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Shear Deformable Super-Convergent Finite Element for Steel Beams Strengthened with GFRP Plate

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Abstract

The present study investigates the flexural behaviour of steel beams strengthened by adhesively bonding a GFRP plate to one of the flanges. The model captures shear deformation effects and partial interaction between the steel and GFRP owing to the relative flexibility of the adhesive. A general closed form solution is first developed for the governing coupled system of differential equations. The solution is then used to formulate mechanics-based shape functions and develop a finite element with superior convergence characteristics. The model is used to investigate the response of multi-span continuous beams, determine the strength gained by GFRP strengthening, and quantify shear deformation effects on the response of strengthened beams. A technique capturing partial interaction effects is devised to characterize the flexural strength of Class 3 strengthened beams. A classification limit for strengthened Class 3 sections is also proposed within the framework of the Canadian Standard CAN-CSA S16 (2014).

Keywords: Steel beams, Glass-fiber reinforced polymer (GFRP), strengthening, finite element formulation, shear deformation, partial interaction.

1. Introduction and Background

Glass-fiber reinforced polymer (GFRP) possesses a high strength-to-weight ratio and a high corrosion resistance. As such, GFRP plates provide a viable alternative for strengthening steel members (Shaat et al. 2004, Harries and El-Tawil 2008). With available plate thicknesses as large as 19mm, GFRP plates can have a significant flexural stiffness and a compressive capacity comparable to carbon-FRP sheets or plates (Mini et al. 2014, Saribiyik and Caglar 2016). GFRP plates are bonded to steel using a thin layer of adhesive with typically low shear modulus, thus providing only partial interaction between the steel beam and the GFRP plate. Such systems can be reliably analyzed using 3D finite element modeling. However, they typically involve a significant effort to build and involve a high computational effort. As such, an effective design of GFRP-strengthened steel members necessitates

the development of analytical and 1-D finite element models that are both accurate and simple. Within this context, the present study develops a super-convergent finite element that captures shear deformation effects and models the partial interaction between the steel beam and the GFRP plate, while avoiding mesh discretization errors common in other finite elements and providing accurate predictions of deformations and stresses.

Analytical models for the analysis of composite beams were developed in various studies including Ditaranto (1973), Gara et al. (2006), Koutsawa and Daya (2007), Pham and Mohareb (2015a), and Bai and Davidson (2016). These studies neglected transverse shear deformation effects and thus may overestimate the stiffness for short-span composite beams. To overcome this limitation, shear deformable theories were developed in Akhras and Li (2007), Ranzi and Zona (2007), Back and Will (2008), Dalir and Shooshtari (2015), Pham and Mohareb (2015b), Daouadji et al. (2016), and Ecsedo and Baksa (2016). While closed form solutions developed in these studies are applicable to single span beams, their extension to systems with multiple spans, and/or general loading becomes impractical. In this respect, finite element solutions offer a practical alternative. Ganapathi et al. (1999) developed a three-noded C1 finite element for rectangular sandwich beams based on Hermittian shape functions. Perel and Palazotto (2001) formulated an element for cylindrical bending of transversely compressible sandwich plates with thick faces and transversely thick cores. Desai and Ramtekkar (2002) proposed a 6-node 2D mixed finite element for analysis of laminated beams. Dall'Asta and Zona (2004) developed a mixed three-field finite element for the non-linear longitudinal-transverse response of steel-concrete beams based on fifth order shape functions. Ranzi et al. (2006) and Ranzi and Zona (2007) developed a 14-degree-of-freedom element. Back and Will (2008) developed a shear deformable finite element formulation for the flexural and buckling analysis of composite I-beams. Their solution was based on two, three, and four-noded elements in which each node possessed 7 nodal unknowns. Lagrangian interpolation shape functions were used. Based on Hermitian cubic and linear interpolation functions, Challamel et al. (2010) developed 10- and 12- degree-of-freedom elements for

the out-of-plane behavior of partially composite or sandwich beams. Pham and Mohareb (2015b) developed a finite element formulation to capture the longitudinal-transverse response of a steel beam strengthened with a GFRP plate through an adhesive layer. Both non-shear and shear deformable elements were developed in the study. A common feature in the above studies is the fact that they involve mesh discretization errors, which reduce as the number of elements increase. Thus, for such elements, a mesh study is essential to ensure that discretization errors are kept to a minimum. Super-convergent finite elements provide an alternative approach to eliminate discretization errors in the solution. The main idea for such elements is to develop the governing equilibrium equations of the system, formulate a closed form solution for the equilibrium and then use the closed form solution to develop shape functions with superior convergence characteristics. Examples of such treatments are provided in Nowzartash and Mohareb (2005) who developed a shear deformable finite element for the analysis of sandwich beams, Faella et al. (2010) and Martinelli et al. (2012) who developed non-shear deformable super-convergent elements. The study in Nguyen et al. (2011) and Keo et al. (2016) developed finite element formulations for multilayered Timoshenko beam columns with partial interaction. Their study captures the partial interaction through the concept of shear bond stiffness to approximate partial interaction between layers. Joao (2013) developed a super-convergent finite element formulation for multilayered composite members with partial interaction. Pham and Mohareb (2015b) developed super-convergent finite element formulations for steel beams reinforced with GFRP plate through an adhesive layer. However, the solution omitted shear deformation effects. Within this context, the present study focuses on (1) providing a closed form solution for coupled differential equation system governing the equilibrium of the beams bonded with GFRP plates through a thin adhesive layer, (2) developing a super-convergent beam finite element that avoids discretization errors, (3) developing a technique to determine the stress profile and associated flexural strength of the strengthened steel section, and (4) proposing a modified local buckling classification scheme for Class 3 sections within the framework of CAN-CSA S16 (2014).

2. Statement of the Problem

A steel wide flange beam reinforced with a GFRP plate bonded to one of the flanges through a thin layer of adhesive is considered (Fig.1a). The beam has general end support conditions at ends A and B and is subjected to tractions $p_z(s_b, n_b, z)$ and $p_v(s_b, n_b, z)$ in the longitudinal and transverse directions, respectively, where (s_b, n_b, z) are the coordinates in a local Cartesian coordinate system (Fig. 1b). The objective of the present study is to develop a finite element capable of capturing partial interaction and shear deformation effects, while accurately predicting the stresses and deformations with a minimal number of nodal unknowns.

Three global left-handed coordinate systems $OXYZ$, $O_pX_pY_pZ$ and $O_aX_aY_aZ$ are chosen for cross-sections of the wide flange beam, GFRP plate, and adhesive layer, respectively, where origins O , O_p and O_a are selected to coincide with the centroids/shear center of the beam, GFRP plate, and adhesive layer, (Figs. 1b-d). Local left-handed coordinate systems $C_bn_b s_b z$, $C_p n_p s_p z$ and $C_a n_a s_a z$ are defined for the beam, plate, and adhesive in which point C_i ($i = b, p, a$) is a generic point lying on the section contour (or mid-surface), the s_i -direction is tangent to the contour, while the n_i -direction is normal to the contour. The z -axis is parallel to the global Z -axis. The geometric definitions for the composite cross-section are also shown in Fig.1.

Fig. 1. Composite beam configuration

3. Assumptions

The kinematic model of the present study is based on Pham and Mohareb (2014). As such, the following similar assumptions are adopted in the present study: (i) section contours for the steel beam and GFRP plate are assumed to remain undeformed in their own plane, (ii) perfect bond is assumed at the adhesive-GFRP and adhesive-steel interfaces, (iii) given that the adhesive layer is thin, the change in transverse displacement within the thickness of the layer is assumed negligible compared to the

transverse displacement of the composite system, (iv) all materials are assumed to deform within their elastic range. and (v) stresses induced by pre-existing loads are not considered. The present study focuses only on the longitudinal-transverse response and thus the lateral and torsional displacement fields are omitted. The following additional assumptions are introduced in the present study:

(iv) In order to capture shear deformation effects, the wide flange beam is assumed to undergo rotations $\theta_{xb}(z)$ about the X – axis of the cross-section which is distinct from the derivatives of the corresponding transverse displacement $V'(z) \neq \theta_{xb}(z)$ in line with the Timoshenko beam assumption (Fig.2). Also, for the GFRP plate, rotations $\theta_{xp}(z)$ are assumed to be distinct from the corresponding displacement $V'(z) \neq \theta_{xp}(z)$.

(v) Due to the fact that the adhesive material possesses a very small modulus of elasticity when compared to those of the beam or GFRP, longitudinal normal stresses within the adhesive layer are considered negligible. Also, only transverse shear stress component is considered to contribute to the longitudinal-transverse response while the other two shear components are negligible.

4. Overview of Relevant Previous Work

Given that the change in transverse displacement within the thickness of the layer is negligible (Assumption iii), the transverse displacements $V(z)$ of the beam centroid and the GFRP plate centroid are considered equal. As a matter of convention, the angle of rotation $\theta_{xb}(z)$ about the x-axis follows the sign conventions shown in Fig. 2. The longitudinal displacement of the beam centroid is $W_b(z)$ while that of the GFRP plate is $W_p(z)$.

Fig. 2. Partial interaction for the system of a GFRP plate bonded to a wide flange beam

(Elevation View)

The total potential energy π is the sum of internal strain energies stored within the steel beam $\bar{U}_b$, the GFRP plate $\bar{U}_p$, the adhesive layer $\bar{U}_a$ minus the load potential energy loss $\bar{V}$ by member tractions

forces $p_v(s_b, n_b, z), p_z(s_b, n_b, z)$ acting on the member undergoing displacements $v(s_b, n_b, z), w(s_b, n_b, z)$, i.e.,

$$\pi = \bar{U}_b + \bar{U}_p + \bar{U}_a - \bar{V} \quad (1)$$

where

$$\begin{aligned} \bar{U}_b &= \frac{1}{2} \int_L \int_{A_b} (E_b \varepsilon_{zb}^2 + G_b \gamma_{szb}^2) dz, & \bar{U}_p &= \frac{1}{2} \int_L \int_{A_p} E_p \varepsilon_{zp}^2 dz, \\ \bar{U}_a &= \frac{1}{2} \int_L \int_{A_a} G_a \gamma_{zna}^2 dz, & \bar{V} &= \int \int_{z,s} (p_v v + p_z w) ds dz \end{aligned} \quad (2)$$

The strain-displacement relations appearing in Eq. (2) are provided in Appendix 1. For the steel beam one has:

$$\begin{aligned} \varepsilon_{zb}(z) &= W'_b(z) - y(s_b) \theta'_{xb}(z) + n_b \cos \alpha(s_b) V''(z) \\ \gamma_{szb}(z) &= [V'(z) - \theta_{xb}(z)] \sin \alpha(s_b) \end{aligned} \quad (3)$$

in which $\alpha(s_b)$ is the angle between the angle between the the tangent coordinate s_b of the beam contour and the x axis, i.e., $\alpha(s_b) = \pi/2$ for the web $\alpha(s_b) = 0$ for the flanges. Also, for the GFRP plate, one has:

$$\begin{aligned} \varepsilon_{zp}(z) &= W'_p(z) + n_p V''(z) \\ \gamma_{szp}(z) &= 0 \end{aligned} \quad (4)$$

The displacement fields within the adhesive layer are linearly interpolated from displacement fields obtained at two interfaces of steel beam and GFRP plate. Only the transverse shear strain component is considered and is obtained given by:

$$\gamma_{zna}(z) = -c_1 W_b(z) + c_1 W_p(z) - c_3 \theta_{bx}(z) - c_2 V'(z) \quad (5)$$

in which constants $c_1 = (1/t_2)$, $c_2 = (t_1 + 2t_2 + t_3)/(2t_2)$, $c_3 = h_b/(2t_2)$ are defined, t_1 is the thickness of the GFRP plate, t_2 is that of the adhesive layer, t_3 is the flange thickness and h_b is the depth of the

steel beam cross-section (Fig. 1). From Eqs. (3)-(5), by substituting into Eq. (2) and then Eq. (1), the total potential energy of the system can be obtained as

$$\begin{aligned} \pi = \int_L & \left[\frac{1}{2} E_b A_b W_b'^2 + \frac{1}{2} c_1^2 G_a A_a W_b^2 + \frac{1}{2} E_p A_p W_p'^2 + \frac{1}{2} c_1^2 G_a A_a W_p^2 + \frac{1}{2} (E_b I_{ssbf} + E_p I_{xyp}) V''^2 \right. \\ & + \frac{1}{2} (G_b A_{bw} + c_2^2 G_a A_a) V'^2 + \frac{1}{2} E_b I_{xxb1} \theta_{xb}'^2 + \frac{1}{2} (G_b A_{bw} + c_3^2 G_a A_a) \theta_{xb}^2 - c_1^2 G_a A_a W_b W_p + c_1 c_3 G_a A_a W_b \theta_{xb} \\ & + c_1 c_2 G_a A_a W_b V' - c_1 c_3 G_a A_a W_p \theta_{xb} - c_1 c_2 G_a A_a W_p V' + (c_2 c_3 G_a A_a - G_b A_{bw}) \theta_{xb} V' \Big] dz \\ & - \int_L [q_{zb}(z) W_b(z) + q_{yb}(z) V(z) + m_{xb}(z) \theta_{xb}(z) + m_{xb1} V'(z)] dz \end{aligned} \quad (6)$$

in which symbols E_b and E_p denote the elastic moduli of steel and GFRP materials, respectively, and G_b , G_a are shear moduli of steel and adhesive, and the following cross-sectional properties are defined:

$$A_b = \sum_{i=1,3} b_i t_i; A_{bw} = h_w t_w; A_p = t_1 b_p; A_a = t_2 b_p; I_{xxb1} = t_w h_w^3 / 12 + 2 b_f t_3 h_b^2 / 4; I_{ssbf} = b_f t_3^3 / 6; I_{xyp} = b_p t_1^3 / 12;$$

Where symbols A_b, A_p, A_a denote the cross-section areas of the wide flange beam, GFRP plate and adhesive layer, respectively and the line loads are defined as

$$\begin{Bmatrix} q_{zb}(z) \\ q_{yb}(z) \\ m_{xb1}(z) \\ m_{xb}(z) \end{Bmatrix} = \int_{s_b} \begin{Bmatrix} p_z(s_b, n_b, z) \\ p_y(s_b, n_b, z) \sin \alpha(s_b) \\ p_z(s_b, n_b, z) n_b \cos \alpha(s_b) \\ -p_z(s_b, n_b, z) y(s_b) \end{Bmatrix} ds_b \quad (7)$$

By evoking the stationarity condition of the total potential energy function (Eq. (6)) and performing integration by parts (Pham and Mohareb (2014)), one recovers the following coupled systems of equilibrium equations:

$$\begin{bmatrix} (-\alpha_b D^2 + 1) & -1 & r_1 D & r_2 \\ -1 & (-\alpha_p D^2 + 1) & -r_1 D & -r_2 \\ -r_1 D & r_1 D & \alpha_I D^4 - (\alpha_s + r_1^2) D^2 & (\alpha_s - r_1 r_2) D \\ r_2 & -r_2 & (r_1 r_2 - \alpha_s) D & -\alpha_J D^2 + (\alpha_s + r_2^2) \end{bmatrix} \begin{Bmatrix} W_b \\ W_p \\ V \\ \theta_{xb} \end{Bmatrix} = \frac{1}{c_1^2 G_a A_a} \begin{Bmatrix} q_{zb}(z) \\ 0 \\ q_{yb}(z) \\ m_{xb}(z) \end{Bmatrix} \quad (8)$$

in which the following parameters have been defined:

$$\begin{aligned} r_1 &= c_2/c_1; r_2 = c_3/c_1; \alpha_a = c_1^2 G_a A_a; \alpha_I = (E_b I_{ssbf} + E_p I_{xsp})/\alpha_a; \alpha_J = E_b I_{xxb1}/\alpha_a; \alpha_p = E_p A_p/\alpha_a; \\ \alpha_b &= E_b A_b/\alpha_a; \alpha_s = G_b A_{bw}/\alpha_a \end{aligned} \quad (9)$$

5. New Closed Form Solution

Equation (8) was obtained and solved numerically in Pham and Mohareb (2014). However, no closed form solution was obtained. The present study complements the previous work by providing first a closed form solution for the system (Appendix 2), which leads to the following expression:

$$\begin{Bmatrix} W_p(z) \\ W_b(z) \\ V(z) \\ \theta_{xb}(z) \end{Bmatrix} = \begin{bmatrix} 0 & f_1 & 2f_1 z & (f_2 + f_3 z^2) & z & 1 & R_7 e^{m_7 z} & R_8 e^{m_8 z} & R_9 e^{m_9 z} & R_{10} e^{m_{10} z} \\ 0 & 0 & 0 & (f_3 - 3f_1) z^2 & z & 1 & S_7 e^{m_7 z} & S_8 e^{m_8 z} & S_9 e^{m_9 z} & S_{10} e^{m_{10} z} \\ 1 & z & z^2 & z^3 & 0 & 0 & e^{m_7 z} & e^{m_8 z} & e^{m_9 z} & e^{m_{10} z} \\ 0 & 1 & 2z & (f_4 + 3z^2) & 0 & 0 & T_7 e^{m_7 z} & T_8 e^{m_8 z} & T_9 e^{m_9 z} & T_{10} e^{m_{10} z} \end{bmatrix} \{C\}_{10 \times 1} \quad (10)$$

in which $\langle C \rangle_{1 \times 10}^T = \langle C_1 \ C_2 \ C_3 \ C_4 \ C_5 \ C_6 \ C_7 \ C_8 \ C_9 \ C_{10} \rangle$ is a vector of integration constants to be determined from the boundary conditions, and the following constants have been defined.

$$f_1 = r_1 + r_2; \quad f_2 = 6 \frac{\alpha_J r_2}{\alpha_s} + 6 \frac{(r_1 + r_2)(\alpha_s + r_2^2) \alpha_b \alpha_p}{\alpha_s (\alpha_b + \alpha_p)}; \quad f_3 = 3 \frac{(r_1 + r_2) \alpha_b}{(\alpha_b + \alpha_p)}; \quad f_4 = 6 \frac{\alpha_J}{\alpha_s} + 6 \frac{(r_1 + r_2) r_2 \alpha_b \alpha_p}{\alpha_s (\alpha_b + \alpha_p)};$$

$$R_k = (r_1 + r_2) m_i + m_i (1 - m_i^2 \alpha_b) \frac{\alpha_J}{\alpha_b r_2} + (\beta_2 - \beta_1 m_i^2) \frac{(\alpha_b r_2^2 + \alpha_b \alpha_s + \alpha_J - m_i^2 \alpha_b \alpha_J) m_i^2 - \alpha_s}{r_2 \alpha_b} m_i;$$

$$S_k = m_i \frac{\alpha_J + (\beta_2 - \beta_1 m_i^2) (m_i^2 \alpha_J - \alpha_s)}{r_2 \alpha_b}; \quad T_k = m_i (1 + \beta_2 m_i^2 - \beta_1 m_i^4); \quad k = 7, 8, 9, 10$$

The values $m_i, (i = 7, 8, 9, 10)$ are the root of the equation $-\beta_1 m^4 + (\beta_2 + \beta_4) m^2 + (1 - \beta_3) = 0$ where

$$\beta_1 = r_1 \alpha_b \alpha_p \alpha_I \alpha_J / \alpha_s \chi; \beta_2 = \left\{ r_1 \alpha_b \alpha_p (\alpha_s + r_1^2) \alpha_J - \alpha_I \left[r_2 (\alpha_s - r_1 r_2) \alpha_b \alpha_p - r_1 (\alpha_b + \alpha_p) \alpha_J \right] \right\} / \alpha_s \chi;$$

$$\beta_3 = (r_1 + r_2) \alpha_b \alpha_p \alpha_I / \chi; \beta_4 = \left[r_1 (\alpha_b + \alpha_p) \alpha_I + (r_1 + r_2) (\alpha_s + r_1^2) \alpha_b \alpha_p \right] / \chi;$$

$$\chi = (r_1 + r_2) (\alpha_s - r_1 r_2) \alpha_b \alpha_p - r_1 (\alpha_b + \alpha_p) \alpha_J$$

6. Mechanics-Based Shape Functions

To relate the displacement fields to the nodal displacement vector, shape functions will be developed based on the closed form solution of the governing equations of equilibrium. From the expression in Eqs. (10), by substituting into the displacement vector

$$\langle \Delta(z) \rangle_{1 \times 5}^T = \langle W_p(z) \quad W_b(z) \quad V(z) \quad V'(z) \quad \theta_{xb}(z) \rangle \quad (11)$$

one obtains

$$\{\Delta(z)\}_{5 \times 1} = [\mathbf{Z}(z)]_{5 \times 10} \{\mathbf{C}\}_{10 \times 1} \quad (12)$$

in which matrix $[\mathbf{Z}(z)]_{5 \times 10}$ is defined as

$$[\mathbf{Z}(z)]_{5 \times 10} = \begin{bmatrix} 0 & f_1 & 2f_1z & (f_2 + f_3z^2) & z & 1 & R_7e^{m_7z} & R_8e^{m_8z} & R_9e^{m_9z} & R_{10}e^{m_{10}z} \\ 0 & 0 & 0 & (f_3 - 3f_1)z^2 & z & 1 & S_7e^{m_7z} & S_8e^{m_8z} & S_9e^{m_9z} & S_{10}e^{m_{10}z} \\ 1 & z & z^2 & z^3 & 0 & 0 & e^{m_7z} & e^{m_8z} & e^{m_9z} & e^{m_{10}z} \\ 0 & 1 & 2z & 3z^2 & 0 & 0 & m_7e^{m_7z} & m_8e^{m_8z} & m_9e^{m_9z} & m_{10}e^{m_{10}z} \\ 0 & 1 & 2z & (f_4 + 3z^2) & 0 & 0 & T_7e^{m_7z} & T_8e^{m_8z} & T_9e^{m_9z} & T_{10}e^{m_{10}z} \end{bmatrix} \quad (13)$$

From Eq. (11), by setting $z = 0$ and $z = L$, one obtains:

$$\{\Delta_N\}_{10 \times 1} = [\mathbf{L}]_{10 \times 10} \{\mathbf{C}\}_{10 \times 1} \quad (14)$$

in which $\{\Delta_N\}_{10 \times 1}$ is the nodal displacement vector of a finite element and is defined as (Fig. 3):

$$\langle \Delta_N \rangle_{1 \times 10}^T = \langle W_p(0) \quad W_b(0) \quad V(0) \quad V'(0) \quad \theta_{xb}(0) \quad \vdots \quad W_p(L) \quad W_b(L) \quad V(L) \quad V'(L) \quad \theta_{xb}(L) \rangle \quad (15)$$

and matrix $[\mathbf{L}]_{10 \times 10}$ is defined as:

$$[\mathbf{L}]_{10 \times 10} = \begin{bmatrix} [\mathbf{Z}(0)]_{5 \times 10} \\ \vdots \\ [\mathbf{Z}(L)]_{5 \times 10} \end{bmatrix} \quad (16)$$

From Eq. (14), one can obtain $\{\mathbf{C}\}_{10 \times 1} = [\mathbf{L}]_{10 \times 10}^{-1} \{\Delta_N\}_{10 \times 1}$. By substituting into Eq.(12), one has:

$$\{\Delta(z)\}_{5 \times 1} = [\mathbf{N}(z)]_{5 \times 10} \{\Delta_N\}_{10 \times 1} \quad (17)$$

in which $[\mathbf{N}(z)]_{5 \times 10}$ is a matrix of 50 shape functions and is given by:

$$[\mathbf{N}(\mathbf{z})]_{5 \times 10} = [\mathbf{Z}(\mathbf{z})]_{5 \times 10} [\mathbf{L}]_{10 \times 10}^{-1} \quad (18)$$

Fig. 3. Nodal displacement fields of an element

It is noted that since the closed form solution provided in Eq. 10 is able to represent rigid body motion and constant strains, the interpolation scheme in Eq. 17 bears the same properties since Equations 10 and 17 are equivalent. It is also noted that the $V(z)$ meets C1 continuity, while functions $W_b(z), W_p(z), \theta_{xb}(z)$ meet C0 continuity requirements in a manner conforming with the highest orders of the displacement derivatives appearing in the total potential energy functional in Equation 6.

The displacement fields $W_p(z), W_b(z), V(z), \theta_{xb}(z)$ extracted from Eq. (17) by pre-multiplying the right hand side by $\langle \rho_i \rangle_{1 \times 5}^T$ $i = 1, 2, 3, 4$, i.e.,

$$[W_p(z), W_b(z), V(z), \theta_{xb}(z)] = [\langle \rho_1 \rangle_{1 \times 5}^T, \langle \rho_2 \rangle_{1 \times 5}^T, \langle \rho_3 \rangle_{1 \times 5}^T, \langle \rho_4 \rangle_{1 \times 5}^T] [\mathbf{N}(\mathbf{z})]_{5 \times 10} \{\mathbf{\Delta}_N\}_{10 \times 1} \quad (19)$$

in which $\langle \rho_1 \rangle_{1 \times 5}^T = \langle 1 \ 0 \ 0 \ 0 \ 0 \rangle$; $\langle \rho_2 \rangle_{1 \times 5}^T = \langle 0 \ 1 \ 0 \ 0 \ 0 \rangle$; $\langle \rho_3 \rangle_{1 \times 5}^T = \langle 0 \ 0 \ 1 \ 0 \ 0 \rangle$; and $\langle \rho_4 \rangle_{1 \times 5}^T = \langle 0 \ 0 \ 0 \ 0 \ 1 \rangle$ have been introduced.

7. Finite Element Formulation

From Eqs. (19), by substituting into the variational principle in Eq.(6), one obtains:

$$\begin{aligned}
\pi = & \langle \Delta_N \rangle_{1 \times 10}^T \left\{ \int_L \left[\frac{E_b A_b}{2} [N'(z)]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \frac{E_b I_{ssbf}}{2} [N''(z)]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N''(z)]_{5 \times 10} + \right. \right. \\
& + \frac{E_b I_{xxbl}}{2} [N'(z)]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \frac{G_b A_{bw}}{2} [N'(z)]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \\
& + \frac{G_b A_{bw}}{2} [N(z)]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} - G_b A_{bw} [N(z)]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} \Big] dz + \\
& + \int_L \left[\frac{E_p A_p}{2} [N'(z)]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_1 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \frac{E_p I_{xyp}}{2} [N''(z)]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N''(z)]_{5 \times 10} \right] dz + \\
& + \int_L \left[\frac{c_1^2 G_a A_a}{2} [N(z)]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_1 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + \frac{c_1^2 G_a A_a}{2} [N(z)]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + \right. \\
& - c_1^2 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + c_1 c_3 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + \\
& + c_1 c_2 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} - c_1 c_3 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + \\
& - c_1 c_2 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \frac{c_2^2 G_a A_a}{2} [N'(z)]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} + \\
& + \frac{c_3^2 G_a A_a}{2} [N(z)]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T [N(z)]_{5 \times 10} + c_2 c_3 G_a A_a [N(z)]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T [N'(z)]_{5 \times 10} \Big] dz \Big\} \{ \Delta_N \}_{10 \times 1} \\
& - \langle \Delta_N \rangle_{1 \times 5}^T \int_L \left\{ [N(z)]_{10 \times 5}^T [q_{zb}(z) \{ \rho_2 \}_{5 \times 1} + q_{yb}(z) \{ \rho_3 \}_{5 \times 1} + m_{xb}(z) \{ \rho_4 \}_{5 \times 1}] + m_{sb1} [N'(z)]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \right\} dz
\end{aligned}$$

From Eq. (20), by setting the first variation of the total potential energy to zero, one obtains:

$$\left([K_b]_{10 \times 10} + [K_p]_{10 \times 10} + [K_a]_{10 \times 10} \right) \{ \Delta_N \}_{10 \times 1} - \{ P \}_{10 \times 1} = 0 \quad (21)$$

in which $[K_b]$ is the stiffness matrix contributed by the wide flange steel beam. It includes the effects of axial deformation, transverse flexural deformation and transverse shear deformations and is defined as:

$$\begin{aligned}
[K_b] = & \int_L \left\{ E_b \left[A_b \left[N'(z) \right]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} + I_{ssbf} \left[N''(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N''(z) \right]_{5 \times 10} + \right. \\
& + I_{xxb1} \left[N'(z) \right]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} \left. + G_b A_{bw} \left[\left[N'(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} + \right. \right. \\
& + \left[N(z) \right]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} - \left[N(z) \right]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} \\
& \left. \left. - \left[N'(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} \right] \right\} dz
\end{aligned}$$

while $[K_p]$ is the stiffness matrix contributed by the GFRP plate. It includes the effects of axial deformation and transverse flexural deformation and is defined as:

$$[K_p] = E_p \int_L \left\{ A_p \left[N'(z) \right]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_1 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} + I_{xpp} \left[N''(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N''(z) \right]_{5 \times 10} \right\} dz$$

Also, $[K_a]$ is the stiffness matrix contributed by the transverse shear deformation in the adhesive layer and defined as:

$$\begin{aligned}
[K_a] = & G_a A_a \int_L \left\{ c_1^2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_1 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + c_1^2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + \right. \\
& - 2c_1^2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_2 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + 2c_1 c_3 \left[N(z) \right]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + \\
& + 2c_1 c_2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_2 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} - 2c_1 c_3 \left[N(z) \right]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + \\
& - 2c_1 c_2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_1 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} + c_2^2 \left[N'(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} + \\
& \left. + c_3^2 \left[N(z) \right]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_4 \rangle_{1 \times 5}^T \left[N(z) \right]_{5 \times 10} + 2c_2 c_3 \left[N(z) \right]_{10 \times 5}^T \{ \rho_4 \}_{5 \times 1} \langle \rho_3 \rangle_{1 \times 5}^T \left[N'(z) \right]_{5 \times 10} \right\} dz
\end{aligned}$$

In Eq.(21), the nodal load vector P is defined as:

$$\{P\} = \int_L \left\{ \left[N(z) \right]_{10 \times 5}^T \left[q_{zb}(z) \{ \rho_2 \}_{5 \times 1} + q_{yb}(z) \{ \rho_3 \}_{5 \times 1} + m_{xb}(z) \{ \rho_4 \}_{5 \times 1} \right] + m_{sb1} \left[N'(z) \right]_{10 \times 5}^T \{ \rho_3 \}_{5 \times 1} \right\} dz$$

The above treatment will subsequently be shown to lead to superior convergence characteristics compared to the element developed in Pham and Mohareb (2015b), also based on the functional in Eq. (6) in conjunction with a more conventional interpolation scheme whereby the lateral displacement $V(z)$ was related to the nodal displacement vector $\langle V(0) \ V'(0) \ V(L) \ V'(L) \rangle$ through

Hermitian interpolation functions, while displacement $W_b(z)$, $W_p(z)$, $\theta_{xb}(z)$ were respectively related to the nodal displacement vectors $\langle W_b(0) \ W_b(L) \rangle$, $\langle W_p(0) \ W_p(L) \rangle$, $\langle \theta_{xb}(0) \ \theta_{xb}(L) \rangle$ through linear interpolation.

8. Moment Resistance based on First Yield

For given loads q_{zb} , q_{yb} , m_{sb1} , and m_{xb} , the nodal displacements $\langle \Delta_N \rangle_{1 \times 10}^T = \langle W_p(0) \ W_b(0) \ V(0) \ V'(0) \ \theta_{xb}(0) \ W_p(L) \ W_b(L) \ V(L) \ V'(L) \ \theta_{xb}(L) \rangle$ for an element are determined from solution of the discretized equilibrium equations. Given the stiffness matrices $\mathbf{K}_b$, $\mathbf{K}_p$, $\mathbf{K}_a$ and the load vector $\{\mathbf{P}\}_{10 \times 1}$, one can determine a nodal force vector $\langle \mathbf{Q} \rangle_{1 \times 10}^T = \langle -N_p(0) \ -N_b(0) \ -Q(0) \ -M_{x1}(0) \ -M_{x2}(0) \ N_p(L) \ N_b(L) \ Q(L) \ M_{x1}(L) \ M_{x2}(L) \rangle$ from the relation

$$\{\mathbf{Q}\}_{10 \times 1} = [\mathbf{K}_b + \mathbf{K}_p + \mathbf{K}_a]_{10 \times 10} \{\Delta_N\}_{10 \times 1} - \{\mathbf{P}\}_{10 \times 1} \quad (22)$$

where internal forces $N_p(z_e)$, $N_b(z_e)$, $Q(z_e)$, $M_{x1}(z_e)$, $M_{x2}(z_e)$, $z_e = 0, L$ follow the beam theory sign convention (Fig. 4).

Fig. 4. Sign convention for nodal forces

Given the internal forces $N_p(z_e)$, $N_b(z_e)$, $M_{x1}(z_e)$, $M_{x2}(z_e)$, the displacement derivatives at the nodes $W_p'(z_e)$, $W_b'(z_e)$, $V''(z_e)$, $\theta_{xb}'(z_e)$ are determined from (Pham 2013) as

$$W_p'(z_e) = \frac{N_p(z_e)}{E_p A_p}, \quad W_b'(z_e) = \frac{N_b(z_e)}{E_b A_b}, \quad V''(z_e) = \frac{M_{x1}(z_e)}{E_b I_{ssbf} + E_p I_{xyp}}, \quad \theta_{xb}'(z_e) = \frac{M_{x2}(z_e)}{E_b I_{xxb1}} \quad (23)$$

From Eqs. (23), by substituting into Eqs. (3)a and (4)a, and using the stress-strain relations $\varepsilon_{zb} = E_b \varepsilon_{zb}$, $\varepsilon_{zp} = E_p \varepsilon_{zp}$, the longitudinal stresses are expressed as

$$\begin{aligned}
 \sigma_{zb}(z_e) &= E_b \left[\frac{N_b(z_e)}{E_b A_b} + n_b \cos \alpha(s_b) \frac{M_{x1}(z_e)}{E_b I_{ssbf} + E_p I_{xcp}} - y(s_b) \frac{M_{x2}(z_e)}{E_b I_{xxb1}} \right] \\
 \sigma_{zp}(z_e) &= E_p \left[\frac{N_p(z_e)}{E_p A_p} + n_p \frac{M_{x1}(z_e)}{E_b I_{ssbf} + E_p I_{xcp}} \right]
 \end{aligned} \quad (24)$$

The corresponding stresses $\sigma_{s,t}$, $\sigma_{s,b}$, $\sigma_{g,t}$, $\sigma_{g,b}$ at the extreme fibers (Fig. 5) at section $z_{\max}$ of maximum moment are evaluated from Eqs. (24) as

$$\begin{aligned}
 \sigma_{s,t}(z_{\max}) &= \frac{N_b(z_{\max})}{A_b} + \frac{t_3}{2} \frac{M_{x1}(z_{\max})}{I_{ssbf} + (E_p/E_b)I_{xcp}} + \frac{h_w + t_3}{2} \frac{M_{x2}(z_{\max})}{I_{xxb1}} \\
 \sigma_{s,b}(z_{\max}) &= \frac{N_b(z_{\max})}{A_b} - \frac{t_3}{2} \frac{M_{x1}(z_{\max})}{I_{ssbf} + (E_p/E_b)I_{xcp}} - \frac{h_w + t_3}{2} \frac{M_{x2}(z_{\max})}{I_{xxb1}} \\
 \sigma_{g,t}(z_{\max}) &= \frac{N_p(z_{\max})}{A_p} - \frac{t_1}{2} \frac{M_{x1}(z_{\max})}{I_{ssbf} + (E_p/E_b)I_{xcp}} \\
 \sigma_{g,b}(z_e) &= \frac{N_p(z_{\max})}{A_p} + \frac{t_1}{2} \frac{M_{x1}(z_{\max})}{I_{ssbf} + (E_p/E_b)I_{xcp}}
 \end{aligned} \quad (25)$$

It is noted that the above stresses correspond to the applied loads q_{zb} , q_{yb} , m_{sb1} , m_{xb} . Thus, in general, the peak stress $\sigma_{s,t}$ at the top fiber will differ from the target yield strength F_y . Thus, the above expressions need to be multiplied by $\lambda = F_y/\sigma_{s,t}$ to recover the stress profile corresponding to the first yield. The neutral axis depth d_{NA} is then evaluated determined from

$$d_{NA} = \frac{\sigma_{s,t}}{\sigma_{s,b} + \sigma_{s,t}} h \quad (26)$$

By summing the moments of the stresses about the neutral axis, the internal moment corresponding to the first yield is obtained as

$$\begin{aligned}
 M_y^{PI} &= \lambda \left\{ \sigma_{s,t} \left[\frac{d_{NA}^2 b}{3} - \frac{(d_{NA} - t_3)^3 (b - t_w)}{3d_{NA}} \right] + \sigma_{s,b} \left[\frac{(h - d_{NA})^2 b}{3} - \frac{(h - d_{NA} - t_3)^3 (b - t_w)}{3(h - d_{NA})} \right] \right. \\
 &\quad \left. + \frac{1}{2} \sigma_{g,t} b t_1 \left(h - d_{NA} + t_2 + \frac{1}{3} t_1 \right) + \frac{1}{2} \sigma_{g,b} b t_1 \left(h - d_{NA} + t_2 + \frac{2}{3} t_1 \right) \right\}
 \end{aligned} \quad (27)$$

where M_y^{PI} accounts for partial interaction effects, and the corresponding equivalent elastic section modulus S_x^{PI} for the GFRP-strengthened section is

$$S_x^{PI} = M_y^{PI} / F_y \quad (28)$$

Fig. 5. Longitudinal normal stress profile of the composite section (a) cross-section, (b) stress profile, and (c) stress resultants

9. Classification considerations

The stress profile in Figure 5 shows that the neutral axis is offset from the steel section centroid. Hence, the stress profile in the web of a strengthened steel member under bending alone resembles to that of a bare steel member under combined compression and bending. This similarity suggests the possibility of adopting present web classification rules for bare steel beam-columns to GFRP-strengthened steel members under bending. For a beam-column member with a wide flange section with a web clear height h_w and web thickness t_w , under a compressive force C_f , present classification rules in CAN-CSA-S16 (2014) for Class 3 sections are

$$\frac{h_w}{t_w} \leq \frac{1900}{\sqrt{F_y}} \left(1 - 0.65 \frac{C_f}{\phi C_y} \right) \quad (29)$$

in which $C_y = A_b F_y$ is the compressive yield resistance and $\phi = 0.9$ is the resistance factor. For a bare steel beam-column, the axial compressive force C_f is obtained from a conventional analysis. In contrast, for a strengthened steel member under pure bending, the net compressive force C_f acting on the steel section alone can be obtained by summing the internal forces acting on the steel section alone, i.e., $C_f = C_s - T_s$ where $C_s = \lambda (2 - t_3/d_{NA}) b t_3 \sigma_{s,t} / 2 + \lambda (d_{NA} - t_3)^2 t_w \sigma_{s,t} / 2 d_{NA}$ and $T_s = \lambda [2 - t_3 / (h - d_{NA})] b t_3 \sigma_{s,b} / 2 + \lambda (h - d_{NA} - t_3)^2 t_w \sigma_{s,b} / 2 (h - d_{NA})$.

Also, the stress distributions in the compression flange of width b and thickness t_f for a bare steel beam-column is identical to that of a strengthened beam, and the Class 3 classification limit of CAN-CSA-S16 (2014)

$$\frac{b}{2t_f} \leq \frac{200}{\sqrt{F_y}} \quad (30)$$

remains valid.

10. Verification and examples

10.1. Example 1 -Verification of Results

A 2m-span continuous composite beam (Fig. 6) has a W150x13 cross-section and is bonded to a GFRP plate with thickness $t_1 = 19mm$ and a width $b_p = 100mm$, through an adhesive of thickness $t_2 = 0.79mm$. The modulus of elasticity of steel is $E_s = 200GPa$, that of GFRP is $E_p = 17.2GPa$ (Abushagur and Damatty 2013) and that of adhesive is $E_a = 3.18GPa$ (Shaath and Fam 2007). Poisson's ratios of all three materials are assumed to be 0.3. Both spans are 1.5m while the length of the GFRP plate is $L_1 = 2.0m$ and is symmetrically placed relative to the intermediate support. Longitudinally, the end $z = 0$ of the beam axis is restrained while at $z = L, 2L$, it is axially free. Both ends of the GFRP plate are longitudinally free. The beam is subjected to a uniform distributed line load $q_y = 15kN/m$ applied at the central line of the beam and a concentrated load $P = 5kN$ applied at $z = L/3$ from the left support. It is required to compare the predictions of the present model to that based on a 3D FEA solution. Also, a comparison of these solutions with a classical solution (i.e., Timoshenko beam theory (TB-No Reinforcement) of the non-strengthened steel beams under the same loading and support conditions is also made to quantify the effect of strengthening.

Fig. 6. Continuous composite beam

Given that the finite element formulation is based on shape functions that satisfy the equilibrium equations, four elements (AB, BC, CD, DE) are found enough to achieve convergence. Finer discretizations are found not to influence the predicted stresses and displacements. This finding contrasts with the element in (Pham and Mohareb 2015b). To obtain a suitable number of intermediate reading, 80 elements were selected for the present example where 14 elements are chosen to model the unstrengthened segments while 52 elements were selected for the reinforced segment. The 3D FEA analysis is based on the eight-node brick element C3D8R with reduced integration. A mesh study has indicated that the predicted displacements and stresses do not change for meshes finer than 20 elements along the flange overhang, 4 elements across the flange thickness, 70 elements along the web height, 4 elements across the web thickness, 4 elements across the adhesive thickness, 8 elements across the GFRP thickness, 500 elements along the GFRP plate length and 1500 elements in the longitudinal direction of the steel beam.

Figure 7 depicts the transverse deflection along the span as predicted by the present model and the 3D FEA solution. In both solutions, the maximum transverse deflection takes place at a distance at about 600 mm from the left support. The present model predicts a peak displacement of 0.590 mm. This compares to 0.604mm based on the 3D FEA solution, a 2.3% difference. Also shown on the figure is the deflection obtained from a shear deformable Timoshenko beam solution for the case of no GFRP reinforcement (TB-No Reinforcement). The predicted displacement is 0.66mm occurred at a distance of 638mm from the left support. The GFRP plate reduces the peak deflection by 11.9%.

Fig. 7. Transverse deflection for a two-span continuous beam

Figures 8a-b present the longitudinal normal stresses at the bottom flange and that at the top of the GFRP plate, respectively. The stress predictions based on the present study and the 3D FEA solutions are in close agreement (Fig. 8a). Based on the present solution, the peak stress is found to be 92.8 MPa. This value compares to 90.6 MPa based on 3D FEA, corresponding to 2.4% difference. This value

also compares to a maximum stress of 113.3MPa for the non-reinforced steel beam as predicted by the Timoshenko beam solution (Fig. 8a), i.e., the GFRP reinforcement reduces the stresses by 25.1%. Close agreement is observed for the maximum normal stress in the GFRP plate (Fig. 8b). The maximum stresses based on the present study is 8.20 MPa. This compares to a stress of 8.41 MPa based on the 3D FEA solution, corresponding to a difference of 2.5%.

Fig. 8. Normal stress distribution for two-span continuous beam at (a) at bottom flange
and (b) at top of GFRP plate

10.2. Example 2- Convergence characteristics of the element

A 3m-single span beam (Fig.9a) has a W150x13 cross-section and is strengthened with a GFRP plate of thickness $t_1 = 19mm$ and width $b_p = 100mm$ through an adhesive of thickness $t_2 = 0.79mm$. Material properties identical to those of the previous example. The GFRP plate is symmetrically placed about mid-span and has a length L_1 of 2.5m. End $z = 0$ of the beam is longitudinally restrained, end $z = L$ is free while both ends of the GFRP plate are longitudinally free. The beam is subjected to a uniformly distributed line load $q_y = 10kN/m$ applied at the centerline of the beam. It is required to compare the results based on the predictions of the present finite element to that in Pham and Mohareb (2015b). The later element was based on the functional in Eq. (6) in conjunction with a more conventional interpolation scheme whereby the lateral displacement $V(z)$ was related to the nodal displacement vector $\langle V(0) \ V'(0) \ V(L) \ V'(L) \rangle$ through Hermitian interpolation functions, while displacement $W_b(z), W_p(z), \theta_{xb}(z)$ were respectively related to the nodal displacement vectors $\langle W_b(0) \ W_b(L) \rangle, \langle W_p(0) \ W_p(L) \rangle, \langle \theta_{xb}(0) \ \theta_{xb}(L) \rangle$ through linear interpolation. The problem is solved using 10, 30, 40 and 60 Hermitian elements. In all cases, 4 elements are used to model the unstrengthened segments and the remaining elements were used to model the strengthened segment using a uniform discretization. Also, the problem is solved using four and six

elements. Figures 9b-d respectively depict the transverse deflection, the longitudinal normal stresses at the top flange of the steel beam, and that at the bottom fiber of the GFRP plate. The displacements and stresses based on the 40-element mesh is observed to nearly coincide with that based on the 60-element mesh. In contrast, the displacements and the stresses based on the 10-element solution are found to depart from the 40-element mesh. For the 30-element mesh, deflections are found to agree with those based on the 40-element mesh while the stresses are observed to depart from those based on the 40-element mesh, suggesting that a 40-Hermitian element mesh is needed for convergence in the present example. The stress distributions predicted by the coarser mesh of Hermitian elements exhibit kinks at the element interfaces (Fig. 9c). These are due to the approximate interpolations adopted within the Pham and Mohareb 2015b formulation. This behaviour contrasts with the perfectly smooth stress distributions predicted by the present super-convergent element, even when a small number of elements are taken to model the beam.

By comparison, the displacements and stresses based on the present finite element are found to exactly agree with the converged results whether 4 or 6 elements are use. Further mesh refinements were found to be unnecessary for convergence, indicating that 4-element solution is enough to achieve convergence. This is a natural outcome of the fact that the present element is based on shape functions which exactly satisfy the equilibrium conditions.

Fig. 9. Comparison of the solution convergence between Hermitian element and present study(*bracketed numbers indicate the number of finite element*)

10.3. Example 3- First yielding moment capacity and Section classification

It is required to determine the section class and the first yield moment for the strengthened beam in Example 2 and compare the yield moment to that of the bare steel section. The steel yielding strength is $F_y = 350 \text{ MPa}$. For the bare beam, the elastic section modulus is $S_x = 80.6 \times 10^3 \text{ mm}^3$, the corresponding first yield moment is $M_y^{bare} = S_x F_y = 28.2 \text{ kNm}$.

Based on Eq. (30), flange slenderness is $b/2t_f = 100/(2 \times 4.9) = 10.2 < 200/\sqrt{350} = 10.7$. Thus the flange meets Class 3 requirements. Since the stress distribution within the web is unknown a-priori, the axial force C_f can be considered to vanish to provide a preliminary web classification. Based on Eq. (29), the web slenderness $h_w/t_w = 138.2/4.3 = 32.1 < 1900/\sqrt{350} = 102$ and thus the web is found to meet Class 3 requirements.

When the strengthened beam is subjected to $q_y = 10 \text{ kN/m}$, the stresses at the extreme fibers of the section of maximum moment (i.e., midspan section) are $\sigma_{s,t} = 131.3$, $\sigma_{s,b} = 116$, $\sigma_{g,t} = 9.2$, $\sigma_{g,b} = 11.9 \text{ MPa}$ as determined from Equation (25). To attain the point of first yield, the applied load must be magnified by a factor $\lambda = 350/131.3 = 2.67$. The corresponding magnified stresses become $\sigma_{s,t} = 350$, $\sigma_{s,b} = 309.2$, $\sigma_{g,t} = 24.5$, $\sigma_{g,b} = 31.7 \text{ MPa}$. Distance d_{NA} as determined from Eq. (26) is $d_{NA} = 82.2 \text{ mm}$. The first yield moment based on Eq. (27) is $M_y^{PI} = 29.9 \text{ kNm}$. This value is marginally lower than that based on the transformed section method of $M_y^{TS} = 30.0 \text{ kNm}$, suggesting that a full interaction has been nearly achieved. This will be the case for thin adhesive layers with a relatively high elasticity modulus. Compared to the bare beam, the GFRP-strengthening is found to increase the moment capacity by 6.0% in the present example.

Irrespective of the magnitude of C_f , flange class remains 3. Now that the stress profile has been determined, it is possible to re-classify the web while accounting for the effect of net axial compression. One has $C_f = C_s - T_s = 221.1 - 183.3 = 37.8 \text{ kN}$ and $\phi C_y = \phi A_b F_y = 0.9 \times 1574.3 \times 350 = 495.9 \text{ kN}$. In this case, based on Eq. (29), the web slenderness $h_w/t_w = 32.1$ remains less than the more refined classification limit $1900(1 - 0.65 \times 37.8/495.9)/\sqrt{350} = 96.5$ and the web remains meeting Class 3 requirements.

For softer and/or thicker adhesives, interaction will typically be partial. Consider a weak adhesive with an elastic modulus 3.38 MPa (Asik and Tezcan 2005) and a 2.0mm thickness. The stresses predicted by the present solution are $\sigma_{s,t} = 350$, $\sigma_{s,b} = 310.4$, $\sigma_{g,t} = 12.1$, $\sigma_{g,b} = 19.6 \text{ MPa}$ and coincide with those based on a 3D FEA solution, while those predicted by the transformation section method are $\sigma_{s,t} = 350$, $\sigma_{s,b} = 280.7$, $\sigma_{g,t} = 25.2$, $\sigma_{g,b} = 32.2 \text{ MPa}$. The first yield moment based on Eq. (27) is $M_y^{PI} = 29.2 \text{ kNm}$. As expected, this value lies in between the flexural strength $M_y^{bare} = S_x F_y = 28.2 \text{ kNm}$ for the steel section alone and $M_y^{TS} = 30.2 \text{ kNm}$ based on the transformed section method (the slight increase in M_y^{TS} compared to that the initial strengthened section is due the slightly larger moment arm due to the thicker adhesive layer). The interaction ratio provided by the soft adhesive ρ can be determined from the relation $M_y^{PI} = (1 - \rho) M_y^{bare} + \rho M_y^{TS}$. In this case the interaction ratio is $\rho = 50\%$, which compares to $\rho = 94\%$ for the stiffer/thinner adhesive.

10.4. Example 4: Effect of shear deformation-1

A W150x13 beam (properties provided in Example 2) is fixed at both ends and is fully reinforced with a GFRP plate with thickness $t_1 = 19\text{mm}$, width $b_p = 100\text{mm}$ through an adhesive of thickness $t_2 = 0.79\text{mm}$. Material properties of steel, GFRP and adhesive are similar to those of Example 2. The beam is subjected to a concentrated transverse load of 15 kN acting at mid-span. It is required to investigate the shear deformation effect by comparing the transverse deflection based on the present shear deformable solution to the non-shear deformable theory (NSDT) developed in Pham and Mohareb (2015b) for various spans.

Figures 10a-d depict the distribution of the transverse deflection d along the span length for four spans: $L = 1.0, 2.0, 3.0$, and 5.0m . For the shortest span of 1.0m span beam (Fig. 10a), with span to depth ratio (L/h) of 6.8 , the present solution predicts a peak displacement of 0.13mm in close agreement with that

predicted by the 3D FEA solution of 0.14mm at mid-span and, a 3.6% difference. The NSDT solution predicts a peak displacement of 0.06mm, a 59% difference. For longest span examined, ($L=5.0\text{m}$ and L/h of 33.8), the peak displacement predicted by the present study (and the 3D FEA) is 7.38mm while that based on NSDT is 6.97mm, a difference of 5.6% (Fig. 10d).

Fig. 10. Deflections predicted by the present solution and non-shear deformable study (a) $L=1.0\text{m}$, (b) $L=2\text{m}$, (c) $L=3\text{m}$, and (d) $L=4\text{m}$.

10.5. Example 5: Effect of shear deformation-2

To further investigate the contribution of shear deformation, a simply supported beam with a W150x13 section subjected to a uniformly distributed line load $q=15\text{kN/m}$ is considered. Spans are varied so that the span to depth ratios vary from 3.1 to 41. Two cases are considered: (a) with GFRP-strengthening with $t_1 = 19\text{mm}$, width $b_p = 100\text{mm}$ and (b) without strengthening. The strengthened beam is solved using the shear deformable element developed in the present study and the non-shear deformable solution (Pham and Mohareb 2015b) and ratio of the shear deformable solution to the non-shear deformable solution is plotted against the span to depth ratio (Fig. 11). For comparison, the un-strengthened beam is solved using Timoshenko beam theory (shear deformable) and the Euler-Bernoulli beam theory (non-shear deformable). A single element is used to model a half of the span. In a manner similar to homogeneous beams, the results show that shear deformation effects tend to be significant for short-span beams, where the ratio of peak deflections significantly exceed unity and tends to approach unity as the span to depth ratio increases. Also, the ratio of the deflection of the fully-reinforced beam is higher than that of the unstrengthened beam. This shows that shear deformation is more important in composite beams. When the shear deformation contribution is 5%, i.e., the ratio of peak deflections 1.05, the corresponding span to depth ratio for the strengthened beam is 15.5 which compares to only 8.78 for the non-strengthened, suggesting that shear deformation for strengthened beams is significantly more important than un-strengthened beams.

Fig. 11. Ratios of the deflection based shear deformable solution to that based on non-shear deformable solution for simply supported beam

10.6 Example 6: Deflections and Stiffness

An existing beam has a W150x13 cross-section has a span $L = 3m$. The beam is to be strengthened by a GFRP plate of thickness $t_1 = 19mm$ and width $b_p = 100mm$, through an adhesive of thickness $0.79mm$. Modulus of elasticity of the steel is $E_s = 200GPa$, that of GFRP is $E_p = 42GPa$ and the shear modulus of the adhesive is $G_a = 0.4GPa$. Poisson's ratio for all three materials is 0.3. Transversely, the wide flange beam is simply supported at both ends. Longitudinally, the end $z = 0$ of the beam axis is restrained while the end $z = L$, it is free. Both ends of the GFRP plate are longitudinally free. If the service live loads are 12 kN/m, it is required to determine the reduction in deflections due to strengthening and quantify the individual stiffness of each of the three components and compare them to that of the strengthened system.

For the bare beam, the mid-span deflection as predicted by the conventional beam theory (including shear deformation effects) is 10.6mm which exceeds the allowable deflection threshold $L/360 = 8.3mm$.

The deflection of the beam after strengthening, as predicted by the present finite element, reduces to 7.52mm and is less than the live load deflection threshold.

A possible indicator of the stiffness is the value of the total load applied divided by the mid-span displacement. The stiffness of the bare beam alone is found to be 3380 kN/mm (Table 1). This stiffness increases to 4520 kN/mm when the beam is strengthened, a 41% increase. For comparison, the individual stiffness values of the GFRP plate and adhesive layer alone are provided, and, as expected, they are orders of magnitude lower than that of the steel beam alone and that of the strengthened system.

Table 1 Deflection and stiffness for strengthened beam in Example 6

10.7 Example 7: Stresses and degree of utilization

A beam similar to that in example 6 is considered in this example. Steel is A36 ($F_y=248\text{MPa}$) and the GFRP strength is 42 MPa. The factored loads are to increase in this case to 17kN/m. It is required to compare the degree of material utilization of the bare beam and that of the strengthened beam.

Before strengthening, the normal stresses at the top and bottom flanges are 232MPa, which correspond to a 93% degree of utilization. This value is larger than that the 90% threshold implied by the resistance factor of 90% and flags the necessity of strengthening the section. After strengthening, the present finite element model predicts a significant reduction in the compressive stresses in the steel to 133MPa, a 43% reduction. The tensile stresses moderately reduce to 212MPa, an 8.8% reduction. The corresponding degree of steel utilization is 85.4%, which is lower than the 90% threshold. The compressive stresses in the GFRP are 37.0 GPa which corresponds to a 88.0% degree of utilization.

Examples 6 and 7 show that, for the example considered, GFRP strengthening is rather effective in reducing deflections, but is only mildly effective in reducing the normal stresses in the steel.

11. Conclusions

- A closed form solution was developed for steel beams strengthened with GFRP plate. The solution captures shear deformation effects and partial interaction between the GFRP and steel section. The solution was shown to predict a flexural resistance that lies in between that of the bare steel bare section and that predicted by the transformed section method.
- A super-convergent finite element formulation was developed based on the closed form solution. The element eliminates discretization errors common in conventional interpolation techniques and allows the accurate prediction of deflections and stresses with a minimal number of elements. Results were shown to agree with those based on 3D FEA solutions.
- A procedure to evaluate the flexural capacity of the strengthened beams based on first yield was proposed based on the stress profile predicted by the present formulation. The procedure is applicable to Class 3 sections.

- A modified classification limit for strengthened Class 3 sections was proposed and it is analogous with that of beam-columns.
- In the examples investigated, while GFRP strengthening was shown to increase the flexural capacity based on first yield only mildly (by 6% in Example 2), it was effective in reducing deflections (by 12% in Example 1).
- Shear deformation effects were shown to be significantly more important when predicting deflections for strengthened beams than for the case of bare beams, even for longer span beams. For instance, Example 5 shows that while the omission of shear deformation effects would under-predict the peak deflection by 5% for a bare beam with span to depth ratio lower than 8.7, the 9-ratio associated with the same error is found lie in the neighborhood of 15.5 for the strengthened beam.

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(a) Profile of the composite beam

(b) Cross-section of steel beam

(c) Cross-section of GFRP plate

(d) Cross-section of adhesive layer

(a)

(b)

(a) Composite beam configuration

(b) Transverse deflection

(c) at the farthest fibre of the unreinforced flange

(d) at the farthest fibre of the GFRP plate

(a) $L=1.0\text{m}$

(b) $L=2.0\text{m}$

(c) $L=3.0\text{m}$

(d) $L=5.0\text{m}$

Table 1 Deflection and stiffness for strengthened beam in Example 6

Component	Deflection (mm)	Stiffness (N/mm)
Steel beam alone	10.6	3380
GFRP plate alone	5272	6.8
Adhesive layer alone	2.80e+09	1.29e-5
Strengthened system	7.52	4520

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