Geometrization of Heat Conduction in Perturbative Spacetimes

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Geometrization of Heat Conduction in Perturbative Spacetimes

Sameerah Jamal

Abstract We prove that the problem of symmetry determination linked to first-order perturbations of a metric, can be elegantly expressed using geometric conditions. In particular, an important feature of this study is that for any spacetime that contains small perturbations, any equation constructed on such a space will inherit the perturbations. Intrigued by this connection between geometry and perturbations, we take the heat conduction equation and explore how the inherited perturbations affect the geometric symmetry conditions.

Keywords Approximate Symmetries · Lie symmetries · Perturbations.

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1 Introduction

These days, the obtention of Lie symmetries of differential equations has been automated by symbolic algebraic software- readers can find such examples in [1–3]. Most symbolic programs are limited by higher dimensions, and adding a perturbation to the equation, only exacerbates the problem. It is then natural to ask whether one can simplify the computations in some analytical way. With this as our motivation, we establish geometric criteria which facilitate the determination of Lie symmetries of perturbative differential equations in

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general spaces. In a recent paper, we have devised such criteria for regular perturbative Lagrangians [4].

Now, there are several methods for the construction of approximate symmetries of perturbative equations. The overwhelming majority of approximate studies have preferred the method proposed in [5], as it is easy to implement. On the other hand, it is well-known that the approach by [6] is computationally tedious, but is preferred from a perturbation theory perspective, as the dependent variables are first expanded in a perturbation series to some order before being substituted back into the equation. A third method exists [7] which is a good simplification of the previous approach for particular differential systems.

In some of our previous investigations [8–10], we have looked at the influence of geometry and the collineations of a metric, in connection with the exact symmetries of a partial differential equation. Roughly speaking, our objective in the present paper is to take a perturbative space which induces an approximate partial differential equation with the following additional properties: a) the approximate equation we select is that of the heat conduction equation with a flux term, b) to preserve generality, we do not specify the metric. Ultimately we prescribe a set of criteria for the construction of approximate symmetries for the multidimensional heat equation. To the best of our knowledge, the key idea of our study of geometry versus perturbations in a general Riemannian space, has not yet appeared in the literature. This enhances the applicability of our approach.

In the text, we prove the main result of the present paper which states the geometric conditions for the approximate Lie point symmetries of the heat equation in a general perturbative space. These conditions can be extended to the Shrödinger equation which may be considered as a special case of the heat equation. In particular, we want to employ some geometric tools to construct approximate symmetry conditions. Notably, the conditions have been generalized so that they may be applied to the heat or Shrödinger equation for any finite dimension.

Assuming the Einstein summation convention, let $M^n$ be a pseudo-Riemannian manifold of dimension $n$ endowed with a pseudo-Riemannian metric $g_{ij}$ given in local coordinates $x^i = (x^0, \ldots, x^n)$. The variable $t$ can also be used to refer to the coordinate $x^0$. The heat conduction equation with flux constructed on such a metric is denoted by

\[ \Box g_{ij} u - u_t = f(x^i, u), \]  

(1)

where

\[ \Box g_{ij} u = g^{ij} u_{,ij} - \Gamma^i_{jk} u_{,k} \]

with $\Gamma^i_{jk} = g^{ik} \Gamma^k_{jk} (x^k)$ and $\Gamma^i_{jk}$ are the Christoffel symbols, $u_{,i} = \frac{\partial u}{\partial x^i}$ and $u_{,ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Note that if $f_0 = V(x^i)u$, Eq. (1) is the Shrödinger (linear diffusion) equation, the constant $\hbar$ and the imaginary unit $i$ is absorbed in the variables $x^k, t$ respectively [11]. This particular choice of $f_0$ indicates that the system exchanges energy with the environment.
The paper is organized as follows. In Section 2 we recall some basic definitions from the approximate theory of differential equations. In Section 3 we state and prove the construction of generalized geometric conditions that give rise to approximate symmetries. Section 4 gives an example illustrating our approach to the construction of approximate symmetries as presented in Section 3, and in Section 5 we conclude.

2 Basic Definitions and Structures

Firstly, to prove the main theorem of the Section 3, we need to recall some intermediate results. A detailed discussion of the material and theory of this section can be found in [5]. We will consider the approximation in the first-order of precision in the perturbation parameter $\varepsilon$. An approximate equation

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0, \quad z = (z^1, \ldots, z^N)$$

(2)

is approximately invariant with respect to the one-parameter approximate transformation group

$$\varepsilon^i \approx h_i(z, \alpha, \varepsilon) \equiv h^0_i(z, \alpha) + \varepsilon h^1_i(z, \alpha), \quad i = 1, \ldots, N,$$

("$\alpha, \varepsilon$" are two infinitesimal parameters) with the generator

$$X = X_0 + \varepsilon X_1 + O(\varepsilon^2),$$

(3)

if and only if,

$$\left( X_0 F_0(z) + \varepsilon \left( X_1 F_0(z) + X_0 F_1(z) \right) \right)_{F_0(z)=0} = O(\varepsilon)$$

(4)

The determining equation (4) can be written as follows:

$$X_0 F_0(z) = \lambda(z) F_0(z)$$

(5)

$$X_1 F_0(z) + X_0 F_1(z) = \lambda(z) F_1(z)$$

(6)

The factor $\lambda(z)$ is determined by (5) and then substituted into (6), where the latter equation holds for $F_0(z) = 0$. Alternatively, one may evaluate (5) to obtain the exact symmetries, then find an auxiliary function $H$ by virtue of (5), (6) and (2), that is

$$H = \frac{1}{\varepsilon} X_0 \left( F_0(z) + \varepsilon F_1(z) \right)_{F_0(z)=0 + \varepsilon F_1(z)=0}.$$

(7)

Thereafter $X_1$ is calculated by solving the determining equation for deformations

$$X_1 F_0(z)|_{F_0(z)=0} + H = 0.$$

(8)
3 The Heat Conduction Equation with a Flux

As an introduction to perturbative metrics, consider a line element decomposed into a sum of an exact and an approximate metric, viz.

\[ g_{ij} = \delta_{ij} + \varepsilon \gamma_{ij} + O(\varepsilon^2). \]

A perturbed heat equation on this space may be constructed to be of the form (2), where the exact or unperturbed heat equation with flux is

\[ F_0 = \Box \delta_{ij} u - u_t - f_0(x^i, u), \quad (9) \]

and the approximate part is

\[ F_1 = \Box \gamma_{ij} u - u_t - f_1(x^i, u). \quad (10) \]

Next we seek to establish general symmetry relations between the Lie symmetries of a second-order partial differential equation of the form (2) with (9) and (10)

Suppose that the symmetry generator is of the form

\[ X = \xi^i (x^i, u) \partial_{x^i} + \eta (x^i, u) \partial_u, \quad (11) \]

and since (9) and (10) are second-order in derivatives, we require from the standard Lie symmetry approach, the second prolongation of the generator (11) viz.

\[ X^{[2]} = X^{[1]} + \eta^{ij} \partial_{u_{,ij}}. \]

\[ X^{[1]} \]

is the first prolongation given by the expression

\[ X^{[1]} = X + \eta^i \partial_{u_i}, \]

where \( \eta^i = D_i \eta - u_j D_i \xi^j = \eta_i + u_i \eta_u - \xi^j u_{,j} - \xi^k u_{,j} u_{,k}, \) if \( D_i = \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_{,ij}} + \ldots \) is the total derivative, and

\[ \eta^{ij} = \eta_{,ij} + (\eta_{uu} u_{,j} + \eta_{uj} u_{,i}) - \xi^{k}_{ij} u_{,k} + \eta_{uu} u_{,ij} - (\xi^k_{,u} u_{,j} + \xi^k_{,j} u_{,i}) + (u_{,ij} u_{,k} + u_{,ik} u_{,j}) \eta^{k}_{u}, \]

The exact Lie point symmetries of (9) are found in the usual way, that is, they are given by condition (5)

\[ \delta^{ij} \eta_{0}^{j} - \eta_{0}^{j} \Gamma^{i} + \xi_{0}^{k} \left( (\delta^{ij})_{,i} u_{,j} - (\Gamma^{i})_{,i} u_{,j} \right) - X_{0} f_{0}(x^i, u) = \lambda \left( \Box \delta_{ij} u - u_t - f_0(x^i, u) \right), \quad (12) \]

where \( \Gamma^{i} = 1. \) Eq. (12) solves to obtain the form of the Lie point symmetry of (9) (see Ibraghimov [12], Bluman [13] or Paliathanasis et. al. [14]), viz. the symmetry coefficients are

\[ \xi_{0}^i = \xi_{0}^i(t), \quad \xi_{0}^k = \xi_{0}^k(x^i), \quad \eta_{0} = a_0(x^i)u + b_0(x^i). \quad (13) \]
Subsequently, condition (6) reads as
\[
\delta^{ij} \eta^{ij} - \eta^{ij}_0 \Gamma^i + \xi^i_1 \left( (\delta^{ij})_{,i} u_{,ij} - (\Gamma^i)_{,i} u_{,i} \right) - X_1 f_0 (x^i, u) \\
+ \gamma^{ij} \eta^{ij}_0 - \eta^{ij}_0 \Gamma^i + \xi^i_0 \left( (\gamma^{ij})_{,i} u_{,ij} - (\Gamma^i)_{,i} u_{,i} \right) - X_0 f_1 (x^i, u) \\
= \lambda \left( \Box_{\gamma^{ij}} u - u_t - f_1 (x^i, u) \right)
\]
Thus, some expressions may need to be appropriately reindexed and we may express the prolonged terms as \( \eta^i_0 = a_0 u + b_0, a_0 u + a_0 u, i - \xi^{i,j}_0 u, j \), and
\[
\eta^{ij}_0 = a_{0,ij} u + b_{0,ij} + a_{0,ij} u, j + a_{0,ij} u, i - \xi^{0,ij}_0 u, k + a_{0,ij} - \left( \xi^{k,ij}_0 u, jk + \xi^{h,ij}_0 u, ik \right).
\]

It remains to expand and evaluate the condition (14) \(^1\) by the substitution of the prolongations \( \eta^i, \eta^{ij}, \eta^{ij}_0 \) and \( \eta^{ij}_0 \).

The above derivation leads to the following theorem.

**Theorem 1** The first-order approximate symmetry generators of (2) by virtue of (9) and (10) are given by
\[
\xi^i_1 = \xi^i_1 (x^i), \quad \eta^i = a_1 (x^i) u + b_1 (x^i),
\]
and the determining conditions
\[
\gamma^{ij} (a_{0,ij} u + b_{0,ij}) - (a_{0,i} u + b_{0,i}) \Gamma^i - \Gamma^i \eta^{ij} + \eta^{i1} \delta^{ij} + \lambda f_1 \\
\xi^i_1 f_{0,i} + \xi^i_0 f_{0,i} + \xi^i_1 f_{1,i} + (a_{0,i} u + b_{0}) f_{1,u},
\]
\[
\delta^{ij} \xi^{k,1}_{1,ij} - 2 \delta^{ik} \eta^{ij}_{1,i} + \eta^{i1}_{1,ij} - \xi^{k,1}_{1,ij} \eta^{ij}_{1,ij} + \xi^i \xi^j + \xi^j \xi^i - 2 \gamma^{ij} \eta^{ij}_{0,ij} \\
= \xi^i f_{0,i} + \xi^i_0 f_{0,i} + \xi^j_0 f_{1,j} + (a_{0,i} u + b_{0}) f_{1,u},
\]
\[
\xi^i_1 \left( (\delta^{ij})_{,i} \right) + \delta^{ij} \left( (\eta^{ij})_{,i} - \xi^{k,1}_{1,ij} \right) + \xi^i \left( (\gamma^{ij})_{,i} \right) - \xi^i \left( \xi^{k,ij}_0 - \xi^{k,ij}_0 \right) = \lambda \gamma^{ij}.
\]

This theorem is quite general since the metric is not specified, which makes the above result very useful. Note that, for an arbitrary metric \( g_{ij} \), using the formula
\[
g^i_j = -\Gamma^i_{ks} g^{ks} - \Gamma^j_{ks} g^{ks},
\]
implies that Eq. (18) is, after some reindexing, equivalent to
\[
\nabla^i \xi^i_0 + \nabla^1 \xi^i_0 + \nabla^i \xi^i_1 + \nabla^1 \xi^i_1 = (a_0 - \lambda) \gamma^{ij} + \eta^{i1} \delta^{ij},
\]
and by definition of the geometric derivative, that is, the Lie derivative operator, we may thus rewrite Eq. (19) as
\[
L_{\xi^i_0} \gamma^{ij} + L_{\xi^i_1} \delta^{ij} = (a_0 - \lambda) \gamma^{ij} + \eta^{i1} \delta^{ij}.
\]

We point out here that, in the absence of any perturbation, there would be far more geometric simplifications. Nevertheless, one can further the above

\(^1\) Details of the full system can be found in the Appendix.
analysis under some physical considerations involving the exact Lie point symmetries of (9). Namely, applying the results contained in [14], we have that the exact symmetry coefficient \( \xi_0^\ell \) is a conformal Killing vector of the metric \( \delta_{ij} \) with conformal factor \( \xi_0^r = a_0(x') - \lambda(x') \), and \( \xi_0^b = T(t)Y^k(x') \) where \( Y^i \) is a Homothetic vector with conformal factor \( \psi = \text{constant} \). Hence, one may simplify the determining conditions, according to two cases, to a certain extent.

**Case 1:** \( Y^i \) is a nongradient Homothetic or Killing vector, then the exact Lie symmetry vector is given by the expression

\[
\begin{align*}
X_0 &= (2c_2\psi t + c_1)\partial_t + c_2Y^i\partial_i + (a_0(t)u + b_0(t,x))\partial_u, \\
\text{where, } c_1, c_2 &\text{ are constants, } a_0(t), b_0(x'), f_0(x', u) \text{ satisfies the constraint equation} \nonumber
\end{align*}
\]

\[\text{In this scenario, Eq. (20) is recast as}
\]

\[
L_{\xi_0^i} \partial_i \xi_0^j + L_{\xi_0^j} \partial_j \xi_0^i = 2c_2\psi \gamma_{ij} + a_1 \delta_{ij},
\]

and Eq. (16) and Eq. (17) simplify considerably too, but are less elegant. They are,

\[
\begin{align*}
H(a_0)u + H(b_0) + K(a_1)u + K(b_1) + 2c_2\psi f_1 &= \xi_0^i f_{0,i} + (a_0 u + b_0) f_1, \\
\delta_{ij} &\text{ satisfies the constraint equation}
\end{align*}
\]

\[
\begin{align*}
\text{Similarly, if } Y^i &= S^j \text{ is a gradient Homothetic or Killing vector, then the exact Lie symmetry vector is given by the expression} \\
X_0 &= \left(2\psi \int T \, dt + c_1\right) \partial_t + TS^j \partial_i + \left(\left(-\frac{1}{2}T,TS^j + F(t)\right) u + b_0(t,x)\right) \partial_u, \\
\text{where } F(t), T(t), b_0(x'), f_0(x', u) \text{ satisfies the constraint equation}
\end{align*}
\]

\[
\begin{align*}
\text{In this situation, Eq. (20) is recast as}
\end{align*}
\]

\[
L_{\xi_0^i} \partial_i \xi_0^j + L_{\xi_0^j} \partial_j \xi_0^i = \left(2\psi \int T \, dt + c_1\right) \gamma_{ij} + a_1 \delta_{ij},
\]

and Eq. (16) and Eq. (17) are

\[
\begin{align*}
H(a_0)u + H(b_0) + K(a_1)u + K(b_1) + (2\psi \int T \, dt) f_1 &= \xi_0^i f_{0,i} + \left(2\psi \int T \, dt + c_1\right) f_{1,i} + \left(2\psi \int T \, dt + c_1\right) \partial_t \xi_0^j, \\
&\text{ and Eq. (16) and Eq. (17) are}
\end{align*}
\]

\[
\begin{align*}
\text{In this situation, Eq. (20) is recast as}
\end{align*}
\]
respectively.

The explicit symmetry conditions may be deduced once we select a specific space, this will significantly reduce the complexity of the geometric conditions.

Finally, we remark that if one wanted the auxiliary function \( H \), it is, in a general form, given by the expression

\[
H = \varepsilon^{-1} \left[ \xi_0^i \left( \delta^i_j u, i, j - \Gamma^i_j u, i, j - f_{0, i} \right) - (a_0 u + b_0) f_{0, u} - \Gamma^i_j (a_{0, i} u + b_{0, i} + a_{0} u, i) \right. \\
- \xi^i_k u, k + \xi_0^i u, j \right] + \delta^i_j \left( a_{0, i} u + b_{0, i} + a_{0} u, j + a_{0, j} u, i - \xi^k_{0, i} u, k + a_{0} u, i, j \right) \\
- \left( \xi^k_{i, j} u, j + \xi_0^k u, j \right) + \varepsilon \left( \xi_0^k \left( \Gamma^i_j u, i - f_{1, i} - (a_0 u + b_0) f_{1, u} \right) + \xi^i_j \left( a_{0, i} u + b_{0, i} + a_{0} u, j + a_{0, j} u, i \right) \right) \\
- \left. \xi^i_k u, k + a_{0} u, i, j - \left( \xi^k_{0, i} u, j + \xi_0^k u, j \right) \right] \\
\mod F_0 + \varepsilon F_1 = 0.
\]

Next we illustrate the above method by way of application to a particular example.

### 4 The Perturbative Heat Equation in Plane Symmetric Static Space

Amongst the many important perturbed geometries [15,16] etc, we choose to consider the perturbed plane symmetric static metric [17]

\[
ds^2 = e^{2x/a} dt^2 - dx^2 - e^{2x/a} \left( dy^2 + dz^2 \right) + \varepsilon \left( \frac{2t}{b} \left( e^{2x/a} dt^2 - e^{2x/a} \left( dy^2 + dz^2 \right) \right) \right),
\]

where \( a, b \) are constants. This metric was proposed to find a resolution to the problem of defining energy in gravitational wave spacetimes. A static spacetime (plane symmetric static space) was taken and then time dependently perturbed to make it slightly non-static, since gravitational waves must be given by non-static metrics.

The perturbative heat equation on this geometry, by (9) and (10), is

\[
u_{tt} = \frac{1}{a^2} \left( a^2 (u, y_0 + u, z_0) (1 + \varepsilon) + 2 \varepsilon u_{xx} \right) + \left( a^2 u_{xx} + au, x + 2xu, x \right) \varepsilon^2 z - \varepsilon u_{tt} a^2 \right) \\
+ \left( a^2 (u, y_0 + u, z_0) \right) (1 + \varepsilon) - f_0 (x', u) - \epsilon f_1 (x', u).
\]

Suppose we apply Theorem 1 for the homogeneous heat equation. We find for (18) and (15), the exact symmetries are (in the interest of brevity, detailed calculations are omitted), \( c_1, \ldots, c_5 \) are constants

\[
\xi_0^i = c_3, \ \xi_0^x = 0, \ \xi_0^y = c_4 z + c_5, \ \xi_0^z = -c_4 y + c_1, \ \eta_0 = c_2 u + b_0,
\]
which we can list as

\[ Z_0^1 = \partial_t, \quad Z_0^2 = \partial_y, \quad Z_0^3 = \partial_z, \quad Z_0^4 = z\partial_y - y\partial_z, \]
\[ Z_0^5 = u\partial_u, \quad Z_0^6 = b_0(t, x, y, z)\partial_u. \]

The exact symmetries \( Z_0^1 - Z_0^4 \) are the Killing vectors connected to (28). This is of course expected - the exact Lie point symmetries of the heat equation constructed on a Riemannian space, are generated from the Killing or Homothetic vectors of the space.

On the other hand, the first-order approximate Lie point symmetries (\( h_1, \ldots, h_5 \) are constants) are given by the coefficients:

\[ \xi_1 = -h_3, \xi_1^y = 0, \xi_1^z = h_2z + h_5, \xi_1^t = -h_2y + h_1, \eta_1 = h_4u + b_1, \]

or listed as the vectors

\[ Z_1^1 = \varepsilon Z_0^1, \quad Z_1^2 = \varepsilon Z_0^2, \quad Z_1^3 = \varepsilon Z_0^3, \quad Z_1^4 = \varepsilon Z_0^4, \quad Z_1^5 = \varepsilon Z_0^5, \quad Z_1^6 = \varepsilon b_1(t, x, y, z)\partial_u. \]

Additionally, the auxiliary function (27) in this case is given by

\[ H = \frac{1}{a^2}e^{2x/a} \left( a^2 b_{0,xx} + ab_{0,x} + 2xb_{0,x} \right). \]

At this juncture, there are two interrelated questions that arise. The first is whether or not an approximate space admits some sort of approximate conformal Killing algebra, and the second question is if the approximate Lie symmetries of the heat equation are generated from the approximate Killing/Homothetic symmetries of the approximate metric. The second question may only be answered if we have a concrete answer to the first one. To our knowledge, in this direction of research, there is the work [17] and references therein. In particular it was inferred (not proved) that the approximate symmetries can be interpreted to give the extent of energy non-conservation, and hence the energy content, in the gravitational wave-like spacetime. However, more general insight is needed on the issue of approximate conformal Killing vectors and their physical interpretation.

To this end, we attempt to offer a possibility to the second question. If indeed the notion of an approximate Killing/Homothetic symmetry vector is concretized and more important physical implications are established in the future, then and only then, can we offer the conjecture that the approximate Lie symmetries of the heat equation are generated from such approximate Killing/Homothetic symmetries of the approximate metric, using the above example as possible evidence. Intuitively, we believe that the presence of a perturbation maintains the connection between geometry of the space and the physics in the space, at least from a geometric study’s point of view via approximate symmetries.
5 Concluding Remarks

Usually it is not difficult to find symmetries, exact or approximate, where the latter merely involves more tedious computations. However working in higher dimensions, and with the added complexity of the perturbative metric space, often increases the difficulty in obtaining rigorous results. It is therefore vital to have geometric criteria which facilitate the determination of Lie symmetries of perturbative differential equations in general spaces of any finite dimension. We showed how the theory of approximate symmetries can be transferred to a differential equation constructed on an arbitrary perturbative metric space, where the problem is transformable into geometric conditions via an appropriate use of differential geometry. To exemplify our theory, we showcased the heat equation on a plane symmetric static metric.

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6 Appendix

Condition (14) may be expressed as

\[
\delta^{ij} \left( \eta_{ij} + (\eta_{1,i} u_{,j} + \eta_{1,j} u_{,i}) - \xi_{1,i} u_{,k} + \eta_{1,uu} u_{,i} u_{,j} - (\xi_{1,u} u_{,i} u_{,j}
+ \xi_{1,k} u_{,i} u_{,j}) + \eta_{1,ui} u_{,j} + \eta_{1,uj} u_{,i} - (\xi_{1,ui} u_{,j} + \xi_{1,uj} u_{,i})
+ u_{,jk} u_{,i} + u_{,ik} u_{,j} \right) \xi_{1,uu} + \xi_{1,ui} u_{,j} + \xi_{1,uj} u_{,i})
- (\eta_{1,i} + u_{,i} \eta_{1,u} - \xi_{1,i} u_{,j} - \xi_{1,ui} u_{,j}) \gamma_{ij} - (\xi_{1,i} + \xi_{1,j}) \gamma_{ij} + \xi_{1,k} u_{,ik}
- (f_{0,i} \xi_{0,i} + f_{0,u} \eta_{1})
+ \gamma^{ij} \left( a_{0,i} u + b_{0,i} + a_{0,j} u_{,j} - \xi_{0,i} u_{,k} + a_{0,j} u_{,j} - (\xi_{0,ui} u_{,j} + \xi_{0,uj} u_{,i})
- (a_{0,i} u + b_{0,i} + a_{0,j} u_{,j} - \xi_{0,i} u_{,k} + a_{0,j} u_{,j} - (\xi_{0,ui} u_{,j} + \xi_{0,uj} u_{,i})
- (f_{1,i} \xi_{0,i} + f_{1,u} a_{0,u} + f_{1,u} b_{0}) = \lambda \left( \gamma^{ij} u_{,ij} - \gamma^{i} u_{,i} - f_{1}(x^i, u) \right)\right),
\]

To find the determining equations (18), we consider the coefficients of 1, u_{,i} and u_{,ij}, respectively. Finally, the solution of \eta_{1,u} = 0 and \xi_{1,u} = 0 gives the result (15).

References