Assessment of Reliability Based Design of Stable Slopes

Journal:	<i>Canadian Geotechnical Journal</i>
Manuscript ID	cgj-2018-0149.R1
Manuscript Type:	Article
Date Submitted by the Author:	29-May-2018
Complete List of Authors:	De Koker, Nico; University of Stellenbosch, Faculty of Engineering, Civil Engineering; Day, Peter; Stellenbosch University, Department of Civil Engineering; Jones & Wagner, Geotechnical Zwiers, Andrew; University of Stellenbosch, Faculty of Engineering, Civil Engineering; Stantec
Keyword:	Design methods, Failure, Reliability, Risk analysis, Slope stability
Is the invited manuscript for consideration in a Special Issue? :	Not applicable (regular submission)

SCHOLARONE™
Manuscripts

1 Assessment of Reliability Based Design of Stable Slopes

2 Nico de Koker^{1,a,†}, Peter Day^{1,b}, and Andrew Zwiers^{1,c}

3 ¹*Department of Civil Engineering, University of Stellenbosch, South Africa*

4 ^a*ndekoker@gmail.com, +27-83-438-3234*

5 ^b*day@jaws.co.za*

6 ^c*andy.zwiers@stantec.com*

7 [†]*corresponding author*

8 Abstract

9 The viability of applying reliability based design in the limit equilibrium evaluation
10 of slope stability is assessed. Two model slopes are designed according to the limit
11 states specification of EN 1997:2004 and reliability assessed using full Monte Carlo
12 calculation with 10^6 trials. Comparison of these results with reliability estimates ob-
13 tained via a spectrum of different analysis techniques, broadly grouped into first order
14 reliability methods (FORM) and performance function moment estimation methods,
15 indicate that FORM-based techniques are superior, with a first order response surface
16 method found to provide the best combination of accuracy, stability, and speed of
17 convergence. In this way, accurate reliability indices for non-closed form problems of
18 the nature considered here can be obtained within 20-30 function calls, depending on
19 the number of free parameters and required precision.

20 Keywords: Design methods, Failure, Reliability, Risk analysis, Slope stability

21 1 Introduction

22 Reliability based geotechnical and structural design allows the practitioner to target
23 the relevant level of risk directly. The approach differs from the more traditional
24 design frameworks by considering the probability of failure directly, rather than via
25 proxy indicators such as partial factors of safety. However, it requires notable addi-
26 tional computational effort, as well as more stringent requirements for data quality
27 (e.g. Phoon 2016). As a result its design office use has remained mostly limited to
28 supplementary inquiry into unusual problems.

29 Advances in computational power and data analysis techniques now permit its appli-
30 cation as a primary means of design (e.g. Phoon 2008; Jonkman and Schweckendiek
31 2015). While this is especially the case for problems that are examinable via closed-
32 form design equations (De Koker and Day 2018), its true utility to advanced practi-
33 tioners would lie in the analysis of more complex structural and geotechnical problems,
34 which require rigorous iterative numerical solution techniques. Computation times for
35 such calculations would typically be of a scale that the modeller can observe, i.e. a
36 couple of seconds at the very least. Very large numbers of computations are therefore
37 not routinely practicable, so that the method chosen to perform the reliability analysis
38 should preferably provide accurate results from a relatively small number of capacity
39 evaluations.

40 An efficient structural design should reflect the tradeoff between the cost of construc-
41 tion and maintenance, and the risk associated with its possible failure (consequences
42 $\times$ failure probability). By considering this tradeoff JCSS (2008) and ISO 2394:2015
43 have compiled guidelines for the range of reliability levels that designs should aim

44 to achieve (Retief et al. 2016). Indeed, most modern design codes are calibrated in
45 some form with these target levels in mind (Ellingwood et al. 1980; Gulvanessian 2010;
46 Retief and Dunaiski 2010; Fenton et al. 2016). The nominal failure probabilities as-
47 sociated with these target reliabilities imply at least $10^5 - 10^6$ Monte Carlo trials to
48 obtain reliability estimates with reasonable precision.

49 Determining the stability of slopes is a common problem in geotechnical design. Anal-
50 yses are generally carried out via limit-equilibrium methods, such as those of Bishop,
51 Spencer, or Morgenstern-Price (e.g. Duncan, Wright, and Brandon 2014). Execution
52 times for these calculations are small enough for full Monte Carlo calculations to be
53 accessible, even if not practical on a routine basis.

54 Previous studies that considered the reliability of slope design (Kirsten 1983; Wong
55 1985; Christian, Ladd, and Baecher 1994; Low et al. 1998; Duncan 2000; Husein
56 Malkawi, Hassan, and Abdulla 2000; Low 2008; Gibson 2011; Li et al. 2016) either
57 consider slope failure probabilistically in the context of the relatively simple moment-
58 based reliability estimates, or focus on the technical details of implementing some
59 variation on the more advanced first order reliability method. Some of these methods
60 are complex and lack the benefits of intuitive understanding and transparency needed
61 for routine design office use while others are simplified to the extent that they yield
62 inaccurate results.

63 The various methods for estimating reliability of a model structure all seek to reduce
64 the computational burden of a brute force Monte Carlo calculation by making more
65 direct use of the underlying statistics of the problem (Ang and Tang 1984; Harr 1987;
66 Baecher and Christian 2003). As a result, these methods range significantly in both

67 the severity of simplifying assumptions and computational cost.

68 The primary objective of this paper is to assess the extent to which the various reliability methods can be practically and routinely used in a geotechnical design office for
69 the design of slopes. This is done by comparing reliability estimates from a spectrum
70 of methods to accurate values determined via full Monte Carlo calculations, in terms
71 of computational effort and stability. Such an analysis is not currently available, and
72 is necessary if a reliability-based geotechnical design standard is to be compiled.

74 An overview of the theoretical basis for reliability analysis and limit equilibrium analysis of slope stability is presented in Section 2, after which reliability analysis of two
75 model slope problems is examined using a variety of reliability analysis techniques
76 (Section 3 and 4). Insights gained from these analyses are discussed in Section 5 in
77 the context of both limit states and reliability based design.

79 **2 Theoretical Development**

80 Any design framework aiming to balance cost and risk seeks to place an optimally
81 sufficient margin between the realised design and the most likely conditions that would
82 bring about failure. A first requirement for this process is to locate the boundary of
83 the set of failure conditions. A common means of defining this boundary is via the
84 performance function g , for which negative values are taken to denote failure. The
85 boundary of the failure region is then given by the condition

$$86 \quad g(\mathbf{x}) = 0, \quad (1)$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]$ is a coordinate vector of physical parameters treated as random variables with specified distributions. Specifically, x_i is characterised by mean μ_i and variance σ_i^2 , and has a distribution that is generally not normal.

2.1 Performance Function for Slope Stability

Limit equilibrium slope analysis methods express stability via the factor of safety $\mathcal{F}$. From the condition for stability $\mathcal{F} \geq 1$, the performance function can then be expressed as

$$g = \mathcal{F} - 1. \quad (2)$$

The approach aims to determine a lower bound on the factor of safety by finding the critical slip surface, i.e. the surface that yields the smallest factor of safety $\mathcal{F}$. The mobilised portion of the slope is discretised into a set of vertical slices that terminate at a continuous slip surface which, for the purpose of this manuscript, is assumed to be of circular geometry (Figure 1), and then evaluates conditions for equilibrium.

The nature of this equilibrium characterisation constrains the choice of formulation. For example, EN 1997:2004 requires the method to satisfy vertical, horizontal, and moment equilibrium of slices (clause 11.5.1). This requirement disqualifies the simpler formulations, including that of Bishop. With this condition in mind, we will use the method of Spencer (Spencer 1967) in this manuscript.

Spencer's formulation introduces the angle λ as an unknown parameter in addition to $\mathcal{F}$ (Figure 1). These two unknowns are determined by solving the equations for

107 resultant force and moment equilibrium (Duncan, Wright, and Brandon 2014),

$$108 \quad \sum S = 0, \quad (3)$$

$$109 \quad \sum S (\Delta x \sin \lambda - \Delta y \cos \lambda) = 0, \quad (4)$$

111 with

$$112 \quad S = \frac{F_T \sin \alpha - c' \Delta \ell / \mathcal{F} - [F_T \cos \alpha - w \Delta \ell] \tan \phi' / \mathcal{F}}{\cos(\alpha - \lambda) + \sin(\alpha - \lambda) \tan \phi' / \mathcal{F}}, \quad (5)$$

113 where $F_T = W + F_V$, Δx and Δy are the horizontal and vertical moment arms at the
114 base of the slice, and w is the pore water pressure. The soil is characterised by the
115 effective friction angle ϕ' , cohesion c' , and density γ .

116 2.2 Statistical Framework to Quantify Reliability

117 Treatment of the reliability problem is most convenient after transforming the param-
118 eter space coordinate vector $\mathbf{x}$ into uncorrelated standard normal space $\mathbf{u}$. This is
119 done by first determining the univariate standard normal value z_i corresponding to
120 each component x_i using the Rosenblatt transformation equations (Table A1) yielding
121 the vector $\mathbf{z}$, and then accounting for correlation via

$$122 \quad \mathbf{u} = \mathbf{T}^{-1} \mathbf{z}(\mathbf{x}), \quad (6)$$

123 where $\mathbf{T}$ is the upper triangular Cholesky decomposition of the correlation matrix
124 $\mathbf{R} = \mathbf{T}^T \mathbf{T}$.

125 After this transformation the contours of the multi-variate probability density function
126 in $\mathbf{x}$ -space become a set of concentric hyper-spheres centred about the origin in $\mathbf{u}$ -space
127 (Figure 2 illustrates this in two dimensions). The failure condition $g(\mathbf{u}) = 0$ forms
128 a hyper-surface that will be tangent and orthogonal to a hypersphere at its point of

129 closest approach to the origin. This tangent point represents the most likely point of
 130 failure (the “design point” $\mathbf{u}_d$), and its distance from the origin is given by (Rackwitz
 131 and Fiessler 1978)

$$132 \quad \beta_{\text{FORM}} = \min \sqrt{\mathbf{u}^T \mathbf{u}}, \text{ such that } g(\mathbf{u}) = 0. \quad (7)$$

133 The primary objective of the reliability problem is to obtain this probability of failure.
 134 Integration of the n -dimensional probability density function over the volume where
 135 the failure condition $g < 0$ is met represents this probability of failure. It is convenient
 136 to express reliability via the reliability index (Hasofer and Lind 1974),

$$137 \quad \beta = \Phi[p_f], \quad (8)$$

138 where Φ is the standard normal cumulative distribution function.

139 To the extent that the performance function boundary $g = 0$ is a linear hyperplane,
 140 we have $\beta = \beta_{\text{FORM}}$, which is the basic assumption of the first order reliability method
 141 (FORM) discussed below.

142 **2.3 Partial Factor Based Limit States Design**

143 The partial factor framework speeds the design process by representing the design
 144 point approximately instead of attempting to find it explicitly. First, characteristic
 145 values are determined as conservative bounds on the mean parameter values,

$$146 \quad x_k = \bar{x} (1 + \eta \delta), \quad (9)$$

147 where δ is the coefficient of variation, and η depends on the confidence level p_k of the
 148 interval on the mean for which x_k represents the bounding value. (Table A1 includes

equations by which η can be determined for the various distributions used in this manuscript).

Next, design values are obtained by scaling the characteristic values by the partial factors specified in the relevant design code (Table 1). These partial factors are typically calibrated to provide designs that broadly satisfy the target reliability values of JCSS (2008) and ISO 2394:2015.

2.4 Methods for Reliability Determination

In principle, the probability of failure can be most accurately evaluated by means of a Monte Carlo counting procedure in which instances of $\mathbf{x}$ are randomly sampled from its multivariate distribution a large number of times N and the number of failure instances N_f counted. The probability p_f failure then simply follows as

$$p_f = N_f/N. \quad (10)$$

As each iteration involves a simple Bernoulli trial, the failure probability has variance

$$V[p_f] = \frac{p_f(1 - p_f)}{N}. \quad (11)$$

A consequence of this relation is that for small values of p_f , large N values are required to obtain acceptable accuracy. For example for $p_f = 1/10^4$, obtaining $V[p_f] \approx p_f/10$ requires 10^6 trials.

A number of methods attempt to circumvent the large number of function evaluations required by the Monte Carlo method. These methods draw on two primary sets of assumptions. The first idealises the shape of the failure region boundary ($g = 0$) to

169 be linear, while the second idealises the shape of the probability density function of
170 the performance function and/or assumes the underlying variables to be normal.

171 The linear failure boundary used in the first assumption reduces the problem to a
172 one dimensional description in terms of the separation between the most likely overall
173 conditions and the most likely failure conditions, represented via a one-dimensional
174 marginal probability distribution. The effect of this assumption is generally small,
175 because the approximation takes effect away from the most likely point of failure, so
176 that parts of the failure boundary where deviations from linearity become large are
177 at small probability densities.

178 Because the performance function is not generally normally distributed, the second
179 assumption can only provide reasonable results at low reliability ($\beta \lesssim 1.5$), where
180 the probability integral is dominated by the area of failure closest to the mean and
181 the shape of the bounding function is of secondary importance. However, its effect
182 becomes increasingly severe as p_f decreases and the failure region retracts into the
183 distribution tail, where the shape of the probability density function becomes critical.
184 FORM does not require any assumptions regarding the nature of parameter distribu-
185 tions. The method only assumes the failure boundary to be linear in the region of
186 the design point, an assumption which is generally robust at high reliability levels, as
187 deviations from linearity are limited to regions of extremely low probability density.

188 FORM evaluates β directly using Equation 7. By contrast, Monte Carlo finds p_f
189 first, with β following via Equation 8. To find the design point, FORM approaches
190 implement a variety of constrained optimisation schemes. One common approach, to
191 which we will refer as NR-FORM, uses a multi-dimensional Newton-Raphson formula-

tion consisting of a series of root-finding and gradient evaluation iterations (Figure 3; Hasofer and Lind 1974; Rackwitz and Fiessler 1978). An alternative approach, collectively known as response-surface methods (e.g. Wong 1985), replaces the performance function with a surrogate, analytical parameterisation of g (the “response surface”). The design point is then found by using the iteratively refined response surface. We will refer to these methods as RS-FORM.

The NR-FORM algorithm is very efficient for problems where the local root and gradient evaluation can be analytically expressed, rather than numerically evaluated. The RS-FORM approaches seek to harness this strength for problems where the performance function is expensive to compute, by substituting it with a “surrogate” closed form parameterisation g^* . In this manuscript we consider the case of first and second order polynomial surrogate functions (RS1 and RS2), that is

$$g_1^* = a_0 + \sum_i^n a_i x_i \quad (12)$$

$$g_2^* = b_0 + \sum_i^n b_{i1} x_i + b_{i2} x_i^2. \quad (13)$$

where the $n + 1$ coefficients a_j and b_j are evaluated by determining g at $n + 1$ points on a constellation about an iteratively refined best estimate of the design point $\mathbf{u}_d^{[k]}$ (Bucher and Bourgund 1990, see Figure 3)

$$\mathbf{u}_j^{[k+1]} = \left[u_{d1}^{[k]} \pm \zeta, u_{d2}^{[k]} \pm \zeta, \dots, u_{dn}^{[k]} \pm \zeta \right], \quad (14)$$

where ζ is a pre-selected sampling factor with values around 1.0 – 2.0. The value of ζ should be carefully chosen, especially in the case of second (or higher) order surrogate functions. If the candidate updated estimate to the design point $\mathbf{u}_d^{[k+1]}$ does not fall within the domain covered by the constellation $\mathbf{u}^{[k+1]}$, spurious extrapolation outside

the region can result in divergence; too large a constellation could be associated with parameter values for which the performance function itself is poorly defined, such as high friction angle or negative cohesion values.

Points where g are evaluated are therefore chosen using an intuitive scheme that can be readily implemented in a design office. While this method is sufficient for many engineering problems, more efficient ways of determining the sampling points have been developed, and would be useful when evaluation of g becomes extremely expensive to compute (e.g. Sudret 2015).

In contrast to the FORM class of methods, which require several iterations of performance function evaluations to reach converged estimates of the design point, the more approximate moment-based methods seek to estimate reliability directly. This is done by simplifying the problem description to a single variable ($\mathcal{F}$) which depends on the underlying stochastic parameters via a model relation (eg. Equation 5), and assigning a new distribution for $\mathcal{F}$ (or g). Of course, characterising the distribution still requires several evaluations of g , depending on the approach taken, which include the first order second moment equation (FOSM) and the point estimate method (PEM) (Ang and Tang 1984; Baecher and Christian 2003).

A common choice in these moment based methods is to simply assume $\mathcal{F}$ to be normally distributed, resulting in the expression (e.g. Christian, Ladd, and Baecher 1994)

$$\beta_N = \frac{E(\mathcal{F}) - 1}{\sqrt{V(\mathcal{F})}} = \frac{E(g)}{\sqrt{V(g)}}, \quad (15)$$

where $E()$ denotes the mean and $V()$ the variance of the argument. Equation 15 is only valid for g normally distributed, and is not suitable as a general definition of β .

3 Investigation Methodology

3.1 Problems Considered

To compare the various reliability analysis methods discussed in Section 2.4, we will analyse two slope problems. The first is the model problem discussed by Lämsivaara and Poutanen (2013), with a drained homogeneous soil. The second is a variation of this model slope, with a phreatic surface at depth D below the plateau, seeping to the slope toe.

Parameters for the problem are summarised in Figure 1 and Table 1. Statistical distributions and variation coefficients for soil parameters and the surcharge load are assigned based on the work of Phoon and Kulhawy (1999) and Retief and Dunaiski (2010).

3.2 Procedure of Analysis

The starting point for each analysis is a limit states design using the partial factors prescribed by NA to BS EN 1997:2004, obtained by determining the slope width B for which $\mathcal{F} = 1$. This optimal design is then analysed using full Monte Carlo, NR-FORM, RS-FORM, FOSM, and PEM. All analyses are performed using Spencer's formulation with 50 slices.

The reliability response to changes in the value of a single parameter or variation coefficient is then explored by repeating the limit states design optimisation and subsequent RS-FORM analyses over a representative range of mean parameter values and coefficients of variation. Values of the remaining parameters are kept at their

mean/reference values (Figure 1, Table 1), while only the parameter of interest is varied.

3.3 Method of Analysis

The limit equilibrium slope analysis method for calculation of g has two nested layers of numerical analysis (Equations 3-5). The inner involves an iterative root-finding algorithm to solve for $\mathcal{F}$ and λ on a given slip surface geometry. A non-linear Newton method (Press et al. 2007) is used for this. The outer requires minimisation of $\mathcal{F}$ over candidate slip surfaces to find the critical surface. This is done using a conjugate gradient algorithm (Press et al. 2007). An example of such a failure surface optimisation search is shown in Figure 4.

The resulting performance function is then used in the various reliability analyses to determine β and p_f values. For the Monte Carlo analyses, N uncorrelated sets of standard normal variables $\mathbf{u}$ are generated by means of the default random engine of the C++11 standard library (ISO/IEC CD 14882:2013). These variables are then transformed to correlated parameter space $\mathbf{x}$ via the Rosenblatt equations in Table A1 and the inverse of Equation 6.

To find the design point, NR-FORM analyses use Brent's root finding algorithm, and numerical derivatives obtained either by simple finite differencing or by Ridder's method (Press et al. 2007), depending on the level of refinement reached by the algorithm. The surrogate performance function (Equation 12) implemented in RS-FORM is parameterised from performance function evaluations on the parameter constellation by means of Gaussian elimination. This parameterisation allows analytical solution

for the design point on the surrogate function g^* .

4 Results

Limit states design and reliability analysis results for four different slope combinations are summarised in Table 2. Reliability results are broadly similar for the optimised drained versus seeping slopes. Very close agreement is found between the β values determined using Monte Carlo and FORM methods, while the moment based methods yield β values that are 20-30% lower than the Monte Carlo equivalents. Monte Carlo and FORM results further indicate the negative correlation between c' and ϕ' has a strong impact on reliability values.

This influence of correlation is further seen in Figure 5, which illustrates the reliability solutions for the seeping slope through marginal projections of the four dimensional density function, together with the FORM-located design point, and the failure boundary line through it. The boundary of the failure region shows very little curvature. Design values used in the limit states slope design calculation are also shown on the figure, and are further compared to the respective design points in Table 3.

Figure 6 compares the variation of FORM-determined β , and factor of safety $\mathcal{F}$, over reasonable ranges of parameter values and their coefficients of variation, respectively. Note that the slope angle is optimised for each parameter value, as previously done using limit states design. Of particular note is the contrast between $\mathcal{F}$ and β : $\mathcal{F}$ remains mostly constant around 1.3, while β varies strongly over the ranges considered. A similar trend is seen with variations in the parameter coefficients of variation.

5 Discussion

5.1 Reliability Analysis

The good agreement between Monte Carlo and FORM can be understood in terms of the lack of strong curvature of the failure boundary (Figure 5). The assumption of a linear boundary plane (the basis of FORM) is therefore reasonable. As such, the reliability techniques that rely on this assumption can be expected to provide accurate reliability estimates for slope problems.

Comparison of the number of performance function evaluations required to locate the design point (Figure 7) indicate that the first and second order polynomial response surface approaches (RS-FORM) are superior to the Newton-Raphson approach (NR-FORM). The latter requires a much higher number of function evaluations – a result of the root finding step in which it aims to return to the failure boundary. In addition, the first order response surface approach (RS1) is found to be more robust than the second order option, as the optimisation algorithm is less susceptible to divergence and instability resulting from spurious extrapolation.

A broader comparison of the number of function evaluations required by the various methods (Figure 8) indicates that there is no significant advantage to performing moment-based reliability estimation calculations. The number of function evaluations required by these methods is comparable to those associated with the RS-FORM approaches, which provide superior reliability estimates when compared to the Monte Carlo results.

This underestimation of β by the moment based methods can be understood by con-

323 sidering the frequency distribution of safety factor values obtained during the full
 324 Monte Carlo calculations for the seeping slope (Figure 9, with correlation included).
 325 The distribution is clearly skewed, with the mode located towards lower $\mathcal{F}$ relative
 326 to the mean. By assuming $\mathcal{F}$ to be normally distributed, these methods can only
 327 be expected to be accurate in cases where the reliability is low and the tail of the
 328 distribution has little effect on the probability of failure. However, at target reliability
 329 values for civil structures, the probability of failure is represented entirely by an area
 330 in the tail of the distribution, and the assumption of normality will have a significant
 331 influence.

332 To illustrate this difference, consider the contrast between Normal and Log-Normal
 333 distribution functions, both computed using the mean and standard deviation of the
 334 Monte Carlo values (Figure 9). Although both distributions follow the cumulative
 335 distribution values close to the mean, the Log-Normal distribution provides a notably
 336 superior representation of the distribution, especially in the tail, where the values
 337 associated with failure are located. $\mathcal{F}$ is located well into the tail, so that the actual
 338 p_f value is almost $10\times$ smaller than that implied by a normal distribution.

339 This comparison suggests that $\mathcal{F}$ would be better represented as Log-Normally dis-
 340 tributed, in which case β follows as

$$341 \quad \beta_{LN} = \frac{\ln(\mu)}{\xi} - \frac{1}{2}\xi, \quad \text{with } \xi = \sqrt{\ln(1 + (\sigma/\mu)^2)}. \quad (16)$$

342 Moment-based β estimates derived in this way compare somewhat more favourably
 343 with the Monte Carlo and FORM results (Figure 8), though the improvement is not
 344 sufficient to justify their use.

345 A full Monte Carlo calculation, in which the critical slip surface is found for every

trial parameter combination, can require a significant amount of time to execute. At around 0.5 seconds per trial, a single 10^6 step calculation needs around six days to complete if run in series. As a result, some slope analysis packages perform Monte Carlo “probabilistic” calculations which keep the critical failure surface fixed at that found for the mean parameters (speeding the calculation by a factor of order 10^4). In principle this approach would result in a distribution located at somewhat higher $\mathcal{F}$ values than that of a full Monte Carlo run, so that the reliability would be overestimated.

Comparison of the frequency distributions of full versus single-slip surface Monte Carlo runs confirms this prediction, though only with a small margin. β determined with a single-slip Monte Carlo run of $N = 10^6$ steps is 3.82, somewhat above the 95% interval on the full Monte Carlo value ($3.69^{+0.05}_{-0.04}$, based on the binomial standard deviation). A similar result was reported by Low (2008). A fixed surface probabilistic run can therefore provide a relatively good estimate of the reliability that a full Monte Carlo run would yield, though such a result will always overestimate β to some extent, and can only be considered as an upper bound.

5.2 Implications for Limit States Design

The contrast between the extent to which reliability β and factor of safety $\mathcal{F}$ vary with changes in parameter mean and coefficient of variation values is concerning (Figure 6). The reliability for a limit states designed slope geometry changes by as much as 35% over the range of parameter values considered, falling to values as low as $\beta = 2.2$ in the case of high cohesion. By contrast, the factor of safety computed using the

characteristic parameter values remains virtually unchanged over the same range of parameter values. This implies that designing to a target factor of safety will yield conservative designs for some soil conditions, and insufficient levels of reliability for others.

Limit states slope design may therefore treat a problem less reliably than desired in some cases. This is further illustrated in Table 3. Comparison between the design point and its limit states representation (design value determined from partial factors) indicates that failure occurs at a much lower surcharge load than that assumed in the limit states design, due to the lower design point cohesion value.

The range of β values obtained over the spectrum of parameter values considered lies mostly below the target value of $\beta = 3.8$ suggested by EN 1997:2004, a contrast which is enhanced if the negative correlation between c' and ϕ' is excluded. Back-analysis of failed slopes (e.g. Kirsten 1983; Santamarina et al. 1992) suggest that the nominal probability of failure associated with slopes tend to be considerably higher than the generic target levels provided by JCSS (2008) and ISO 2394:2015. However, comparison of the failure probability of a structure to these target values must be done in the context of the use of the structure. Slopes associated with mining-operations have relatively short design lifetimes and are often monitored; slopes associated with public infrastructure have long lifetimes and cannot be expected to be regularly monitored. The low slope reliabilities found by these studies can therefore not be used to justify obtaining reliabilities below the JCSS (2008) and ISO 2394:2015 target values.

5.3 Reliability Based Design

To enable reliability based design to be meaningfully performed, the formulation used in the design should not overestimate the reliability and thus result in a relatively unsafe structure. However, the technique should also not underestimate reliability to such an extent that designs are unnecessarily conservative and uneconomical.

While comparison to the Monte Carlo results indicates all the methods considered to result in safe designs, the moment-based methods (FOSM, PEM) will yield overly conservative and uneconomical designs, as the corresponding “true” β value (Monte Carlo) will be quite a bit higher than the target value used in the design. The FORM-based approaches are therefore preferable from an accuracy perspective.

Two practical considerations concerning the numerical implementation of non-closed form reliability calculation should be pointed out. Firstly, because any non-closed form analysis will involve an iterative numerical convergence, a number of nested layers result when combined with the FORM-type reliability methods. To ensure numerical compatibility, tolerances must narrow towards inner layers, so that achieving satisfactory precision on reliability can require very high precision on these layers, implying an increase in execution time.

Secondly, some optimisation algorithms used in slope analysis programs evaluate the objective function over a discrete grid, and simply return the point yielding the lowest value without refining it. This approach results in a piecewise variation of $\mathcal{F}$ with changes in material parameters, which can destabilise outer level algorithms used to locate the design point. The response-surface methodology (RS-FORM) circumvents

411 this difficulty for the most part, by choosing a continuous surrogate performance
412 function parameterised using values at discrete points.

413 Taking accuracy, stability, speed of convergence, and adaptability into account, the
414 RS-FORM methodology stands out as the most suitable to reliability analysis of non-
415 closed form and iterative problem types of the nature considered in this manuscript.

416 6 Conclusion

417 A detailed assessment of the viability of routinely applying reliability analysis as a
418 design tool for slope stability was carried out. Two model slopes were analysed using
419 Spencer's formulation of the limit equilibrium method, with slope dimensions designed
420 to satisfy the limit states specification of EN 1997:2004. Benchmark reliability values
421 were then determined using a full Monte Carlo calculation with 10^6 trials.

422 A spectrum of reliability analysis techniques was then applied to the same slope struc-
423 tures. These techniques can be broadly grouped into first order reliability methods
424 (FORM) and performance function moment estimation methods (PEM and FOSM).
425 Comparison to the Monte Carlo benchmarks indicates FORM-based techniques to
426 be superior, with a first order response surface method found to provide the best
427 combination of accuracy, stability, and speed of convergence.

428 These results suggest that accurate reliability indices for non-closed form problems
429 of the nature considered here can be obtained within 20-30 function calls, depending
430 on the number of free parameters and required precision, indicating that the routine
431 use of reliability analysis in slope design is practical. However, a similar study for

432 geotechnical design using finite element analysis is required to evaluate the utility of
433 reliability analysis over a more representative range of non-closed form problems. Such
434 a study would also be able to account for the extent to which spatial variability in soil
435 properties should be explicitly included in geotechnical reliability analysis problems.

436 **Acknowledgements**

437 The authors are indebted to Massyn Harmse and Nico Vermeulen for insightful dis-
438 cussions on the topic.

Draft

References

- Ang, A.H., and Tang, W.H. 1984. Probability Concepts in Engineering Planning and Design: Volume II Decision, Risk, and Reliability. John Wiley and Sons, New York, 1st edn.
- Baecher, G.B., and Christian, J.T. 2003. Reliability and Statistics in Geotechnical Engineering. John Wiley and Sons, New York, 1st edn.
- Bucher, C.G., and Bourgund, U. 1990. A fast and efficient response surface approach for structural reliability problems. *Structural Safety*, **7**(1):571–66.
- Christian, J.T., Ladd, C.C., and Baecher, G.B. 1994. Reliability Applied to Slope Stability Analysis. *Journal of Geotechnical Engineering*, **120**(12):2180–2207.
- De Koker, N., and Day, P.W. 2018. Assessment of Reliability Based Design for a Spectrum of Geotechnical Design Problems. *Proceedings of the ICE - Geotechnical Engineering*, **171**(2):147—159.
- Duncan, J.M., Wright, S.G., and Brandon, T.L. 2014. Soil Strength and Slope Stability. John Wiley and Sons, New York, 1st edn.
- Duncan, J.M. 2000. Factors of Safety and Reliability in Geotechnical Engineering. *Journal of Geotechnical and Geoenvironmental Engineering*, **126**(4):307–316.
- Ellingwood, B., Galambos, T.V., MacGregor, J.G., and Cornell, C.A. 1980. Development of a Probability Based Load Criterion for American National Standard A58 - Building Code Requirements for Minimum Design Loads in Buildings and Other Structures. US Department of Commerce, Washington DC.

- 460 European Committee for Standardization 2004. Eurocode 7 - Geotechnical Design,
461 EN 1997:2004. CEN, Brussels.
- 462 Fenton, G.A., Naghibi, F., Dundas, D., Bathurst, R.J., and Griffiths, D.V. 2016.
463 Reliability-based Geotechnical Design in 2014 Canadian Highway Bridge Design
464 Code. Canadian Geotechnical Journal, **53**(2):236251.
- 465 Gibson, W. 2011. Probabilistic Methods for Slope Analysis and Design. Australian
466 Geomechanics, **46**(3):1–12.
- 467 Gulvanessian, H. 2010. Eurocode: Basis of structural design. *In* The essential guide to
468 Eurocodes transition. *Edited by* J. Roberts. British Standards Institution, London,
469 pp. 25–41.
- 470 Harr, M. 1987. Reliability-Based Design in Civil Engineering. McGraw-Hill, New
471 York, 1st edn.
- 472 Hasofer, A.M., and Lind, N.C. 1974. Exact and invariant second-moment code format.
473 Journal of the Engineering Mechanics Division, ASCE, **100**(1):111–121.
- 474 Husein Malkawi, A.I., Hassan, W.F., and Abdulla, F.A. 2000. Uncertainty and relia-
475 bility analysis applied to slope stability. Structural Safety, **22**(2):161–187.
- 476 International Organisation for Standardization 2013. Programming Languages – C++,
477 ISO/IEC CD 14882:2013. ISO, Geneva.
- 478 International Organisation for Standardization 2015. General principles on reliability
479 for structures, ISO 2394:2015. ISO, Geneva.
- 480 Joint Committee on Structural Safety 2008. Risk Assessment in Engineering:

- Principles, System Representation and Risk Criteria [online]. Available at
<http://www.jcss.ethz.ch/> [cited 26 January 2018].
- Jonkman, S.N., and Schweckendiek, T. 2015. Developments in Levee Reliability and
Flood Risk Analysis in the Netherlands. *In* Proceedings of the 5th International
Symposium on Geotechnical Safety and Risk, Rotterdam, 13-16 October 2015. IOS
Press, Amsterdam, pp. 50–60.
- Kirsten, H.A.D. 1983. Significance of the probability of failure in slope engineering.
The Civil Engineer in South Africa, **25**(1):17–23.
- Lämsivaara, T., and Poutanen, T. 2013. Safety concepts for slope stability. *In*
Landslides in Sensitive Clays: From Geosciences to Risk Management. *Edited by*
J. L'Heureux, A. Locat, S. Leroueil, D. Demers, and J. Locat, Springer, Dordrecht,
pp. 395–407.
- Li, D-Q., Zheng, D., Cao, Z-J., Tang, X-S., and Phoon, K.K. 2016. Response surface
methods for slope reliability analysis: Review and comparison. *Engineering Geology*,
203:3–14.
- Low, B.K. 2008. Practical reliability approach using spreadsheet. *In* Reliability-Based
Design in Geotechnical Engineering: Computations and Applications. *Edited by*
K.K. Phoon. Taylor & Francis, New York, pp. 134–169.
- Low, B.K., Gilbert, R.B., Wright, S.G. 1998. Slope Reliability Analysis Using General-
ized Method of Slices. *Journal of Geotechnical and Geoenvironmental Engineering*,
124(4):350–362.

- 502 British Standards Institution 2007. UK National Annex for Eurocode 7 - Geotechnical
503 Design, NA to BS EN 1997:2004. BSI, London.
- 504 Phoon, K.K. 2008. Numerical Recipes for Reliability Analysis - a Primer. *In*
505 Reliability-Based Design in Geotechnical Engineering: Computations and Appli-
506 cations. *Edited by* K.K. Phoon. Taylor & Francis, New York, pp. 1–75.
- 507 Phoon, K.K. 2016. Reliability as a basis for geotechnical design. *In* Reliability of
508 Geotechnical Structures in ISO2394. *Edited by* K.K. Phoon and J. Retief. CRC
509 Press, London, pp. 1–25.
- 510 Phoon, K.K., and Kulhawy, F.H. 1999. Characterization of geotechnical variability.
511 Canadian Geotechnical Journal, **36**(4):612–624.
- 512 Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P. 2007. Numerical
513 Recipes: The Art of Scientific Computing. Cambridge University Press, Cambridge,
514 3rd edn.
- 515 Rackwitz, R., and Fiessler, B. 1978. Structural reliability under combined random
516 load sequences. Computers and Structures, **9**:484–494.
- 517 Retief, J.V., and Dunaiski, P.E. 2010. The Limit States Basis of Structural Design for
518 SANS 10160-1. Background to SANS 10160 - Basis of structural design and actions
519 for buildings and industrial structures, African Sun Media, Stellenbosch.
- 520 Retief, J.V., Dithinde, M., and Phoon, K.K. 2016. General principles on reliability
521 according to ISO2394. *In* Reliability of Geotechnical Structures in ISO2394. *Edited*
522 *by* K.K. Phoon and J. Retief. CRC Press, London, pp. 33–48.

- 523 Santamarina, J.C., Altschaeffl, A.G., and Chameau, J. L. 1992. Reliability of Slopes:
524 Incorporating Qualitative Information. *Transportation Research Record*, **1343**:1–5.
- 525 Spencer, E. 1967. A Method of analysis of the Stability of Embankments Assuming
526 Parallel Inter-Slice Forces. *Géotechnique*, **17**(1):11–26.
- 527 Sudret, B. 2015. Polynomial chaos expansions and stochastic finite-element methods.
528 *In Risk and Reliability in Geotechnical Engineering. Edited by K.K. Phoon and*
529 *J. Ching*, CRC Press, London, pp. 265–300.
- 530 Wong, F.S. 1985. Slope reliability and response surface method. *Journal of Geotech-*
531 *nical Engineering*, **111**(1):32–53.

Draft

532 Notation

B dimension determined from limit states design

c cohesivity

D depth of phreatic surface

E expected value

F load

$\mathcal{F}$ factor of safety

g performance function

g^* surrogate performance function

N number of Monte Carlo trials

n number of random variables

p probability

533 $\mathbf{R}$ correlation matrix

S resultant load on a slice

$\mathbf{T}$ Cholesky decomposition of $\mathbf{R}$

u uncorrelated standard normal variable

$\mathbf{u}$ uncorrelated standard normal vector

$\mathbf{u}_d$ design point in $\mathbf{u}$ space

w pore water pressure

W weight of a slice

V variance

x random variable of any distribution

$\mathbf{x}$ correlated vector of components x

$\mathbf{z}$ correlated standard normal vector

	Φ	standard normal cumulative distribution function
	α	slice base inclination
	β	reliability index
	δ	coefficient of variation
	ϕ	soil friction angle
	γ	soil density
	γ_x	partial factor for property x
	λ	resultant load angle
	μ	mean
534	σ	standard deviation
	η	characteristic multiplier
	ξ	statistical measure of spread
	ζ	response surface sampling factor

Subscripts

T	total
V	vertical
d	design value
f	failure
k	characteristic value

Table 1: Statistical parameters used for the surcharge load and soil properties.

	Distribution	δ	η	γ_x
Surcharge				
q_V	Log-Normal	0.25	1.818	1.3
Material parameters				
γ	Normal	0.05	-0.5	1.0
c'	Log-Normal	0.40	-0.5	1.25
ϕ'	Log-Normal	0.10	-0.5	1.25
Correlation: $\rho_{c\phi} = -0.3$				

Table 2: Summary of analysis results at the reference mean parameter values noted in Figure 1 and Table 1

	Limit States Design		Reliability Index β					
	Slope Geometry	Factor of Safety (characteristic)	Monte Carlo	FORM			Moment-Based	
				NR	RS1	RS2	FOSM	PEM
Drained								
$\rho_{c\phi} = 0$	$B = 19.92 \text{ m}$	1.29	3.11	3.02	3.01	3.02	2.14	2.36
$\rho_{c\phi} = -0.3$	(inclined at 26.7°)		3.63	3.59	3.58	3.59	2.54	2.80
Seeping								
$\rho_{c\phi} = 0$	$B = 27.97 \text{ m}$	1.27	3.14	3.11	3.11	3.11	2.23	2.48
$\rho_{c\phi} = -0.3$	(inclined at 19.7°)		3.69	3.64	3.65	3.64	2.64	2.94

536

Table 3: Comparison of FORM-derived design points with the characteristic and design values used in limit states design of the two model slopes.

		q_v	γ	c	ϕ'
		(kN/m ²)	(kN/m ³)	(kPa)	(deg)
<u>Limit States Design</u>					
Characteristic values		50.0	17.6	6.40	24.7
Design values		65.0	17.6	5.12	20.2
<u>Reliability Analysis</u>					
Design point					
Drained	$\rho_{c\phi} = 0$	38.1	18.1	3.60	20.6
	$\rho_{c\phi} = -0.3$	39.9	18.3	3.52	20.9
Seeping	$\rho_{c\phi} = 0$	37.6	17.6	3.76	20.2
	$\rho_{c\phi} = -0.3$	39.5	17.4	4.16	20.0

Figure 1: Slope geometry, loading and soil parameters for the model slope problem considered.

538

Draft

Figure 2: Geometrical relationship between parameter space $\mathbf{x}$ and uncorrelated standard normal space $\mathbf{u}$. Contours of the multi-variate probability density function become spherical, with the most likely failure vector normal to the boundary of the failure region.

539

Draft

Figure 3: Schematic illustration of the iterative path used to find the design point using Newton-Raphson vs response surface FORM. Point colours group iterations of each technique.

540

Draft

Figure 4: Surfaces probed by the factor of safety minimisation algorithm to locate the critical failure surface (drawn in red).

541

Draft

Figure 5: Marginal projections of the four dimensional density function (pdf) with the FORM-determined design point and projected failure boundary line of the seeping slope. LSD indicates limit states design. Note the effect of negative correlation on the volume in the failure region.

542

Figure 6: Variation of FORM-determined β and characteristic value factor of safety with variations in parameter mean and coefficient of variation values, determined for the seeping slope.

Figure 7: Comparison of function evaluations required by the different FORM methodologies to locate the design point and determine the reliability index (seeping slope).

544

Figure 8: Function evaluations required by the various methods to determine the reliability index (seeping slope). Green line indicates Monte Carlo value (10^6 trails) together with the 95% confidence interval based on Binomial standard deviation.

Figure 9: Frequency distribution of a 10^6 trial full Monte Carlo computation (seeping slope with correlation), compared to normal (N) and log-normal (LN) distributions determined from its mean and standard deviation.

546

547 **Appendix**

Table A1: Rosenblatt equations for transforming random variables between parameter space x and standard normal space z , and for relating the characteristic multiplier η to the appropriate confidence interval p_k value. After Ang and Tang (1984). $\gamma_e = 0.577216\dots$ is the Euler-Mascheroni constant.

$x \sim N(\mu, \sigma)$	$z = \frac{x-\mu}{\sigma}$ $x = \mu + z\sigma$ $\eta = \Phi^{-1}(p_k)$ $p_k = \Phi(\eta)$
$x \sim LN(\mu, \sigma)$	$z = \frac{\ln x - \lambda}{\xi}$ $x = \exp(\lambda + \xi z)$ $\lambda = \ln(\mu/\sqrt{1+\delta^2})$ $\xi = \sqrt{\ln(1+\delta^2)}$ $\eta = \frac{\exp[\Phi^{-1}(p_k)\xi - 1/2\xi^2] - 1}{\delta}$ $p_k = \Phi\left(\frac{\ln(1+\eta\delta)}{\xi} + \frac{1}{2}\xi\right)$ $\delta = \frac{\sigma}{\mu}$
$x \sim \text{Gumbel}(\mu, \sigma)$	$z = \Phi^{-1}\left(\exp\left(-\exp\left(\frac{\lambda-x}{\xi}\right)\right)\right)$ $x = \lambda - \xi \ln(-\ln(\Phi(z)))$ $\lambda = \mu - \xi\gamma_e$ $\xi = \frac{\sqrt{6}}{\pi}\sigma$ $\eta = -\frac{\sqrt{6}}{\pi}[\gamma_e + \ln(-\ln p_k)]$ $p_k = \exp\left[-\exp\left(-\left(\frac{\pi\eta}{\sqrt{6}} + \gamma_e\right)\right)\right]$

548