion future worlds, unencumbered by any explicit goal orientations. Even the more well known technique of linear programming has been utilized only sparingly by geographers in spite of its origins in an essentially geographical problem.

This study then ignores the apparent bias of geography against normative models. The dynamic programming approach is first outlined in its various formulations. Secondly, some of the significant geographical applications of dynamic programming are discussed in some detail. Finally some of the advantages and limitations of the approach are briefly considered.

In this review, emphasis is placed on the substantive applications of dynamic programming. Computational difficulties are frequently mentioned, but strategies by which these can be overcome are not discussed in detail. Only discrete time problems are considered. Thus, the dynamics of all the problems are expressed in terms of simple difference equations rather than differential-difference equations. The relationships between dynamic programming and other control theory models are not discussed.
2. **Basic Concepts of Dynamic Programming**

2.1 The Systems Approach

In recent years, there has been a growing movement in geography and other disciplines towards the development of common frameworks which might stimulate research having broad applicability in the study of a wide variety of phenomena. Increasing emphasis is being placed on models which may describe the behaviour of many otherwise unrelated processes. This search for theory or theoretical frameworks common to a wide range of phenomena is one of the characteristics of the systems approach which has become increasingly fashionable in the past few years. Important aspects of the systems approach include feedback, feed-forward, control, information, entropy, goal-seeking and multidimensional dynamic relationships. Although an increasing number of geographers use these and other systems concepts, very few have explicitly adopted mathematical systems approaches in their research. Among the simplest of such approaches is dynamic programming.

In both its formulation and solution procedures, dynamic programming is markedly different from other types of mathematical programming. On the one hand, it is extremely general so that a wide variety of problems can be formulated as dynamic programming problems. On the other hand, there are no computer programming packages which can be used to solve all or most of the problems so formulated.*

As Nemhauser (28) states, "Multistage analysis is a problem solving approach rather than a technique." The researcher must translate his problem

*The program of Bellmore, Howard, and Nemhauser (9) is perhaps the most useful program currently available.
into a dynamic programming format, and even then, it is within his discretion to specify the optimization technique which is to be used for each stage of the process. This technique may be complete enumeration, linear or nonlinear programming, Fibonacci or some other search technique. In summary, dynamic programming is an approach which specifies a general procedure whereby some complex and/or dynamic control problems may be solved sequentially, combining optimal sub-problems in such a way that an optimum solution to the total problem is obtained.

2.2 Deterministic Models

A more thorough discussion of the various formulations of dynamic programming problems can be found in the ever increasing number of fine textbooks; for example, Bellman (3), Bellman and Dreyfus (6), Beckmann (2), Jacobs (21), Nemhauser (28) and White (30). The following all too brief summaries are presented to make the subsequent review of applications more meaningful.

The dynamic programming approach solves a decision-making problem in a series of stages. In its simplest discrete, deterministic form, the following aspects of the process are known:

(i) the initial state \( X \) of the system or process (this is a numerical descriptor which may be either a scalar or a vector);

(ii) the set of possible decisions \( d \) which may be taken at each stage of the process;

(iii) the transfer function \( T \) which maps a state-decision pair into new system states;

(iv) the reward (or cost) function \( r \) which summarizes the immediate payoff or cost resulting from a given transformation;

(v) a criterion function \( f \), a composition of all of the individual stage rewards, which is to be maximized or minimized;
(vi) \( N \), the number of stages in the process.

Wherever possible, the above notation is used in the remainder of this paper.

![Diagram](image)

\[ f_t = f_{t+1} \circ r(t) \]

\[ d(t), t=1, 2, ..., N \]

Figure 1

*Note that this conforms with the convention of numbering the stages in reverse order i.e., \( X(t) \) is \( t \) stages from the end of the process.

The problem as summarized in Figure 1 is to select the sequence of feasible \( d(t), t=1, 2, ..., N \), such that the criterion function \( f \) is maximized or minimized. Such a sequence of decisions is called the optimal policy.

The solution procedure depends upon Bellman’s Principle of Optimality (3) which states that "an optimal set of decisions has the property that whatever the first decision is, the remaining decisions must be optimal with respect to the outcome which results from the first decision." Thus, with only one stage remaining, the problem becomes a single stage optimization problem, the solution of which is in the form of a function of the input state \( X(1) \). Decision \( d(1) \) is chosen such that \( r_1(X(1), d(1)) \) is maximized. With two stages remaining, \( d(2) \) is chosen as a function of \( X(2) \) so that the composition of the return from that stage and the subsequent stage is maximized.
In general then, the recursion equations as adapted from Nemhauser (28) are the following:

\[
\begin{align*}
    f_t(X(t)) &= \max_{d(t)} \{r_t(X(t),d(t)) \text{ of } T_t(X(t),d(t)) \}, \quad t=2,2,\ldots,N \\
    d(t) &= \max \{r_t(X(t),d(t))\}, \quad t=1
\end{align*}
\]

where "o" is a composition operator (generally addition, multiplication, or selecting the maximum or minimum of \((r_t,f_{t-1})\)).

In the continuous case, the transfer function takes the form of a system of differential equations; in the discrete, analytic case, it is in the form of first order difference equations. For example, \(X(t) = X(t+1) - d(t+1)\). The form of the recurrence relation should be interpreted much more generally, however. Note that the transfer function \(T\) and the reward function \(r\) are both subscripted. This implies that neither of these functions need be invariant throughout the entire process. Indeed, the relationships may be in the form of tabulated data. Thus, many systems which cannot be completely described analytically by differential and/or difference equations may be optimized using a dynamic programming approach.

Note that the solution to equation set (1) yields a sequence of \(d(t)\) for a given value of \(X(N)\). Moreover, once the equations have been solved, different values of \(X(N)\) can be postulated to determine the sensitivity of the optimal policy and criterion function to different initial states, budget levels for example. By solving for a particular initial state, we can obtain with little additional effort the solution for the same system in all feasible initial states.
2.3 Stochastic Models

In its simplest form the dynamic programming problem under risk is very similar to deterministic formulations. In addition to the state and decision variables, a set of random variables $\xi(t)$, $t=1, \ldots, N$ with independent and known probability distributions is introduced. These variables enter into the transfer and reward functions and the objective is modified so that the expected value of the criterion function is to be maximized or minimized. Thus the recurrence relations are now the following:

$$f_t(X(t)) = \left\{ \begin{array}{l}
\max_{d(t)} \sum_{\xi(t)} \left[ P_t(\xi(t)) \{ r_t(X(t), d(t), \xi(t)) \} \circ f_{t-1}(T_t(X(t), d(t), \xi(t)) \} \right] \\
t=2, \ldots, N \\
\max_{d(1)} \sum_{\xi(1)} P_1(\xi(1)) \{ r_1(X(1), d(1), \xi(1)) \}, \ t=1.
\end{array} \right.$$ 

where $P_t(\xi(t))$ is the probability of the random variable taking on value $\xi(t)$ in stage $t$ and "\(\circ\)" is a composition operator (addition or multiplication).

Note that the solution to equation set (2) is in the form of (a) total expected rewards and (b) conditional decisions. Only the initial decision $d(N)$ is determined since only the initial state $X(N)$ is known with certainty. The optimal policy is thus not a rigid plan, but rather a sequence of "if-then" statements which allow the planner to respond to the future states of the system as they come apparent (or indeed allow such responses to be completely automated).

A special case of stochastic dynamic programming may be described as

*It is interesting to note that Bellman (5) admits that he first formulated dynamic programming as a stochastic problem. Only later did he discover its deterministic form and its relation to the calculus of variations.
Markovian decision processes. The decision maker chooses the probabilistic transfer function (a Markov chain transition probability matrix) at each stage of the process in such a way that the total expected rewards are minimized. Howard (19) developed an ingenious alternative method of solution to this class of problems. His famous illustrative example of the taxicab problem is essentially a locational decision-making problem of some interest to geographers. Marble (46) suggests that some aspects of individual travel behaviour could be described using this framework.

2.4 "Adaptive" Models

Some processes are characterized by uncertainty rather than risk, i.e., the true probabilities or the parameters of the probability distribution are not known. In some of these cases, it is possible and potentially useful to adopt a dynamic programming approach (4,109). An initial decision is made on the basis of a priori probabilities. That is, the problem is assumed to be a stochastic dynamic programming problem. These estimates are then revised on the basis of the results of that stage. Yet another decision is made, the results monitored, and estimates revised. By continually updating parameter estimates on the basis of working with the system, the planner or controller gradually transforms the problem from one of making decisions under uncertainty to one of decision making under risk.

This framework is intuitively appealing and one might argue very close to implicit decision frameworks which are actually employed by planners and controllers. Dynamic programming describes the problem in a general formalized manner. The operational applicability of this approach to real decision problems has been severely limited, however, since each unknown parameter or
probability adds another state variable, and thus the limits of computational feasibility are quickly encountered. Computational problems are briefly discussed in a later section.

2.5 Nonserial Systems

All of the previous and most of the subsequent discussion assumes a purely sequential process. The outputs of one stage become the inputs of the following stage. This assumption ignores important processes in which two sequential systems converge or diverge at a given stage, or systems in which an output of stage \( t \) initiates a parallel process which is fed-back or forward to become an input to the main process at stage, \( t+k \) or \( t-k \) where \( k > 1 \).

These more complex multistage decision problems are now amenable to solution (28). Meier and Beightler (82), have described and optimized branching multiple stage water resource systems using these relatively recent techniques of nonserial dynamic programming.

3. Applications of Dynamic Programming

3.1 General Applications

Because of the inherent generally of the dynamic programming approach, a wide variety of decision processes have been formulated as dynamic programming problems. Many of the references listed in Part A of the bibliography give some idea of the vast number of systems which have been so described. Aris (1), Beckmann (2), Bellman (4,6,8), Hadley (17), Jacobs (21), Kaufmann (23), Kaufmann and Cruon (24), and Nemhauser (28) are especially notable in this respect.
Inventory control models in which current stock levels are the state variables, quantities ordered are the decision variables, and sales levels are random variables are particularly suitable to formulate as dynamic programming problems (95, 100, 102). In addition, however, the approach has been used to describe mathematically the following decision problems: component replacement, allocation of resources between alternative sub-systems or over time, bottleneck situations, control of competitive processes, curve fitting, control of economic trends, the knapsack problem, missile trajectory problems, and many others.

It is possible that several of these topics may in certain cases have some geographically interesting implications. For the purpose of this paper, however, only those problems which relate directly to the spatial, regional, and/or the man-environment traditions of geography have been considered.

3.2 Transportation Systems

Transportation problems have been among the most intensively studied in operations research or management science. It is therefore not surprising that many transportation problems have been studied within the dynamic programming framework. The following discussion considers three somewhat arbitrary categories of transportation topics relating to paths, construction, and flows respectively.

3.2.10 Optimal Path Problems

3.2.11 The Shortest Path Problem

The familiar problem of determining the shortest path through a network (or the kth shortest path) can be formulated and solved using the dynamic
programming approach. For the deterministic case, the recurrence equation is

\[ f_i = \min\{d_{ij} + f_j\} \quad i = 0, \ldots, N-1 \]

\[ f_N = 0 \]

where \( f_i \) is the distance of the shortest path between nodes \( i \) and \( N \). Dreyfus (37) notes, however, that much more efficient methods have been developed to solve this problem. Only in cases where negative values of \( d_{ij} \) are permitted should the dynamic programming formulation be employed.

Dynamic programming has the additional advantage, however, that it can be readily extended to the stochastic case. In one of the more interesting extensions, Kalaba (43) formulates such a problem so that the criterion function is the probability of reaching a destination within a specified time period; the recurrence relation is

\[ f_i(t) = \max \int_0^t P_{ij}(t-s) f_j(s) \, ds \]

\[ f_N(t) = 1 \]

where \( f_i(t) \) is the probability of reaching destination \( N \) in \( t \) time units or less, given the process is initiated in \( i \) and an optimal policy is adopted, and \( P_{ij}(s) \) is the probability density function of moving from state \( i \) to state \( j \) in \( s \) time units.

Note that the solution of these equations would yield an optimal feedback control policy of which only the first move would be deterministically specified. Subsequent moves would depend upon the random outcomes of actual travel times. Such a framework could conceivably have practical applications in the automated routing of commodity and passenger vehicle systems.

3.2.12 Generalized Euler Paths

The first paper on graph theory, written more than two centuries ago, considers a problem which is essentially geographic in nature. Given a river, islands and a set of bridges connecting the islands and main river banks, the problem is to describe a route starting at any point which passes over each
bridge exactly once and returns to the initial point. Either a feasible solution to the problem exists or one does not exist. All feasible solutions are optimal.

Bellman and Cooke (34) have generalized the problem using dynamic programming so that the objective is to devise a cyclic route which passes over each bridge (i.e. link or edge) at least once so that the number of repetitions is minimized. The state vector is defined to be a list \((Q,E)\) where \(Q\) is the node at which the tracing point currently lies and \(E\) is composed of the set of edges remaining to be traversed. The decision is of course the node at which the tracing point will be at the next stage. The recurrence relation is then

\[
f(Q,E) = \min_{Q_1, Q_2} \left[ (\text{link} QQ_1 \notin E, \text{link} QQ_2 \in E, \{Q_2\}) \right] + f(Q_1, E_2)\]

where (1) \(Q_1\) and \(Q_2\) are nodes directly connected to \(Q\)

(2) \(\text{link} QQ_1 \notin E\)

(3) \(\text{link} QQ_2 \in E\)

and thus (4) \(E_2 = E - \{Q_2\}\).

The authors outline their adaptation of the basic dynamic programming algorithm which could be used to solve this problem. They admit, however, that the procedure is currently computationally infeasible for graphs of high complexity because of the vast number of possible combinations and permutations of edges and nodes.

3.213 Travelling Salesman Problem

Among the most famous problems in network analysis, as well as one of
the most resistant to adequate solution, is the travelling salesman problem. Given a set of cities (points) and the distance between each pair, the problem consists of constructing the cyclic graph of minimum length which passes through every city. This problem has stimulated a vast amount of research and for large numbers of points, it is still computationally infeasible. (See Bellmore and Nemhauser (35) for review of the many approaches to this problem).

Bellman (33) and Gonzalez (39) offer dynamic programming approaches as solution procedures for the travelling salesman problem. For more than fifteen points, however, Gonzalez found the number of computations and storage requirements to be excessive. This approach, while certainly more effective than exhaustive enumeration, is clearly dominated by other techniques (35).

3.22 Transportation Flow Problems

The most widely known and used transportation model is of course the Hitchcock-Koopmans Transportation Problem which determines those shipments $X_{ij}$ which minimize the total cost of transportation subject to the constraints that all resources are used and all demands are met.

That is,

$$\text{MIN } C = \sum_{i=1}^{n} \sum_{j=1}^{N} C_{ij} X_{ij}$$

subject to

$$\sum_{j=1}^{N} X_{ij} = X_{i}$$

$$\sum_{i=1}^{n} X_{ij} = Y_{j}$$

$$X_{ij} \geq 0$$
Bellman (32) has shown that this and related problems can be readily formulated as a dynamic programming problem. Using the Principle of Optimality the demands of the \(N\)th destination are determined as a function of the resources at the various supply points; the demands of destination N-1 are then determined as a function of the remaining resources at the n supply points; etc. The state variables are the resources currently available at each of the supply points, i.e., \(X_1(t), X_2(t), \ldots, X_n(t)\). The dynamics of the process are simply

\[
X_i(t) = X_i(t+1) - X_{i,t+1} \quad i = 1, 2, \ldots, n \\
X_i(N) = X_i
\]

The recurrence relations is thus

\[
f_t(X_1(t), X_2(t), \ldots, X_n(t)) = \min_{X_{it}} \left[ \sum_{i=1}^{n} c_{it} X_{it} + f_{t-1}(X_1(t) - X_{it}, X_2(t) - X_{2t}, \ldots, X_n(t) - X_{nt}) \right]
\]

As Bellman notes, the computational feasibility of such a problem depends almost entirely upon the number of sources since computation increases only linearly with the number of stages (i.e., number of destinations). Moreover, the number of state variables can be reduced by one since

\[
\sum_{i=1}^{n} X_i = \sum_{j=1}^{N} Y_j
\]

Thus a problem with 4 or 5 supply points and a very large number of destinations can be solved. The advantage of this formulation is of course that it is no longer necessary to assume proportional costs.

Bellman (32) considers two elaborations on this basic problem. The first optimizes a process where the \(t^{th}\) set of destinations becomes the original set for the next set of demand points. The second considers problems
which explicitly take network structure into account and thus imposes capacity constraints on links and/or nodes.

Midler (47) has developed a dynamic programming model which determines the optimal flow of different commodities through a multimodal transportation system with stochastically variable demands. The model determines conditionally the combination of modes to be used, the assignment of commodity classes to modes, the supply points which should serve each destination and the rerouting of carriers from destinations to sources. The criterion function is a quadratic user cost function.

The model is essentially an augmented inventory control model which uses a moderately sophisticated matrix algebraic formulation. The precise formulation is much too complex to discuss here in any detail. It does demonstrate very clearly, however, the flexibility of the dynamic programming approach in that there are fewer limitations on the form of relationships than with other mathematical programming models. Computational difficulties, however, limit the size of the problem. Midler states, for example, that a problem with four origins and destinations, two modes and six commodity classes would under certain circumstances be susceptible to solution.

A natural gas network flow problem is considered by Wong and Larson (55). The problem is to determine the optimum suction and discharge pressure for each compression station such that total compressor horsepower is minimized subject to specified steady state flow and pressure constraints. The simple single pipeline case is readily formulated and solved using straightforward serial dynamic programming. Single junction and multiple junction networks are optimized using non-serial techniques described in Nemhauser (28). At
any junction the number of state variables of the process increases according to the number of pipelines emanating from that point.

Nemhauser (50), in recent years one of the most frequent contributors to both the theory and applications of dynamic programming, uses a dynamic programming model to determine an optimal scheduling policy for local and express transit service. Net revenues as determined from schedule-dependent usage equations and operating costs are maximized. The model assumes among other things that the relation between usage and required waiting times is known precisely. This of course is an important characteristic of dynamic programming and mathematical programming approaches in general—the functional relationships must be known and specified precisely. Mathematical programming formulations can thus be used as heuristic devices which suggest areas in which valuable research is needed. In dynamic programming, the dynamics (real or artificial) and reward structure of the process must be known.

3.23 Network Construction Problems

3.23.1 Optimal Staging of Transportation Construction

Roberts (52) and Roberts and Funk (53) have suggested that a combination of dynamic and linear programming approaches be used to formulate and solve the problem of when and where to add links to an existing transportation network. Morlok (48) has made a similar suggestion and is currently operationalizing a mixed integer problem in which dynamic programming is utilized for the choice of binary developmental variables while linear programming methods are used to select the best operational policy for each possible configuration. Because of its relative accessibility and simplicity, however, only the study of Funk and Tillman (38) is considered here in detail.
Funk and Tillman have demonstrated the potential usefulness of dynamic programming in scheduling the sequence of links to be added to an existing highway network. The highway planning problem is viewed not simply as a choice between a finite number of alternative network configurations, but rather as a choice between alternative permutations as well as combinations of links to added.

The state of the system is identified by the links which have already been added to the network. Associated with each state is a set of feasible decisions, i.e. those links which can still be added. Each state-decision pair is mapped into an immediate cost (amortized construction, maintenance and travel). These relationships are summarized by a set of hypothetical numerical data. Two four-stage problems are solved for the simple numerical example so that total system costs are minimized subject to the constraint that at most, and then exactly, one link is to be added in each stage of the plan implementation process.

Several comments can be made about this illustrative problem which also apply to many of the other examples considered in this paper:

1. Rarely can all additions to a system be made simultaneously; thus some means to discover the optimal spatio-temporal ordering of transportation links or other planning actions is a potentially useful planning tool (38).

2. The final solution is in the form of a sequence of planning actions, but in many cases a firm commitment need be made only to the first $k$ stages. While those decisions are being implemented, more accurate and additional information may be forthcoming so that cost and/or reward functions can be revised. The remaining $N-k$ stage problem could then be optimized using these revisions (45).

3. Suppose as in (2) a firm commitment need be made only for the first $k$ stages. Moreover, assume that there are many alternative, uncertain future environments, each of which implies a different cost/reward structure. The dynamic programming model is applied to each of these
alternatives. We can say then that plan selection is concerned with the identification of "optimal" sequences whose first k actions are "similar." By assumption, only the first k actions must be selected at stage N. During the first k stages, decisions about the following set of actions can be made in a similar manner. A sequence of first k decisions which is not common to many plans may be excluded if the criterion function is not very sensitive to its substitution by another sequence have a greater commonality (45).

(4) The final physical configuration may be significantly different depending upon whether a static minimum cost solution or a sequential decision-making framework is adopted (38).

(5) Computational difficulties abound because not only are the different combination of actions considered but also the different permutations; thus large transportation network and other planning problems tend to be unmanageable if a direct dynamic programming approach is used.

Gulbrandsun (41) considers a somewhat different problem of optimally allocating resources to 77 "independent" groups of highway projects over four or five year periods. Independence in this case implies that investment in one project will not influence the efficiency of investment in any other project. Using Lagrangian multipliers and dynamic programming, an allocation of resources to projects over time is calculated. It is interesting to note, however, that in order to make the problem feasible, the stages of the problem consist of the 77 projects, the decision variables d(t) are the ordered 4-triple of resources allocated to the tth project in each of the four time periods, and the state variables X(t) are the total resource budgets of each of the time periods after N-t projects have been considered. The problem could not be solved if the decision vector consisted of the resources committed to the 77 projects in the tth time period.

3.232 Location of a Routeway Connecting Two Points

Many problems which are not intrinsically dynamic can be artificially
assumed to be sequential in order to utilize the dynamic programming approach. For example, Werner's (54) multivariate refraction problem of connecting two cities, located in a region where costs are inhomogeneous, such that the joint flow and construction costs are minimized would seem, conceptually at least, to be a dynamic programming problem.*

Kaufmann (23) and others have considered a discrete version of the above problem as a special case of the shortest path problem, and therefore susceptible to solution by dynamic programming. An interesting variation on this problem is considered by Groboillot and Gallas (40). The objective is to connect two cities so that total amortized investment operating and maintenance costs are minimized subject to maximum curvature and gradient constraints. The problem is viewed as a special case of the shortest path problem so that the recurrences relation is

\[
f_k = \min_{j \in E_{ik}} [f_j + C_{jk}]
\]

where \(f_j\) is the total cost associated with the optimal route from the initial point to some intermediate point \(j\).

\(E_{ik}\) is the set of points in section \(i\) from which point \(k\) can be reached

\(C_{jk}\) is the cost of reaching point \(k\) from point \(j\).

The curvature constraints are achieved simply by limiting the possible edges in the graph of the decision tree. Similarly, using a three dimensional graph (the third dimension being elevation), the gradient constraint is ensured by not permitting large changes in altitude from one section (stage) to another.

The authors have used this method with some success in planning the

*This was suggested in conversation by A. J. Scott and is mentioned in Scott (106). Each cost region is a stage of the process. The locational coordinates of the intersection of the routeway and regional boundaries are the system states. The angles of refraction are the decision variables.
location of roadways; in spite of (and perhaps because of) this experience with this approach, they are fully cognizant of the severe operational limitations arising from excessive storage and computational requirements. Apparently they have not utilized any other shortest path solution procedures.

3.3 Regional and Locational Allocation Problems

The assignment of people or things to a set of regions and the location of a set of service facilities so that some objective function is optimized are two of the central problems in normative geography. These are significant problems that increasingly are occupying certain economists, operation researchers, city planners, and geographers.

The regional assignment problem in its simplest form where there is no spatial dependence of returns is readily formulated as a dynamic programming problem. Given a fixed quantity $Q$ of a resource (water, capital, personnel, voting power, etc.) and a return function for each region, what is the optimal allocation of that resource to the $N$ different regions. The problem is to

$$\text{MAX} \sum_{i=1}^{N} r_i (d_i)$$

subject to

$$\sum_{i=1}^{N} d_i \leq Q$$

$$d_i \geq 0 \quad i = 1, 2, \ldots N.$$  

Using dynamic programming, the "dynamics" of the allocation process are simply,

$$X(t) = X(t+1) - d(t+1)$$

$$X(N) = Q$$
and the recursion relation is

\[ f_t(X(t)) = \text{MAX} \left\{ r_t(d(t)) + f_{t-1}(X_{t-1}) \right\} \]

\[ 0 \leq d(1) \leq Q \]

\[ f_1(X(1)) = \text{MAX} \left\{ r_1(d(1)) \right\} \]

\[ 0 \leq d(1) \leq Q \]

The basic model could be used to determine the optimal assignment of salesmen to sales regions (Nemhauser (28)), capital to water resource development sites (Hall and Buras (74)) and many other simple regional assignment problems. Hall (70, 71) uses a model with slightly different dynamic equations which reflect first order spatial dependence to allocate water to linear regions along a water supply canal. Burt and Harris (57) adapt this basic model in order to assign voters to U. S. Congressional districts so that a measure of equal representation is optimized subject to the constraint that districts are as compact as possible.

An important dynamic regional investment problem which takes inter-regional dependencies into account is considered by Erlenkotter (58). The problem is simply to determine the regional allocation of plant investment so that all demands are satisfied and the present value of shipping and capital are minimized given the constant rates of regional demand increases, interregional production-shipment costs, and plant investment cost functions for each region. The dynamics of the process simply observe that current excess capacity (possibly negative) is equal to previous excess capacity plus plant investment in the previous time period minus the regional growth in demand. The author notes that for more than two producing areas, the straightforward dynamic programming approach would become infeasible. By
redefining the state vector and assuming a concave investment cost function, the problem may be reformulated so that a two region problem becomes a one-dimensional dynamic programming problem. The author concludes that a four region problem is the largest which is computationally feasible.

In recent years there seems to be a growing recognition that the location-allocation problem is of significant practical as well as theoretical importance. Much of the recent concern has been with devising efficient exact or approximation algorithms for computing solutions to location-allocation problems (59). Among these is the presentation of Bellman (56) which formulates the problem within a dynamic programming framework.

Of perhaps greater interest are dynamic location-allocation problems—i.e. situations in which account must be taken of growth and/or changing patterns of demand and resource availability. The possibility of adding new facilities to the system in response to such changes should be considered. Teitz (60) discusses some of the conceptual considerations involved in this dynamic planning problem. The initial location decision cannot be made in isolation from predicted future system states and inputs and possible subsequent decisions.

Consider a problem which is apparently similar in nature to Funk and Tillman’s network link addition problem (38) in which point facilities are to be added to a current system over a series of stages so that total discounted travel costs are to be minimized over the entire length of the process. The problem is different in a number of respects the most important of which is that the solution space is not finite. In the static case this
does not present a major problem, but in the dynamic case account must be taken of the possibility that one consumer may be assigned to one facility in one stage and another one in a subsequent stage; thus the respective travel times must be weighted by their durations. At this stage, it is not clear how a dynamic programming approach could be used to resolve this perplexing and significant problem.

3.4 Water Resource Management

Of all the areas in which dynamic programming has been applied to geographically related topics, water resource management problems are certainly the most numerous. This arises in part from the fact that many aspects of these systems can be readily specified in terms of simple difference equations in which at least one of the components is a decision or control variable. Rivers in particular may be considered as one directional and one dimensional spatial systems. Moreover, the processes which operate on these systems (weather and man-initiated controls) may be assumed to be one directional lag-one processes.

In water resource systems, the relationships between exogenous inputs such as streamflow, rainfall, and evaporation rates, decision variables such as water releases and transfers, and outputs or consequences such as new water levels can often be approximated by systems of first order difference equations usually linear and often stochastic. Exogenous inputs are probabilistically predictable because of the long time series which are often available for particular streams and rivers.

The difference equations usually are simply mass balance equations such as the following:
\[ X(t) = X(t+1) - d(t+1) - \xi(t+1) + \lambda(t+1) + \pi(t+1) \]

where \( X \) = water in reservoir (state variable)

\( d \) = water released (decision variable)

\( \xi \) = water loss by evaporation (exogenous, stochastic variable)

\( \lambda \) = streamflow into reservoir (exogenous, stochastic variable)

\( \pi \) = precipitation (exogenous stochastic variable)

and the probability density functions for \( \xi, \lambda \) and \( \pi \) are known. The objective then is to find a conditional sequence of releases which maximize the annual expected net return subject to certain physical and perhaps socio-economic constraints.

This general framework with modifications is used by Hall and Howell (77), Buras (65), Burt (66), Burt (67), Young (86), Sweig and Cole (83), Hall, Butcher and Esogbue (76), and Butcher (69).

Buras (65) uses an interesting variation of such a framework in modelling the optimal joint operating policy for aquifers and reservoirs. This model has the advantage that it can be readily understood, yet still provides some insight into the systems aspects of the dynamic programming approach. There are three state variables in the process and therefore three difference equations:

\[ X^1(t) = X^1(t+1) + \lambda^1(t+1) - d^1(t+1) - d^2(t+1) \]
\[ X^2(t) = X^2(t+1) + X^3(t+1) - d^3(t+1) \]
\[ X^3(t) = d^1(t+1) + \lambda^2(t+1) \]

where \( X^1 \) = water in surface reservoir

\( X^2 \) = water in aquifer

\( X^3 \) = water in recharge facility

\( \lambda^1 \) = streamflow

\( \lambda^2 \) = natural inflow to aquifer
\( d_1 \) = water release from surface reservoir for groundwater
\( d_2 \) = water release from reservoir to irrigate land \( A_s \)
\( d_3 \) = water pumpage from aquifer to irrigate land \( A_g \)

The recurrence relation is

\[
f_t(X^1(t), X^2(t), X^3(t)) = \text{MAX}[\phi(d_2, d_3) + \beta \sum_{j=1}^{M} P_j f_{t-1}[X^1(t-1), X^2(t-1), X^3(t-1)]]
\]

where \( \phi(X^2, X^3) \) is the \( t \)th stage return from irrigation.

and \( \beta \) is an appropriate discount factor.

In addition to the optimal timing of resource utilization, dynamic programming has been used in the allocation problem discussed in a previous section. Hall and Buras (74) for example consider the problem of selecting resource development sites from a finite number of possibilities and the extent to which those sites should be developed. Hall (70, 71) optimally allocates water to regions along a water supply canal. Hall (73) uses the approach to determine the optimal allocation of water to different uses.

There seems to be no reason why these models which have been developed to optimize water resource systems could not be modified and used in the modelling of other resource management problems. Burt (66) states that these approaches are applicable to any temporal resource allocation problem in which the resource is either fixed in supply or partially renewable, thus allowing a difference equation model formulation.

3.5 Agricultural Economics

Agricultural economics would seem to be of interest to geographers on at least two counts. First, agricultural topics have been among the most
popular in economic geography, perhaps because the interaction between man's activities and his physical environment is of obvious importance in farming. Secondly, agricultural economics has mixes and levels of theoretical, empirical and technical orientation which appear to be of particular relevance in geography's current stage of development.

Dynamic programming models are relatively few in agricultural economics, but there seems to be a growing awareness of the potential significance of control theoretic approaches in agricultural studies (92). Burt and Allison (89) consider the problem of crop rotation as a Markovian decision process. The objective is to maximize discounted expected returns by choosing an appropriate conditional policy of planting wheat or leaving the land in fallow. The states of the process are different moisture levels \((i = 1, 2, \ldots, M)\). By selecting one of the two values of the decision variable \(d\), a choice of a corresponding transition probability matrix \(P(ij)\) is also made, associated with each of which is a matrix of rewards \(r_{ij}^d\) - the immediate returns arising from a movement from the \(i^{th}\) to the \(j^{th}\) soil moisture levels. From these two matrices, the expected immediate returns \(r_i^d\) can be obtained for each state-decision pair. The recurrence relation is thus

\[
f_t(i) = \max_d \sum_{j=1}^{M} P_{ij} f_{t-1}(j) + \beta \sum_{d} \sum_{j=1}^{M} r_{ij}^d
\]

where \(\beta\) is appropriate discount factor.

In most cases, a constant policy is optimal for large values of \(N\), and the expected present value can be approximated by

\[
\lim_{N \to \infty} F(N) = (I - \beta P)^{-1} R
\]

which is readily obtained by solving \(M\) simultaneous linear equations.
In a recent paper, Burt (1969) gives a macro-economic policy application of dynamic programming in which the decision variable is $P^f(t)$ the government controlled price of fluid milk. The dynamic aspect of the problem arises from the distributed lag form of the supply equation:

$$q^s(t) = b_0 + b_1 (P^f(t+1) + b_2 P^m(t+1) + b_3 q^s(t+1))$$

where $q^s(t)$ is the total amount of milk supplied and $P^f(t)$ is the price of milk for manufacturing.

The state variables are $P^f(t+1)$ and $q^s(t+1)$; thus the difference equations are the one given above and $S(t) = P^f(t+1)$, where $S(t)$ is a dummy variable. Burt subsequently modifies his model so that a measure of social value is maximized subject to some minimal farmer income constraint.

Burt (87) summarizes some of the other actual and potential applications of dynamic programming in agricultural economics including decisions about farm expansion and the replacement of livestock, machinery and other assets.

In currently developing research areas, there is often much confusion and ambiguity concerning terminology. The dynamic programming literature itself is remarkably free of such ambiguities. There is some apparent confusion, however, in the discussion of related programming approaches. For example, Loftsgard and Heady (93), Day (90) and Day and Tinney (91) use linear programming in a recursive manner. These studies are distinguishable from dynamic programming since they do not utilize Bellman's Principle of Optimality and the process is not optimized over its entire duration by composing the individual stage rewards.
4. Some Fundamental Difficulties in the Application of Dynamic Programming

4.1 Computational Difficulties

Frequently in the above discussion the problem of computational feasibility has been mentioned. A considerable amount of research has been undertaken in order to modify straightforward dynamic programming methods so that increasingly complex problems have become susceptible to solution. A detailed discussion of these methods is not undertaken here; this review has sought to illustrate the basic dynamic programming approach with simple geographical planning and control problems, rather than to provide an exhaustive summary of all aspects of dynamic programming methodology.

Dynamic programming does not a priori determine the method whereby the optimal decision function for any stage is to be attained. This optimum may be derived using the calculus (taking partial derivatives of the criterion function with respect to the decision variable(s)), by a mathematical programming method, or by exhaustive enumeration and comparison of all the possible stage decisions. Where many decisions are possible, the latter alternative may be unwieldly. In such cases optimal search techniques such as Fibonacci search may be employed (6, 28).

The most severe limitation of dynamic programming is imposed by the number of state variables in the process. In most cases, three or four state variables are the most which can be handled computationally by dynamic programming. Bellman (5) suggests that this number will increase to fifteen or twenty as computers become larger and more sophisticated.

Lagrangian multipliers can be utilized to eliminate one of the state variables. Gulbrandson (41) for example has done this in his highway
investment problem.

In the Hitchcock-Koopmans transportation problem with n supply points, the dimension of the state vector can be reduced to n-1 because by assumption,

\[ X_n = \sum_{j=1}^{N} Y_j - \sum_{i=1}^{n-1} X_j. \]

Yet another way to reduce the problem of dimensionality is to increase the grid size i.e. to reduce the range of values which the state variables may assume.

These and other computational refinements are discussed in several of the references in Part A of the bibliography. Of particular interest in this regard are Bellman and Dreyfus (6) and Nemhauser (28).

The "curse of dimensionality" is particularly severe in dynamic geographical problems since the typical situation involves many spatially dependent units each with at least one state variable which change over time. One possible alternative is to redefine the problem so that each spatial unit constitutes a stage while there is a component of the state vector for each time period (41).

4.2 Informational Requirements

Dynamic programming demands a considerable awareness on the part of the researcher of the nature of the process he is attempting to optimize. The dynamics of the process should be specifiable in terms of difference or differential-difference functional equations in which the state of the process at any stage is dependent upon the preceding state, the preceding decision, and perhaps some exogenous (and stochastically predictable)
variables. Moreover the relationships between system states, decisions, and rewards must be well known.

Very few geographers have studied processes in a format which is amenable to optimization within a dynamic programming framework. This review has demonstrated, however, that many geographic processes can be so formulated.

5. **Significance of the Dynamic Programming Approach for Geographic Problems**

Dynamic programming offers both a set of very general computational procedures and a theoretical framework in which the control of dynamic processes can be studied. As a computational procedure, it has certain advantages and disadvantages over other optimization methods. It is not restricted to the optimization of sets of linear equations. It is admirably appropriate for certain kinds of sensitivity analyses since it gives the optimal policy for the entire set of initial states. Computational effort and storage requirements, while very sensitive to the number of state variables are extremely tolerant with respect to the number of stages. Highly constrained problems in general have smaller computational times. Finally, note that the dynamics of the process need not be formulated in terms of mathematically analytic transfer functions. Many of the applications of dynamic programming use data which are in tabular form.

More important perhaps than the computational aspects of the approach are the potential theoretical implications. In order to utilize dynamic programming, the researcher must think of problems in terms of rigorously defined sequential feedback processes. In recent years there has been an increasing amount of discussion about systems, goals and dynamics in geography.
and in science in general. This review has shown that dynamic programming provides a relatively simple framework within which to study the dynamics of certain geographic, goal-oriented systems.
PART A
GENERAL REFERENCES


**PART B**

**TRANSPORTATION**


PART C
REGIONAL AND LOCATION
ALLOCATION PROBLEMS


PART D
NATURAL RESOURCE
MANAGEMENT


PART E
AGRICULTURAL ECONOMICS


PART F

MISCELLANEOUS

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