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Two-photon decay of P-wave positronium: a tutorial

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A detailed exposition of two-photon decays of P-wave positronium is given to fill an existing gap in the pedagogical literature. Annihilation decay rates of P-wave positronium are negligible compared to the rates of radiative electric dipole transitions to the lower energy *S*-states. This circumstance makes such decays experimentally inaccessible. However the situation is different for quarkonium and the experimental and theoretical research of two-photon and two-gluon decays of P-wave quarkonia is a still flourishing field.

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I. INTRODUCTION

Two-photon decay rates of positronium in the P-state were calculated long ago [1, 2]. A well-known textbook in quantum field theory [3] offers this problem as an exercise after presenting a basic tenets and calculation tools of this theory.

Being indeed an excellent exercise in quantum field theory, however we are afraid that most students will find it too complicated. Even if they can find the original papers about this problem [1, 2], more modern presentation in [4], or its quarkonium counterpart in [5], this will not help much, we think.

This feeling is strengthened by the observation that in the unofficial solutions manual [6] of the textbook [3], the decay rates of P-wave positronium are calculated incorrectly.

A detailed derivation of the two-photon amplitudes of various quarkonium states can be found in [7]. Although very useful, this paper uses the Jacob-Wick helicity formalism [8], not covered in any detail in [3] (however this formalism is briefly considered in older QFT textbook [9]), and therefore can seem somewhat esoteric for novices in quantum field theory.

In this paper we attempt to fill this seeming gap in pedagogical literature and provide a detailed calculation of the two-photon decay rates of P-wave positronium along general style of the first five chapters of [3].

The decay width of S-wave positronium in non-relativistic approximation can be obtained by elementary means [10]. Namely, the probability of electron-positron annihilation in S-wave positronium per unit time is

$$\Gamma = \rho v \sigma, \quad (1)$$

where v is electron-positron relative velocity, $\rho = |\psi(0)|^2$ gives a probability that the electron and positron meet each other in the positronium, and σ is their annihilation cross-section when they meet. The later can be related to the annihilation cross-section of the free electron-positron pair as follows. We must multiply the free cross-section by four, because it was averaged over the four possible spin-states of the incident electron and positron. Besides we must take into account the selection rules that only spin-singlet S-wave positronium can decay into two-photons, and only spin-triplet positronium can decay into three-photons. This selection rule can be enforced by taking $v \rightarrow 0$ limit which in the free cross-section leaves only s-wave contribution. Finally we must average over positronium polarization states which brings $1/(2J+1)$ factor in the formula. In this way we get the Pirenne-Wheeler formula [11]:

$$\Gamma(Ps \rightarrow n\gamma) = \frac{1}{2J+1} |\psi(0)|^2 \lim_{v \rightarrow 0} [4v\sigma(e^+e^- \rightarrow n\gamma)]. \quad (2)$$

In the case of P-wave positronium the wave-function at the origin vanishes and the decay amplitude becomes proportional to the spatial derivatives of the wave function at the origin. Correspondingly we need the free annihilation

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cross-section beyond the $v \rightarrow 0$ limit and things become much more complicated. As a result there is no Pirenne-Wheeler like simple way to get annihilation cross-section of P-wave positronium. The only thing which we can predict from the beginning is that in this case the annihilation rate will be suppressed compared to the S-wave positronium annihilation rate by a factor $(|\vec{p}|/m)^2 \sim \beta^2$ since this is a relative magnitude of the leading term in an expansion for small momenta [10]. For positronium the suppression is quite substantial: P -states annihilation lifetimes exceed 100 μs while their fluorescence lifetimes (that is lifetimes for decays to the lower energy states via spontaneous emission of a photon) are only 3.19 ns [33].

For light quarks the suppression goes away and the non-relativistic approximation breaks down completely. Even for heavy quarkonia, such as charmonium, where $\beta^2 \sim 0.3$, and bottomonium, where $\beta^2 \sim 0.1$, the relativistic corrections are important and these corrections were studied in the frameworks of the Bethe-Salpeter equation [12, 13], two-body Dirac equation [14], covariant light-front approach [15], sophisticated quarkonium potential model [16], using an effective Lagrangian and QCD sum rules [17], lattice QCD [18], non-relativistic QCD (NRQCD) [19], to name a few. Regarding experimental situation, see, for example, [20, 21].

Note that, depending on the charge-conjugation parity, some P -wave states (namely 3P_0 and 3P_2) can annihilate into two-photons, while others (namely 1P_1) — into three-photons.

We hope that a detailed understanding of a more simple positronium case will help students to navigate the vast literature devoted to the two-photon and two-gluon decays of quarkonia.

II. POSITRONIUM STATE VECTOR

A correct framework for relativistic bound state problem is provided by the Bethe-Salpeter equation [22, 23] (for pedagogical discussions of this equation see [9, 24]). Fortunately, for weakly bound non-relativistic systems, like positronium, this notoriously difficult formalism simplifies considerably. It was shown [25, 26] that the relativistic two-fermion Bethe-Salpeter equation for such systems allows a systematic perturbation theory and the corresponding lowest-order exactly solvable approximation essentially coincides to the Schrödinger equation for a single effective particle.

At the lowest-order in fine structure constant α , and in its rest frame, the positronium state vector can be approximated by the quantum state

$$|^{2S+1}L_J; M\rangle = \sqrt{2M_{Ps}} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{S_z=-S}^S \begin{bmatrix} l & S & J \\ M - S_z & S_z & M \end{bmatrix} \tilde{\psi}_{lm}(\vec{p}) |S, S_z\rangle, \quad (3)$$

where $m = M - S_z$, $\tilde{\psi}_{lm}(\vec{p}) = \int e^{-i\vec{p}\cdot\vec{x}} \psi_{lm}(\vec{x}) d\vec{x}$ is the momentum space Schrödinger wave function of positronium (the principal quantum number is not indicated) giving the probability amplitude of finding the electron and positron with relative momentum $\vec{p}$ in the positronium, $M_{Ps} \approx 2m$ is the positronium mass (m being the electron mass, not to be confused with the magnetic quantum number m in $\tilde{\psi}_{lm}$) and the $\sqrt{2M_{Ps}}$ factor ensures a proper normalization of the positronium state vector consistent to the normalization of one-particle states adopted in [3]. The Clebsch-Gordan coefficients $\begin{bmatrix} l & S & J \\ m & S_z & M \end{bmatrix}$ (a square bracket notation of [27] is used for these coefficients) couple the angular momentum eigenstates $\tilde{\psi}_{lm}(\vec{p})$ with the total spin eigenstates $|S, S_z\rangle$ to form the total momentum eigenstates $|^{2S+1}L_J; M\rangle$. The total spin eigenstates by themselves are the result of quantum addition of electron and positron spins:

$$|S, S_z\rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ s & S_z - s & S_z \end{bmatrix} a^+(s, \vec{p}) b^+(S_z - s, -\vec{p}) |0\rangle. \quad (4)$$

Here $a^+(s, \vec{p})$ is the creation operator of electron with spin-projection s and momentum $\vec{p}$, while $b^+(S_z - s, -\vec{p})$ is the creation operator of positron with spin-projection $S_z - s$ and momentum $-\vec{p}$.

Note that, since particle number is not conserved in relativistic quantum field theory, in general positronium state vector may contain contributions from Fock states that have particles other than “valence” electron and positron, as in (3). However, in positronium, thanks to its non-relativistic nature, such admixtures are very small. For example, the probability of finding relativistic relative momenta, $p \sim m$ or higher, in positronium is only $O(\alpha^5) \sim 10^{-11}$ [28].

We use standard spectroscopic notation in (3). In particular, L refers to the orbital angular momentum quantum number l written as $S, P, D, F, \dots$ for $l = 0, 1, 2, 3, \dots$

Electron and positron spins can combine to give either a total spin zero singlet state or a total spin one triplet

states. The corresponding non-zero Clebsch-Gordan coefficients are [29]

$$\begin{aligned} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix} &= -\begin{bmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}, \quad \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix} = 1, \\ \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} &= \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}, \quad \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & -1/2 & -1 \end{bmatrix} = 1. \end{aligned} \quad (5)$$

Using them, we easily get S -wave positronium state vectors

$$\begin{aligned} |^1S_0; 0\rangle &= 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) \frac{1}{\sqrt{2}} \left[a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) - a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) \right] |0\rangle, \\ |^3S_1; 0\rangle &= 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) \frac{1}{\sqrt{2}} \left[a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) + a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) \right] |0\rangle, \\ |^3S_1; 1\rangle &= 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) |0\rangle, \\ |^3S_1; -1\rangle &= 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) |0\rangle \end{aligned} \quad (6)$$

Slightly abusing a notation (by using s, s' as matrix indexes) and changing the overall signs of some state vectors then necessary (in quantum theory state vectors are defined up to a phase), we can express (6) state vectors in a more compact way:

$$|^2S+1 S_J; M\rangle = 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) \sum_{s s'} a^+(s, \vec{p}) [A^{(JM)}(-i\sigma_2)]_{s s'} b^+(s', -\vec{p}) |0\rangle, \quad (7)$$

where $A^{(JM)}$ matrices are expressed through the Pauli matrices and the triplet state polarization vectors

$$\vec{n}_1 = \frac{1}{\sqrt{2}} (1, i, 0), \quad \vec{n}_{-1} = \frac{1}{\sqrt{2}} (1, -i, 0), \quad \vec{n}_0 = (0, 0, 1), \quad (8)$$

in the following way

$$A^{(00)} = \frac{1}{\sqrt{2}}, \quad A^{(1M)} = \frac{1}{\sqrt{2}} \vec{n}_M \cdot \vec{\sigma}. \quad (9)$$

To deal with P -wave positronium states, it is convenient instead of $\tilde{\psi}_{1m}(\vec{p})$ eigenstates of the third component of the angular momentum, to introduce Cartesian states

$$\tilde{\psi}^1(\vec{p}) = \frac{1}{\sqrt{2}} (\tilde{\psi}_{1,-1}(\vec{p}) - \tilde{\psi}_{1,1}(\vec{p})), \quad \tilde{\psi}^2(\vec{p}) = \frac{i}{\sqrt{2}} (\tilde{\psi}_{1,-1}(\vec{p}) + \tilde{\psi}_{1,1}(\vec{p})), \quad \tilde{\psi}^3(\vec{p}) = \tilde{\psi}_{1,0}(\vec{p}). \quad (10)$$

We also will need the following non-zero $1 \otimes 1$ Clebsch-Gordan coefficients [29]:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} &= -\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}, \\ \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \end{bmatrix} = 1, \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{3}}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{6}}. \end{aligned} \quad (11)$$

Using (10) and (11), we get for the scalar 3P_0 state

$$\begin{aligned} |^3P_0\rangle &= \frac{2}{\sqrt{6}} \sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \left(\tilde{\psi}^1(\vec{p}) \left[a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) - a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) \right] - \right. \\ &\quad i\tilde{\psi}^2(\vec{p}) \left[a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) + a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) \right] - \\ &\quad \left. \tilde{\psi}^3(\vec{p}) \left[a^+ \left(\frac{1}{2}, \vec{p} \right) b^+ \left(-\frac{1}{2}, -\vec{p} \right) + a^+ \left(-\frac{1}{2}, \vec{p} \right) b^+ \left(\frac{1}{2}, -\vec{p} \right) \right] \right). \end{aligned} \quad (12)$$

Analogous expressions can be obtained easily for three 3P_1 vector states and for five 3P_2 tensor states and it can be checked that all of them has an equivalent compact expression (up to an overall phases) of the form

$$|^3P_J; M\rangle = 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{i=1}^3 \tilde{\psi}^i(\vec{p}) \sum_{s, s'} a^+(s, \vec{p}) [B^{(JM)i}(-i\sigma_2)]_{s, s'} b^+(s', -\vec{p}) |0\rangle, \quad (13)$$

where

$$B^{(00)i} = \frac{1}{\sqrt{6}} \sigma^i, \quad B^{(1M)i} = \frac{1}{2} \epsilon^{ijk} n_M^j \sigma^k, \quad B^{(2M)i} = \frac{1}{\sqrt{2}} h_M^{ij} \sigma^j. \quad (14)$$

Here h_M^{ij} are the polarization tensors for the $J = 2, J_z = M$ states and it is possible to construct them from the $\vec{n}_M$ polarization vectors [30]:

$$h_{\pm 2}^{ij} = n_{\pm 1}^i n_{\pm 1}^j, \quad h_{\pm 1}^{ij} = \frac{1}{\sqrt{2}} \left(n_{\pm 1}^i n_0^j + n_0^i n_{\pm 1}^j \right), \quad h_0^{ij} = \frac{1}{\sqrt{6}} \left(n_1^i n_{-1}^j + n_{-1}^i n_1^j - 2n_0^i n_0^j \right). \quad (15)$$

These h_M^{ij} polarization tensors are traceless, symmetric, mutually orthogonal, and normalized to one:

$$\sum_{i=1}^3 h_M^{ii} = 0, \quad \sum_{i,j=1}^3 h_M^{ij} (h_{M'}^{ij})^* = \delta_{MM'}. \quad (16)$$

Besides, $h_{-M}^{ij} = (h_M^{ij})^*$.

At last, for the remaining three 1P_1 vector states we find equally easily

$$|^1P_1; M\rangle = 2\sqrt{m} \int \frac{d\vec{p}}{(2\pi)^3} \sum_i \tilde{\psi}^i(\vec{p}) \sum_{s, s'} a^+(s, \vec{p}) [C^{(1M)i}(-i\sigma_2)]_{s, s'} b^+(s', -\vec{p}) |0\rangle, \quad (17)$$

where

$$C^{(1M)i} = \frac{1}{\sqrt{2}} n_M^i. \quad (18)$$

It is clear from (7), (13) and (17) that the positronium $\langle 2\gamma |^{2S+1}L_J \rangle$ decay amplitude can be expressed through two-photon annihilation amplitude $M(e^-e^+ \rightarrow 2\gamma) = \langle 2\gamma | e^-(s, \vec{p}) e^+(s, -\vec{p}) \rangle$ of the free electron and positron. So our next task is to study this annihilation amplitude.

III. TWO-PHOTON ANNIHILATION AMPLITUDE OF FREE ELECTRON AND POSITRON

At the lowest order of the perturbation theory, $M(e^-e^+ \rightarrow 2\gamma)$ is described by two Feynman diagrams shown in Fig. 1 and equals to (e is electron's charge, $\epsilon_1 \equiv \epsilon_1(k_1)$ and $\epsilon_2 \equiv \epsilon_2(k_2)$ are photon polarization vectors and we temporarily suppress spin labels on spinors in this section)

$$M(e^-e^+ \rightarrow 2\gamma) = -e^2 \bar{v}(p_2) \left[\frac{\hat{\epsilon}_1^*(\hat{p}_1 - \hat{k}_1 + m) \hat{\epsilon}_2^*}{(p_1 - k_1)^2 - m^2} + \frac{\hat{\epsilon}_2^*(\hat{p}_1 - \hat{k}_2 + m) \hat{\epsilon}_1^*}{(p_1 - k_2)^2 - m^2} \right] u(p_1). \quad (19)$$

Note that

$$(\hat{p}_1 + m) \hat{\epsilon}^* u(p_1) = [2p_1 \cdot \epsilon^* - \hat{\epsilon}^*(\hat{p} - m)] u(p_1) = 2p_1 \cdot \epsilon^* u(p_1),$$

and (19) is equivalent to

$$M(e^-e^+ \rightarrow 2\gamma) = e^2 \bar{v}(p_2) \left[\frac{2p_1 \cdot \epsilon_1^* \hat{\epsilon}_2^* - \hat{\epsilon}_2^* \hat{k}_1 \hat{\epsilon}_1^*}{2p_1 \cdot k_1} + \frac{2p_1 \cdot \epsilon_2^* \hat{\epsilon}_1^* - \hat{\epsilon}_1^* \hat{k}_2 \hat{\epsilon}_2^*}{2p_1 \cdot k_2} \right] u(p_1). \quad (20)$$

But $p_1 \cdot k_1 \approx m^2 - \vec{p} \cdot \vec{k}_1$, $p_1 \cdot k_2 \approx m^2 + \vec{p} \cdot \vec{k}_1$, and $p_1 \cdot k_1 p_1 \cdot k_2 \approx m^4$. This allows to rewrite (20) as follows

$$\begin{aligned} M(e^-e^+ \rightarrow 2\gamma) &= \frac{e^2}{2m^4} \bar{v}(p_2) \left[(m^2 + \vec{p} \cdot \vec{k}_1) (2p_1 \cdot \epsilon_1^* \hat{\epsilon}_2^* - \hat{\epsilon}_2^* \hat{k}_1 \hat{\epsilon}_1^*) + (m^2 - \vec{p} \cdot \vec{k}_1) (2p_1 \cdot \epsilon_2^* \hat{\epsilon}_1^* - \hat{\epsilon}_1^* \hat{k}_2 \hat{\epsilon}_2^*) \right] u(p_1) \\ &\approx \frac{e^2}{2m^4} \bar{v}(p_2) \begin{pmatrix} 0 & 2m^2 a - m^2 \bar{b}_+ + \vec{p} \cdot \vec{k}_1 \bar{b}_- \\ -2m^2 a - m^2 b_+ + \vec{p} \cdot \vec{k}_1 b_- & 0 \end{pmatrix} u(p_1), \end{aligned} \quad (21)$$

FIG. 1: Lowest order Feynman diagrams describing two-photon annihilation of free electron and positron.

where

$$a = \vec{p} \cdot \vec{\epsilon}_1^* \vec{\sigma} \cdot \vec{\epsilon}_2^* + \vec{p} \cdot \vec{\epsilon}_2^* \vec{\sigma} \cdot \vec{\epsilon}_1^*, \quad b_{\pm} = \vec{\epsilon}_1^* \cdot \vec{\sigma} k_2 \cdot \vec{\sigma} \vec{\epsilon}_2^* \cdot \vec{\sigma} \pm \vec{\epsilon}_2^* \cdot \vec{\sigma} k_1 \cdot \vec{\sigma} \vec{\epsilon}_1^* \cdot \vec{\sigma}, \quad \bar{b}_{\pm} = \vec{\epsilon}_1^* \cdot \vec{\sigma} k_2 \cdot \vec{\sigma} \vec{\epsilon}_2^* \cdot \vec{\sigma} \pm \vec{\epsilon}_2^* \cdot \vec{\sigma} k_1 \cdot \vec{\sigma} \vec{\epsilon}_1^* \cdot \vec{\sigma}, \quad (22)$$

and we have used the Coulomb gauge $\epsilon^\mu = (0, \vec{\epsilon})$ for photon polarization vectors, and the chiral representation for gamma matrices.

It is convenient, following Alekseev [2], to express (21) in terms of two-component Pauli spinors. In the chiral representation, adopted in [3],

$$u(p_1) = \begin{pmatrix} \sqrt{\sigma \cdot p_1} \xi \\ \sqrt{\bar{\sigma} \cdot p_1} \xi \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \left(1 - \frac{\vec{\sigma} \cdot \vec{p}}{2m}\right) \xi \\ \left(1 + \frac{\vec{\sigma} \cdot \vec{p}}{2m}\right) \xi \end{pmatrix}, \quad v(p_2) = \begin{pmatrix} -\sqrt{\sigma \cdot p_2} \zeta \\ -\sqrt{\bar{\sigma} \cdot p_2} \zeta \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} -\left(1 + \frac{\vec{\sigma} \cdot \vec{p}}{2m}\right) \zeta \\ -\left(1 - \frac{\vec{\sigma} \cdot \vec{p}}{2m}\right) \zeta \end{pmatrix}, \quad (23)$$

where the approximate equalities, which are valid up to linear in $\vec{p}$ terms, follow from $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ and

$$\sqrt{\sigma \cdot p} = \frac{\sigma \cdot p + m}{\sqrt{2(E+m)}} \approx \frac{1}{2\sqrt{m}} (2m - \vec{\sigma} \cdot \vec{p}), \quad \sqrt{\bar{\sigma} \cdot p} = \frac{\bar{\sigma} \cdot p + m}{\sqrt{2(E+m)}} \approx \frac{1}{2\sqrt{m}} (2m + \vec{\sigma} \cdot \vec{p}). \quad (24)$$

Substituting (23) into (21) and discarding quadratic and higher in $\vec{p}$ terms, we get

$$M(e^- e^+ \rightarrow 2\gamma) \approx \frac{e^2}{2m^3} \zeta^+ \left(-4m^2 a + m^2 (\bar{b}_+ - b_+) + \vec{p} \cdot \vec{k}_1 (b_- - \bar{b}_-) + \frac{m}{2} [b_+ + \bar{b}_+, \vec{\sigma} \cdot \vec{p}] \right) \xi. \quad (25)$$

But, with required precision,

$$k_1 \cdot (\sigma + \bar{\sigma}) = k_2 \cdot (\sigma + \bar{\sigma}) = 2m, \quad k_1 \cdot (\sigma - \bar{\sigma}) = -2\vec{k}_1 \cdot \vec{\sigma}, \quad k_2 \cdot (\sigma - \bar{\sigma}) = -2\vec{k}_2 \cdot \vec{\sigma} = 2\vec{k}_1 \cdot \vec{\sigma},$$

and therefore

$$b_{\pm} - \bar{b}_{\pm} = 2(\vec{\epsilon}_1^* \cdot \vec{\sigma} \vec{k}_1 \cdot \vec{\sigma} \vec{\epsilon}_2^* \cdot \vec{\sigma} \mp \vec{\epsilon}_2^* \cdot \vec{\sigma} \vec{k}_1 \cdot \vec{\sigma} \vec{\epsilon}_1^* \cdot \vec{\sigma}), \quad b_+ + \bar{b}_+ = 2m(\vec{\epsilon}_1^* \cdot \vec{\sigma} \vec{\epsilon}_2^* \cdot \vec{\sigma} + \vec{\epsilon}_2^* \cdot \vec{\sigma} \vec{\epsilon}_1^* \cdot \vec{\sigma}) = 4m \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*. \quad (26)$$

Hence $[b_+ + \bar{b}_+, \vec{\sigma} \cdot \vec{p}] = 0$. To simplify further, we use the equality

$$\vec{\sigma} \cdot \vec{a} \vec{\sigma} \cdot \vec{b} = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}), \quad (27)$$

from which it follows that

$$\vec{\epsilon}_1^* \cdot \vec{\sigma} \vec{k}_1 \cdot \vec{\sigma} \vec{\epsilon}_2^* \cdot \vec{\sigma} = i \left[\vec{\epsilon}_2^* \cdot (\vec{\epsilon}_1^* \times \vec{k}_1) + i\vec{\sigma} \cdot [(\vec{\epsilon}_1^* \times \vec{k}_1) \times \vec{\epsilon}_2^*] \right] = -i\vec{k}_1 \cdot (\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) - \vec{\sigma} \cdot \vec{k}_1 \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*, \quad (28)$$

because $\vec{k}_1 \cdot \vec{\epsilon}_1^* = 0$ and $\vec{k}_1 \cdot \vec{\epsilon}_2^* = -\vec{k}_2 \cdot \vec{\epsilon}_2^* = 0$. Using (28) in (26), we get

$$b_+ - \bar{b}_+ = -4i\vec{k}_1 \cdot (\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*), \quad b_- - \bar{b}_- = -4\vec{\sigma} \cdot \vec{k}_1 \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*, \quad (29)$$

and the final form of the two-photon annihilation amplitude, valid up to linear in $\vec{p}$ terms:

$$M(e^- e^+ \rightarrow 2\gamma) \approx -\frac{2e^2}{m} \zeta^+ \left[\vec{p} \cdot \vec{\epsilon}_1^* \vec{\sigma} \cdot \vec{\epsilon}_2^* + \vec{p} \cdot \vec{\epsilon}_2^* \vec{\sigma} \cdot \vec{\epsilon}_1^* - i\vec{k}_1 \cdot (\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) + \frac{1}{m^2} \vec{p} \cdot \vec{k}_1 \vec{\sigma} \cdot \vec{k}_1 \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^* \right] \xi. \quad (30)$$

This result is consistent with the ones given in [31] and [7] (after a typo is corrected in [7] which lead to mutual interchange of the photon polarization vectors).

IV. TWO-PHOTON DECAYS OF S -WAVE POSITRONIUM

To warm up, let's calculate two-photon decay width of S -wave positronium. It is clear from (7) that the decay amplitude has the following form

$$M(^{2S+1}S_J \rightarrow 2\gamma) = \frac{1}{\sqrt{m}} \int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) \sum_{ss'} [A^{(JM)}(-i\sigma_2)]_{ss'} M(e^-(s, \vec{p}) e^+(s', -\vec{p}) \rightarrow 2\gamma). \quad (31)$$

In deriving prefactor in (31), we have taken into account the relativistic normalization of the one-particle states $|e^-(s, \vec{p})\rangle = \sqrt{2E_p} a^-(s, \vec{p})|0\rangle$ and that $E_p \approx m$ at desired accuracy.

It follows from (30) that

$$M(e^-(s, \vec{p}) e^+(s, -\vec{p}) \rightarrow 2\gamma) = \zeta^{s'} + \Lambda \xi^s, \quad (32)$$

where Λ is some 2×2 matrix acting on spinor indices. But

$$\sum_{ss'} [A^{(JM)}(-i\sigma_2)]_{ss'} \zeta^{s'} + \Lambda \xi^s = \text{Tr}(\hat{\pi}_A^{(JM)} \Lambda), \quad (33)$$

with

$$\hat{\pi}_A^{(JM)} = \sum_{ss'} [A^{(JM)}(-i\sigma_2)]_{ss'} \xi^s \zeta^{s'} + \quad (34)$$

as a selector operator — a 2×2 matrix in spinor space which discriminates between the singlet $J = 0$ and triplet $J = 1$ states.

To calculate $\hat{\pi}_A^{(JM)}$, let's recall that the particle and antiparticle two-component spinors are (note the flipped nature of antiparticle spinors) [3]

$$\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \zeta^\uparrow = \xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \zeta^\downarrow = -\xi^\uparrow = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (35)$$

Then, because the only non-zero components of $-i\sigma_2$ are $(-i\sigma_2)_{\uparrow\downarrow} = -1$ and $(-i\sigma_2)_{\downarrow\uparrow} = 1$, we will have

$$\hat{\pi}_A^{(JM)} = \sum_{ss's''} A_{ss''}^{(JM)} (-i\sigma_2)_{s''s'} \xi^s \zeta^{s'} + \sum_s \left(-A_{s\uparrow}^{(JM)} \xi^s \zeta^{\downarrow+} + A_{s\downarrow}^{(JM)} \xi^s \zeta^{\uparrow+} \right),$$

and

$$\hat{\pi}_A^{(JM)} = A_{\uparrow\uparrow}^{(JM)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) + A_{\downarrow\uparrow}^{(JM)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1, 0) + A_{\uparrow\downarrow}^{(JM)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0, 1) + A_{\downarrow\downarrow}^{(JM)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0, 1) = \begin{pmatrix} A_{\uparrow\uparrow}^{(JM)} & A_{\downarrow\uparrow}^{(JM)} \\ A_{\uparrow\downarrow}^{(JM)} & A_{\downarrow\downarrow}^{(JM)} \end{pmatrix}. \quad (36)$$

Note that $\hat{\pi}_A^{(JM)}$ and $A^{(JM)}$ act in different spaces — the first one acts on spinor indices and the second one acts on spin labels. However (36) indicates that in these spaces they act identically, that is

$$\hat{\pi}_A^{(00)} = \frac{1}{\sqrt{2}}, \quad \hat{\pi}_A^{(1M)} = \frac{1}{\sqrt{2}} \vec{n}_M \cdot \vec{\sigma}. \quad (37)$$

According to (30), at zeroth order

$$\Lambda^{(0)} = \frac{2ie^2}{m} \vec{k}_1 \cdot (\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) \quad (38)$$

doesn't depend on $\vec{p}$. Then the remaining integral

$$\int \frac{d\vec{p}}{(2\pi)^3} \tilde{\psi}_{00}(\vec{p}) = \psi(\vec{x} = 0)$$

just gives the position space positronium wave function at the origin. Therefore

$$M(^{2S+1}S_J \rightarrow 2\gamma) = \frac{2ie^2}{m\sqrt{m}} \vec{k}_1 \cdot (\vec{\epsilon}_1^* \times \vec{\epsilon}_2^*) \psi(\vec{x} = 0) \text{Tr}(\hat{\pi}_A^{(JM)}). \quad (39)$$

To calculate the decay rate, we should module square the amplitude (39), sum over the final state photon polarizations, average over the initial state positronium polarizations, and integrate over the Lorentz invariant final state phase space according to the general formula (overbar indicates the above mentioned summation and averaging over polarizations, $P = (M_{Ps}, \vec{0})$ is the positronium 4-momentum)

$$d\Gamma(^{2S+1}L_j \rightarrow 2\gamma) = \frac{1}{2M_{Ps}} \overline{|M(^{2S+1}L_j \rightarrow 2\gamma)|^2} \frac{d\vec{k}_1}{(2\pi)^3 2|\vec{k}_1|} \frac{d\vec{k}_2}{(2\pi)^3 2|\vec{k}_2|} (2\pi)^4 \delta^{(4)}(P - k_1 - k_2). \quad (40)$$

In Coulomb gauge, the photon polarization sums can be performed by using

$$\sum_{\epsilon} \epsilon^{*i}(\vec{k}) \epsilon^j(\vec{k}) = \delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}. \quad (41)$$

Then (from here, it is assumed that repeated indices are implicitly summed over)

$$\sum_{\epsilon_1 \epsilon_2} k_1^i k_1^j \epsilon^{imn} \epsilon^{im'n'} \epsilon_1^{*m} \epsilon_2^{*n} \epsilon_1^{m'} \epsilon_2^{n'} = \epsilon^{imn} \epsilon^{jmn} k_1^i k_1^j = 2|\vec{k}_1|^2,$$

and

$$\overline{|M(^{2S+1}S_j \rightarrow 2\gamma)|^2} = \frac{1}{2J+1} \sum_{\epsilon_1 \epsilon_2} |M(^{2S+1}S_j \rightarrow 2\gamma)|^2 = \frac{8e^4}{m^3} |\vec{k}_1|^2 |\psi(\vec{x}=0)|^2 |\text{Tr}(\hat{\pi}_A^{JM})|^2. \quad (42)$$

Since Pauli matrices are traceless and $\hat{\pi}_A^{(1M)} = \vec{n}_M \cdot \vec{\sigma}/\sqrt{2}$, we immediately get that the spin-triplet S -wave positronium (orthopositronium) doesn't decay into two photons. Of course this is just what is expected from the C -parity conservation in electromagnetic decays: for the two-photon final state $C = (-1)^2 = 1$ while for the $L = 0, S = 1$ orthopositronium $C = (-1)^{L+S} = -1$.

For the spin-singlet S -wave positronium (parapositronium) $\hat{\pi}_A^{(00)} = 1/\sqrt{2}$ and $\text{Tr}(\hat{\pi}_A^{(00)}) = \sqrt{2}$. Then from (40) and (42) we get (the first $1/2$ factor accounts for the identity of the final state photons)

$$\Gamma(^1S_0 \rightarrow 2\gamma) = \frac{1}{2} \frac{e^4 |\psi(\vec{x}=0)|^2}{\pi m^4} \int_0^\infty |\vec{k}_1|^2 \delta(2m - 2|\vec{k}_1|) d|\vec{k}_1| = \frac{e^4 |\psi(\vec{x}=0)|^2}{4\pi m^2} = \frac{4\pi\alpha^2}{m^2} |\psi(\vec{x}=0)|^2. \quad (43)$$

What remains is to use

$$|\psi(\vec{x}=0)|^2 = \frac{m^3 \alpha^3}{8\pi},$$

valid for the positronium ground state (for radial excitations with the principal quantum number n this quantity is n^3 times less), and obtain a well known result of Pirenne and Wheeler [32, 33]

$$\Gamma(^1S_0 \rightarrow 2\gamma) = \frac{m\alpha^5}{2}. \quad (44)$$

V. TWO-PHOTON DECAYS OF P -WAVE POSITRONIUM

Hydrogen-like wave function of positronium has the form [34]

$$\psi_{nlm}(\vec{r}) = \left[\frac{1}{2n} \left(\frac{2}{na_0} \right)^3 \frac{(n-l-1)!}{(n+l)!} \right]^{1/2} \left(\frac{2r}{na_0} \right)^l e^{-r/na_0} L_{n-l-1}^{2l+1} \left(\frac{2r}{na_0} \right) Y_{lm}(\theta, \phi), \quad (45)$$

where $a_0 = 2/(m\alpha)$ is the Bohr radius for positronium and

$$L_n^m(x) = (n+m)! \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!(k+m)!} x^k \quad (46)$$

are associated Laguerre polynomials [35].

Some words of caution is perhaps appropriate here. In the physical and mathematical literature one encounters two commonly used definitions of Laguerre and associated Laguerre polynomials. This constitutes a possible source of confusion to many students [34, 36]. In this paper we adopt the conventions of Arfken and Weber [35], but one should bear in mind that conventions used can change from book to book. For example, Landau and Lifshits [37] use different conventions (namely, that of Spiegel [38]) that changes the normalization coefficient, as well as the lower index from $n - l - 1$ to $n + l$. Griffiths [39] follow Arfken and Weber when relating associated Laguerre polynomials to the ordinary Laguerre polynomials but uses different normalization in the definition of the latter, and this leads to different normalization coefficient than in (45).

It is clear from (45) that $\psi_{nlm}(\vec{0}) = 0$, if $l \neq 0$. Therefore the zeroth-order approximation of the previous section cannot be applied in the case of P -wave positronium decay and here our work in the pre-previous section pays off: as (30) shows

$$\Lambda = \Lambda^{(0)} + p^j \Lambda^{(1)j}, \quad (47)$$

where $\Lambda^{(0)}$ is given by (38) and it doesn't contribute to the P -wave positronium decay, while

$$\Lambda^{(1)j} = -\frac{2e^2}{m} \left[\vec{\sigma} \cdot \vec{\epsilon}_2^* \epsilon_1^{*j} + \vec{\sigma} \cdot \vec{\epsilon}_1^* \epsilon_2^{*j} + \frac{k_1^j}{m^2} \vec{\sigma} \cdot \vec{k}_1 \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^* \right]. \quad (48)$$

Positronium in the 1P_1 state with $S = 0, L = 1$ cannot decay into two photons due to C -parity conservation. It is reassuring that our formalism confirms this: $M(^1P_1 \rightarrow 2\gamma) \sim p^j \tilde{\psi}^i(\vec{p}) \text{Tr}(\hat{\pi}_C^{(1M)i} \Lambda^{(1)j}) = 0$, because $\Lambda^{(1)j}$ is traceless and

$$\hat{\pi}_C^{(1M)i} = \sum_{ss'} [C^{(1M)i}(-i\sigma_2)]_{ss'} \xi^s \zeta^{s'+} = \frac{1}{\sqrt{2}} n_M^i$$

is proportional to the unit matrix.

As for the 3P_J states, from (13) we have

$$M(^3P_J \rightarrow 2\gamma) = \frac{1}{\sqrt{m}} \text{Tr}(\hat{\pi}_B^{(JM)i} \Lambda^{(1)j}) \int \frac{d\vec{p}}{(2\pi)^3} p^j \tilde{\psi}^i(\vec{p}), \quad (49)$$

where

$$\hat{\pi}_B^{(JM)i} = \sum_{ss'} [B^{(1M)i}(-i\sigma_2)]_{ss'} \xi^s \zeta^{s'+} = \begin{cases} \frac{\sigma^i}{\sqrt{6}}, & \text{if } (J, M) = (0, 0), \\ \frac{1}{2} \epsilon^{ijk} n_M^j \sigma^k, & \text{if } (J, M) = (1, M), \\ \frac{1}{\sqrt{2}} h_M^{ij} \sigma^j, & \text{if } (J, M) = (2, M). \end{cases} \quad (50)$$

Let us first calculate the integral in (49). We have

$$\int \frac{d\vec{p}}{(2\pi)^3} p^j \tilde{\psi}^i(\vec{p}) = -i \nabla^j \psi^i(\vec{x})|_{\vec{x}=0}. \quad (51)$$

Since

$$\frac{1}{\sqrt{2}} (Y_{1,-1}(\theta, \phi) - Y_{1,1}(\theta, \phi)) = \sqrt{\frac{3}{4\pi}} \frac{x}{r}, \quad \frac{i}{\sqrt{2}} (Y_{1,-1}(\theta, \phi) + Y_{1,1}(\theta, \phi)) = \sqrt{\frac{3}{4\pi}} \frac{y}{r}, \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \frac{z}{r},$$

it follows from (10) and (45) that Cartesian wave function of the P -wave positronium is

$$\psi^i(\vec{x}) = x^i \sqrt{\frac{3}{4\pi} \frac{1}{2n} \left(\frac{2}{na_0} \right)^3 \frac{(n-2)!}{(n+1)!} \frac{2}{na_0}} e^{-r/na_0} L_{n-2}^3 \left(\frac{2r}{na_0} \right), \quad (52)$$

and therefore

$$-i \nabla^j \psi^i(\vec{x})|_{\vec{x}=0} = -i \delta^{ij} \sqrt{\frac{3}{4\pi} \frac{1}{2n} \left(\frac{2}{na_0} \right)^3 \frac{1}{(n-1)n(n+1)} \frac{2}{na_0}} L_{n-2}^3(0). \quad (53)$$

But from (46)

$$L_{n-2}^3(0) = \frac{(n+1)!}{(n-2)!3!}. \quad (54)$$

Combining (51), (53) and (54), we finally get

$$\int \frac{d\vec{p}}{(2\pi)^3} p^j \tilde{\psi}^i(\vec{p}) = -i\delta^{ij} \sqrt{\frac{n^2-1}{96\pi n^5}} (m\alpha)^5, \quad (55)$$

and

$$M(^3P_J \rightarrow 2\gamma) = -im^2 \sqrt{\frac{(n^2-1)\alpha^5}{96\pi n^5}} \text{Tr}(\hat{\pi}_B^{(JM)i} \Lambda^{(1)i}). \quad (56)$$

Now it's time to calculate traces in (56) and this can be easily done by using (27). The results are

$$\text{Tr}(\hat{\pi}_B^{(00)i} \Lambda^{(1)i}) = -2\sqrt{6} \frac{e^2}{m} \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*, \quad \text{Tr}(\hat{\pi}_B^{(2M)i} \Lambda^{(1)i}) = -2\sqrt{2} \frac{e^2}{m} \left(2h_M^{ij} \epsilon_1^{*i} \epsilon_2^{*j} + \frac{\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*}{m^2} h_M^{ij} k_1^i k_1^j \right), \quad (57)$$

and

$$\text{Tr}(\hat{\pi}_B^{(1M)i} \Lambda^{(1)i}) = -\frac{2e^2}{m} \left[\vec{\epsilon}_2^* \cdot (\vec{\epsilon}_1^* \times \vec{n}_M) + \vec{\epsilon}_1^* \cdot (\vec{\epsilon}_2^* \times \vec{n}_M) + \frac{\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*}{m^2} \vec{k}_1 \cdot (\vec{k}_1 \times \vec{n}_M) \right] = 0. \quad (58)$$

The last result implies that 3P_1 state doesn't decay into two-photons although it has positive C -parity. This time the decay is forbidden by the Landau-Yang theorem [40, 41] which states that two real photons, when referred to their center of mass frame, cannot be in a state of total angular momentum one. The proof of the theorem is simple and only makes use of such basic concepts as the superposition principle, Bose statistics and transversality of real photons, and rotational invariance [42]. As we see, our formalism correctly reproduces this selection rule too.

In the case of 3P_0 state, the decay amplitude equals to

$$M(^3P_0 \rightarrow 2\gamma) = i \frac{me^2}{2} \sqrt{\frac{(n^2-1)\alpha^5}{\pi n^5}} \vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*. \quad (59)$$

To calculate the decay rate, we should module square this amplitude and sum over the photon polarizations using (41). In this way we get

$$\sum_{\epsilon_1, \epsilon_2} |\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*|^2 = \sum_{\epsilon_1, \epsilon_2} \epsilon_1^{*i} \epsilon_1^j \epsilon_2^{*i} \epsilon_2^j = 2, \quad (60)$$

and

$$|\overline{M(^3P_0 \rightarrow 2\gamma)}|^2 = \sum_{\epsilon_1, \epsilon_2} |M(^3P_0 \rightarrow 2\gamma)|^2 = 8\pi m^2 \frac{n^2-1}{n^5} \alpha^7. \quad (61)$$

Eq.(40) indicates that the decay rate and the corresponding squared amplitude are related by

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2 M_{Ps}} |\overline{M}|^2. \quad (62)$$

In our case the squared amplitude (61) is just a constant, the integration of (62) is trivial and we finally get (the factor 1/2 accounts for the identical final state photons)

$$\Gamma(^3P_0 \rightarrow 2\gamma) = \frac{1}{2} \int \frac{d\Gamma(^3P_0 \rightarrow 2\gamma)}{d\Omega} d\Omega = \frac{n^2-1}{8n^5} m\alpha^7. \quad (63)$$

The amplitude for the $^3P_2 \rightarrow 2\gamma$ decay is more complicated: from (56) and (57) we get

$$M(^3P_2 \rightarrow 2\gamma) = ie^2 m \sqrt{\frac{(n^2-1)\alpha^5}{12\pi n^5}} \left(2h_M^{ij} \epsilon_1^{*i} \epsilon_2^{*j} + \frac{\vec{\epsilon}_1^* \cdot \vec{\epsilon}_2^*}{m^2} h_M^{ij} k_1^i k_1^j \right). \quad (64)$$

The next step is to module square this amplitude, perform the photon polarization sums, and average over the initial state positronium polarization (that is sum over M and divide by $2J + 1 = 5$). This requires some algebra, somewhat simplified by the fact that h_M^{ij} polarization tensors are traceless, symmetric and normalized to one. Besides, $\vec{k}_1^2 \approx m^2$. After the dust settles we find

$$|M(^3P_2 \rightarrow 2\gamma)|^2 = \frac{8\pi(n^2 - 1)m^2\alpha^7}{15n^5} \sum_M \left(2 - 4h_M^{ij}h_M^{*im} \frac{k_1^j k_1^m}{m^2} + h_M^{ij}h_M^{*mn} \frac{k_1^i k_1^j k_1^m k_1^n}{m^4} \right). \quad (65)$$

Then from (62) we get

$$\Gamma(^3P_2 \rightarrow 2\gamma) = \frac{m\alpha^7}{32\pi} \frac{n^2 - 1}{15n^5} \sum_M \left(8\pi - 4h_M^{ij}h_M^{*im} \int \frac{k_1^j k_1^m}{m^2} d\Omega + h_M^{ij}h_M^{*mn} \int \frac{k_1^i k_1^j k_1^m k_1^n}{m^4} d\Omega \right). \quad (66)$$

The integrals involved give the completely symmetric second and forth rank tensors respectively. As the integrands don't contain any external vector or tensor, these tensors should be expressible in terms of δ^{ij} tensors only. Thus (the last expression is obtained by symmetrization of $\delta^{ij}\delta^{mn}$)

$$\int \frac{k_1^j k_1^m}{m^2} d\Omega = A\delta^{jm}, \quad \int \frac{k_1^i k_1^j k_1^m k_1^n}{m^4} d\Omega = B(\delta^{ij}\delta^{mn} + \delta^{im}\delta^{jn} + \delta^{in}\delta^{jm}).$$

To find the unknown coefficients A and B , we simply contract these tensors:

$$4\pi = \int \frac{k_1^j k_1^j}{m^2} d\Omega = A\delta^{jj} = 3A, \quad 4\pi = \int \frac{k_1^i k_1^i k_1^m k_1^m}{m^4} d\Omega = B(\delta^{ii}\delta^{mm} + \delta^{im}\delta^{im} + \delta^{im}\delta^{im}) = 15B.$$

Therefore $A = 4\pi/3$, $B = 4\pi/15$ and

$$\int \frac{k_1^j k_1^m}{m^2} d\Omega = \frac{4\pi}{3} \delta^{jm}, \quad \int \frac{k_1^i k_1^j k_1^m k_1^n}{m^4} d\Omega = \frac{4\pi}{15} (\delta^{ij}\delta^{mn} + \delta^{im}\delta^{jn} + \delta^{in}\delta^{jm}). \quad (67)$$

It remains to substitute (67) integrals into (66) and remember that h_M^{ij} tensors are traceless and normalized to one. As a result, we get

$$\Gamma(^3P_2 \rightarrow 2\gamma) = \frac{m\alpha^7}{32\pi} \frac{n^2 - 1}{15n^5} \sum_M \left(8\pi - \frac{16\pi}{3} + \frac{8\pi}{15} \right) = \frac{n^2 - 1}{30n^5} m\alpha^7. \quad (68)$$

Our final results (63) and (68) are precisely the ones obtained by Tumanov [1] and Alekseev [2]. In particular, for the $n = 2$ states the decay widths are

$$\Gamma(2^3P_0 \rightarrow 2\gamma) = \frac{3}{256} m\alpha^7, \quad \Gamma(2^3P_2 \rightarrow 2\gamma) = \frac{1}{320} m\alpha^7. \quad (69)$$

VI. CONCLUDING REMARKS

We hope that this rather detailed presentation of two-photon decays of P -wave positronium will be helpful for quantum field theory students. The standard approach used in this note is lucid enough and well motivated. However thoughtful students can feel a necessity in a more powerful and complete framework.

Our main assumption was a factorization of the bound state dynamics from the annihilation process. However such an approach violates energy conservation: electron and positron that annihilate are on-shell and thus their total energy is greater than M_{Ps} . Of course for non-relativistic systems, like positronium, the difference is of the order of α^2 and can be neglected at lowest order. However thoughtful students can wonder how the off-shellness of constituents can be reintroduced perturbatively at higher order calculations (an example can be found in [43]).

The ratio $\Gamma(2^3P_0 \rightarrow 2\gamma)/\Gamma(2^3P_2 \rightarrow 2\gamma) = 15/4$, which follows from the previous section, is often quoted in the context of quarkonium decays. However it should be beared in mind that for quarkonia relativistic corrections are important and can lead to significant modification of this ratio [31].

Another potential source of concern is subtle consequences of gauge invariance. As was shown by Low [44], gauge invariance and analyticity implies that the decay amplitude of neutral bosons vanishes in the soft photon limit. Since the standard factorization treats intermediate charged states as on-shell, emission of soft photons by this particles

is accompanied by well known infrared singularities. Although the standard treatment ensures the cancellation of these infrared singularities, the amplitude, for example, for the $Ps \rightarrow 3\gamma$ decay, being finite, doesn't vanish in the soft photon limit, in contradiction to the Low's theorem [45]. Special efforts to correctly account for binding energy corrections are needed to reinforce the theorem [45].

We hope that the two-photon decays of P -wave positronium can serve for attentive students as a vista to a vast new land called a bound state problem in relativistic quantum field theory. As an initial guidebook into this interesting land, we recommend relatively recent PhD thesis [11].

Acknowledgments

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