LONGITUDINAL PATTERNING OF STRUCTURED LIGHT AND ITS APPLICATIONS IN SENSING

by

Ahmed H. Dorrah

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto

© Copyright 2019 by Ahmed H. Dorrah
Abstract

Longitudinal Patterning of Structured Light and its Applications in Sensing

Ahmed H. Dorrah
Doctor of Philosophy
Graduate Department of Electrical and Computer Engineering
University of Toronto
2019

Recent advances in laser beam shaping using tools such as Spatial Light Modulators (SLMs) or phase plates has made it possible to engineer various properties of light in a sophisticated manner, thus opening many new possibilities in Structured Light. In particular, much attention has been dedicated to engineering the properties of non-diffracting light beams, mainly due to their long depth-of-field and self-healing behavior. Notably, controlling the intensity profile of non-diffracting beams has led to many advances in optical tweezers, micro-manipulation, and atom guiding. In addition to intensity, beams with helical phase fronts have been constructed such that they carry Orbital Angular Momentum (OAM); a conserved degree-of-freedom that can be transferred to micro-particles or deployed as means of optical communications. Furthermore, the wavelength and state of polarization are two additional properties of light that play a decisive role in applications such as materials processing, microscopy, and optical communications.

While much effort has been dedicated to structuring the aforementioned properties of non-diffracting light beams — namely its intensity, OAM, polarization, and wavelength — at a single 2D plane, controlling such properties at multiple planes along the beam’s axis has scantily been investigated. In this dissertation, we establish a systematic method to design, create, and analyze a new generation of non-diffracting Structured Light beams in which various properties of light can be controlled, separately or simultaneously, along the beam’s longitudinal axis of propagation. This includes longitudinal control over the intensity profile, OAM, polarization, wavelength, and even the propagation trajectory of the beam (almost at-will); thus creating a new optical toolkit with many degrees-of-freedom to meet the rising demand for advanced Structured Light. In addition, we report on unusual physical phenomena; for instance, one in which the OAM of light can be reversed/changed locally without violating any conservation laws. Finally, we show that Longitudinally Structured Light can be exploited to measure the index of refraction of fluids; providing high resolution measurements over a wide dynamic range, and thus solving an old trade-off that has hitherto been untackled in refractive index sensing.
Acknowledgements

There are many people that deserve acknowledgment for their contributions to my dissertation. I wish to thank my friends and colleagues at the University of Toronto, Lawrence Berkeley Lab, and the University of Witwatersrand for providing great advice and fun memories. I also wish to thank my mentors, colleagues, collaborators, friends, and family as follows:

Firstly, I would like to express my sincere gratitude to my research advisor, Prof. Mo Mojahedi, who has been my mentor throughout my graduate studies at the University of Toronto since I started my M.A.Sc. program in 2012 until the end of my Ph.D. program in 2018. Prof Mojahedi: I sincerely thank you for your continuous support, patience, mentorship, technical advice and knowledge, constructive feedback, and valuable discussions both in science and in life as well. Throughout my years of graduate studies you have significantly developed my research skills and scientific writing. The successful completion of this thesis would not have been possible without your advice and guidance.

I would also like to thank the members of my Ph.D. thesis committee: I start by thanking Prof. Robert W. Boyd (co-affiliated with University of Ottawa and University of Rochester) for serving as the external examiner of my thesis, for providing valuable insights on my dissertation, and for suggesting visionary directions for future research. I am also grateful to Prof. J. Stewart Aitchison, Prof. Peter R. Herman, and Prof. Costas D. Sarris not only for their insightful comments and feedback on my thesis, but also for their stimulating discussions which helped improve the final presentation of my dissertation from various perspectives. In addition, I am grateful to Prof. Adrian I. Nachman for chairing my Ph.D. departmental oral examination and to Prof. G. Andrew Woolley for smoothly administering my Ph.D. final oral examination. My sincere thanks also go to Prof. John E. Sipe for participating in my proposal committee and for providing very useful advice at an early stage of my thesis.

In addition to the valuable mentorship I received at the University of Toronto, I have been fortunate to collaborate with many brilliant researchers from other international institutions. In particular, I wish to thank Prof. Michel Zamboni-Rached from the State University of Campinas in Brazil for all his input in the theory of non-diffracting beams and for his technical feedback and thorough analysis of my experimental results. I also wish to acknowledge Prof. Hoi-Ying Holman — the director of the Berkeley Synchrotron Infrared Structural Biology program — for hosting me in her research group at the Lawrence Berkeley National Lab to study Structured Light at the sub-diffraction limit. In this regard, I wish to thank Dr. Arthur Montazeri who made my stay at Berkeley very rewarding and smooth. Getting exposed to such diverse environment of researchers has helped me work on tackling complex problems in science and certainly improved my communication skills with scientists from outside my field. I am also grateful to Prof. Andrew Forbes from the Physics department at the University of Witwatersrand in South Africa for hosting me in his Structured Light lab. Working in the Structured Light Lab was arguably one of the most fun and rewarding experiences in my Ph.D. program.
Furthermore, the support of my colleagues and friends in the Department of Electrical and Computer Engineering at the University of Toronto had a remarkable impact on the completion of this thesis. It also provided me with the motivation to keep going. For this reason, I would like to thank my fellow graduate students for providing both a friendly environment in addition to many useful technical discussions as well. In this regard, I wish to thank my colleagues and friends: Dr. Arnab Dewanjee, Dr. Xiao Sun, Dr. Muhammad Z. Alam, Dr. Farshid Barhami, Dr. Niklas Caspers, Dr. Pisek Kultavewuti, Maroua Alarbi, Kevin Joseph, Ehsan Alimohammadian, Moein Shayegannia, Ali Zeinedine, Dave Jeong, and Ayman Dorrah.

Last but not least, I am indebted to my dear family for their unconditional love and perpetual support without which I would have never reached where I am now. Thank you very much!

Finally, I wish to acknowledge the financial support received from the Natural Sciences and Engineering Research Council of Canada (NSERC CGS – Grant Number: 601055), Michael Smith Foreign Study Supplement Scholarship (Grant Number: 607547), Ontario Graduate Scholarship, OSA Robert S. Hilbert Grant, and SPIE Education Scholarship programs.
# Contents

List of Figures viii  
List of Abbreviations xi  
Nomenclature xiii  

1 Introduction 1  
   1.1 Historical Background 1  
   1.2 Thesis Objectives 4  
      1.2.1 Engineering the Degrees-of-Freedom of Non-Diffracting Beams along the Axis of Propagation 5  
      1.2.2 Investigating the Physical Dynamics Associated with LSL 6  
      1.2.3 Exploiting LSL in Refractive Index Sensing 6  
   1.3 Thesis Organization 7  

2 Bessel Beams and the Frozen Wave Method 10  
   2.1 Non-Diffracting Propagation 10  
   2.2 Bessel Beam Generation 11  
   2.3 Modulating the Envelope of Bessel Beams 13  
      2.3.1 The Bessel Beam Profile 13  
      2.3.2 Self-Imaging of Bessel Beams 14  
      2.3.3 The Frozen Wave Method 14  

3 Shaping the Longitudinal Intensity of Diffraction-Attenuation-Resistant Beams 17  
   3.1 Motivation 17  
   3.2 Concept 18  
   3.3 Experimental Setup 19  
   3.4 Results 21  
      3.4.1 Constant Intensity Profile 21
3.4.2 Multi-Foci Intensity Pattern ........................................... 23
3.4.3 Growing Intensity Profile ............................................. 24
3.5 Summary ........................................................................... 26

4 Evolution of Orbital Angular Momentum in Longitudinally Structured Light 27
4.1 Overview ........................................................................... 28
4.2 Controlling the Topological Charge of Vortex Beams along the Propagation Direction . 30
  4.2.1 Concept ....................................................................... 30
  4.2.2 Experimental Setup ....................................................... 32
4.3 Generating and Measuring Longitudinally Structured Vortex Beams with Varying OAM and Charge ................................................................. 32
  4.3.1 Pattern 1: Reversing the Sign of the Topological Charge ............ 33
    Modal Decomposition for Pattern 1 ........................................ 33
    Mechanism of Topological Charge Inversion ............................ 35
    Conservation of OAM and Charge ........................................... 38
  4.3.2 Pattern 2: Changing the Magnitude of the Topological Charge .......... 40
    Modal Decomposition for Pattern 2 ........................................ 41
    Mechanism of Topological Charge Transition .......................... 42
    Conservation of OAM and Charge ........................................... 44
  4.3.3 Discussion ..................................................................... 46
4.4 Controlling the Topological Charge in Rotating Petal-Like Structures .......... 46
  4.4.1 Overview ..................................................................... 46
  4.4.2 Pattern 1: Reversing the Sense of Rotation ............................ 48
  4.4.3 Pattern 2: Changing the Order of Phase-Twist ....................... 49
4.5 Summary ........................................................................... 51

5 Controlling the Polarization State and Intensity of Diffraction-Attenuation-Resistant Beams along their Propagation Direction 53
5.1 Overview ........................................................................... 54
5.2 Concept ............................................................................ 54
5.3 Experimental Procedure ...................................................... 56
5.4 Experimental Results .......................................................... 58
  5.4.1 Pattern 1: Longitudinal Control Over the Linear Polarization State .......... 58
  5.4.2 Pattern 2: Simultaneous Control Over Polarization and Intensity Level ...... 60
  5.4.3 Pattern 3: Evolving from Linear to Radial Polarization ................... 61
  5.4.4 Pattern 4: Attenuation-Resistant Beam with Longitudinally Varying SoP ...... 63
  5.4.5 Pattern 5: Attenuation-Resistant Beam with Longitudinally Varying SoP and Charge 65
6 Engineering the Wavelength and Topological Charge of Non-Diffracting Beams along their Axis of Propagation

6.1 Overview

6.2 Concept

6.2.1 Theory

6.2.2 Experimental Setup

6.3 Experimental Results

6.3.1 Pattern 1: Multi-Chromatic Beam with Two Foci

6.3.2 Pattern 2: Multi-Chromatic Beam with Alternating Color

6.3.3 Pattern 3: Multi-Chromatic Vortex Beam with Varying Charge $\ell$

6.3.4 Pattern 4: Multi-Chromatic Petal-Like Beam with Varying Charge

6.4 Summary

7 Curved Frozen Waves Following Arbitrary Spiral and Snake-Like Trajectories in Air

7.1 Overview

7.2 Concept

7.3 Experimental Setup

7.4 Experimental Results

7.4.1 Pattern 1: Spiraling Beam Reversing its Helicity

7.4.2 Pattern 2: Snake-Like Beam Evolving from On-Axis to Off-Axis Propagation

7.5 Summary

8 Experimental Demonstration of Wide-Range Tunable Refractometer Based on OAM of Longitudinally Structured Light

8.1 Overview

8.2 Background

8.3 Concept

8.4 Theoretical Formulation

8.5 Experimental Setup

8.6 Experimental Results

8.7 Improving Sensitivity by Increasing the Interaction Length

8.8 Expanding the Dynamic Range of RI Sensing

8.9 Discussion

8.10 Summary
9 Discussion

9.1 Thesis Contributions and Significance ........................................ 104
9.2 Generalized Frozen Wave Method ............................................. 107

10 Summary and Outlook

10.1 Thesis Summary ................................................................. 109
10.2 Future Work ................................................................. 110
  10.2.1 Longitudinally Structured Light in Arbitrary Media ............... 110
  10.2.2 Engineering Structured Light Over Micro-Metric Regions (Non-Paraxial Regime) . 110
  10.2.3 Enhancing the Performance of the Proposed OAM Sensor ............ 111
  10.2.4 Developing a Generalized Methodology to Create Arbitrary Curved Beams .... 111
  10.2.5 Exploiting the Developed Beam Manipulation Tool-Set in Micro-manipulation . 112
  10.2.6 Longitudinally Structured Light in the Single Photon Limit ............ 112

Appendices

A Modal Decomposition and Reconstruction .................................... 114
  A.1 Procedure ................................................................. 114
  A.2 Choice of Bessel Functions ............................................. 116

B Cross Talk Analysis for Bessel Modes ......................................... 117
  B.1 Cross-Talk Measurements .................................................. 117

C Supplementary information for Chapter 8 .................................... 120
  C.1 Shift of Spatial Frequencies (k-comb) in the Sensed Medium .......... 120
  C.2 Sensor’s tolerance to deviations in $\theta$, $\Delta Q$, and $z$ .................. 121

D Thesis Publications .................................................................. 122
  D.1 Articles Published in Refereed Journals .................................. 122
  D.2 Papers Published in International Conference Proceedings .......... 123
List of Figures

2.1 Schematic diagram showing a basic scheme for generating a Bessel beam.  

2.2 Schematic diagram illustrating the mechanism of Bessel beam self-reconstruction after 
encountering an obstruction.  

2.3 Transverse intensity of Bessel beams with orders $\ell = 0$, $\ell = 1$, and $\ell = 3$. a) Simulated 
2D transverse intensity profiles. b) Intensity profiles sampled along the radial direction, $\rho$.  

3.1 The experimental setup used to generate and detect attenuation-resistant Frozen Waves.  

3.2 Simulated and measured intensity profiles of an ordinary FW (not compensated for losses) 
and the attenuation-resistant FW inside the lossy fluid; both FWs are designed with the 
constant intensity profile.  

3.3 Simulated and measured intensities of an ordinary FW and the attenuation-resistant FW 
inside the lossy fluid; the FWs are designed with the on-off-on-off intensity profile.  

3.4 Simulated and measured intensities of the ordinary FW and the attenuation-resistant FW 
inside the lossy fluid; the FWs are designed with the increasing intensity profile.  

4.1 Schematic diagram illustrating the helical wavefront, transverse intensity and phase pro-
files of vortex modes with different topological charges, $\ell$.  

4.2 Wavefront reversing its handedness and increasing its topological order with propagation.  

4.3 A holographic setup to create, digitally propagate and then detect 3D structured vortex 
beams.  

4.4 Measured and simulated intensity of the beam reversing its charge.  

4.5 Spectral decomposition of the generated 3D Structured Light into Bessel basis.  

4.6 Theoretical and experimental evolution of the Poynting vector, phase, and OAM density 
of our 3D Structured Light reversing its charge.  

4.7 Movement of charges in the vicinity of the beam’s center.  

4.8 Evolution of the OAM and topological charge considering two aperture diameters: $D = 
300 \, \mu m$, and $D = 5 \, mm$.  

4.9 Measured and simulated intensity of the beam varying its topological charge.  

ix
4.10 Spectral decomposition of the generated 3D Structured Light into Bessel basis.

4.11 Theoretical and experimental evolution of the Poynting vector, phase, and OAM density of our 3D Structured Light varying its charge.

4.12 Movement of charges in the vicinity of the beam’s center.

4.13 Evolution of the OAM and topological charge considering two aperture diameters: \( D = 400 \, \mu m, \) and \( D = 5 \, mm \).

4.14 Transverse profiles generated by adding two waveforms with different topological charges.

4.15 Evolution of the rotating light structure as it reverses its sense of rotation.

4.16 Intensity pattern of the petal structure as it changes its phase twist.

5.1 Experimental setup used to demonstrate longitudinal control of the polarization state and intensity.

5.2 Normalized intensity profiles of the measured beam for the first example.

5.3 Normalized intensity profiles of the measured beam for the second example.

5.4 Normalized intensity profiles of the measured beam for the third example.

5.5 Normalized intensity profiles of the measured beam for the fourth example.

5.6 Normalized intensity profiles of the measured beam for the fourth example taken at three consecutive planes around the SoP transition.

5.7 Normalized intensity profiles of the measured beam for the fifth example.

6.1 The holographic setup used to generate beams with varying wavelength and topological charge.

6.2 Measured intensity profile of the generated multi-foci color beam.

6.3 Evaluated amplitudes and phases of the coefficients \( A_{\lambda,\ell,m} \).

6.4 Measured intensity profile of the generated multi-chromatic beam with alternating color.

6.5 Transverse intensity profiles of the multi-chromatic vortex beam captured at different planes along the propagation direction, \( z \).

6.6 Transverse intensity profiles of the multi-chromatic petal-like beam captured at different planes along the propagation direction, \( z \).

7.1 Experimental setup used to generate and detect the Curved Frozen Waves.

7.2 Simulated and measured intensity patterns of the spiral beam.

7.3 The 3D trajectory of the central spot of the spiraling beam obtained from the CCD camera measurements.

7.4 Simulated and measured intensity patterns of the snake-like beam in air.

8.1 Schematic diagram illustrating the longitudinal wavenumbers of OAM modes \( \psi_{-1} \) and \( \psi_{1} \).
8.2 Experimental setup used to generate and detect the rotating structured beam patterns for sensing. ................................................................. 94

8.3 Measured and simulated transverse beam profiles of the rotating beams in air, water, vegetable oil, and cinnamon oil at propagation distance $z = 22$ cm for different cases of $\Delta Q$. 95

8.4 Performance of the proposed sensing scheme at $z = 22$ cm. ............................................................... 96

8.5 Performance of the proposed sensing scheme at $z = 27$ cm. ............................................................... 98

8.6 Extending the dynamic range of sensing by exploiting higher order OAM modes. ............ 101

A.1 A holographic setup to create, digitally propagate and then detect 3D structured vortex beams. .............................................................................. 115

B.1 Cross-talk measurement for Bessel beams with different values of $\ell$. ...................... 118

B.2 Cross-talk measurements for Bessel beams as function of $\delta k_p$. ...................... 119
**List of Abbreviations**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC</td>
<td>Beam Combiner</td>
</tr>
<tr>
<td>BE</td>
<td>Beam Expander</td>
</tr>
<tr>
<td>BS</td>
<td>Beam Splitter</td>
</tr>
<tr>
<td>CCD</td>
<td>Charge Coupled Device</td>
</tr>
<tr>
<td>CFW</td>
<td>Curved Frozen Wave</td>
</tr>
<tr>
<td>CGH</td>
<td>Computer Generated Hologram</td>
</tr>
<tr>
<td>FWs</td>
<td>Frozen Waves</td>
</tr>
<tr>
<td>HWP</td>
<td>Half-Wave Plate</td>
</tr>
<tr>
<td>OAM</td>
<td>Orbital Angular Momentum</td>
</tr>
<tr>
<td>Pol</td>
<td>Polarizer</td>
</tr>
<tr>
<td>RI</td>
<td>Refractive Index</td>
</tr>
<tr>
<td>RIU</td>
<td>Refractive Index Unit</td>
</tr>
<tr>
<td>SAM</td>
<td>Spin Angular Momentum</td>
</tr>
<tr>
<td>SLM</td>
<td>Spatial Light Modulator</td>
</tr>
<tr>
<td>SoP</td>
<td>State of Polarization</td>
</tr>
</tbody>
</table>
Nomenclature

$\Delta \theta$  Intermodal phases between the Bessel modes.

$\Delta x$  Pixel pitch of the Spatial Light Modulator.

$\delta$  Kronecker delta function.

$\ell$  The order (or topological charge) of the Bessel beam.

$\ell_{\text{eff}}$  Effective topological charge of the beam.

$\epsilon_0$  Free space permittivity.

$\epsilon_n$  Accuracy of the sensing scheme.

$h$  Reduced planck’s constant.

$\lambda$  Wavelength.

$S$  Poynting vector.

$\nu_{\theta}$  Mean absolute error in estimating $\theta$.

$\omega$  Angular frequency ($2\pi f$).

$\Phi$  Angular orientation of a rotating beam structure.

$\Psi(\rho, \phi, z, t)$  A superposition of Frozen Waves $\psi(\rho, \phi, z, t)$.

$\psi(\rho, \phi, z, t)$  Scalar representation of the Frozen Wave (engineered superposition of Bessel beams).

$\sigma_r$  Resolution of the sensing scheme.

$\theta$  Differential angular orientation between a rotating beam inside the medium and the orientation of the same beam in air.

$A_m$  Complex coefficients of each Bessel beam in the superposition.

$B$  Magnetic flux density.
$c$  Speed of light in vacuum.

$D$  Aperture diameter for the encoded hologram.

$E$  Electric field intensity.

$F(z)$  Morphological function that describes the desired longitudinal intensity profile of a given Frozen Wave.

$G(\phi)$  Morphological function that describes the azimuthal dependency in a Frozen Wave.

$H(x,y)$  The hologram equation to be addressed onto the Spatial Light Modulator.

$J_z$  Total orbital angular momentum per unit length.

$j_z$  Orbital angular momentum density in the z-direction.

$J_\ell(k_\rho \rho)$  Bessel beam of order $\ell$ with transverse wavenumber $k_\rho$.

$k$  Wavenumber in the medium ($k = \omega n/c$).

$k_\rho$  The transverse wavenumber.

$k_i$  Imaginary part of $k$.

$k_r$  Real part of $k$.

$k_z$  The longitudinal wavenumber.

$L$  Longitudinal extent of the Frozen Wave.

$N$  Limit of summation for the Bessel beam superposition; i.e, total number of Bessel beams is $2N+1$.

$n$  Refractive index of a medium.

$n_i$  Imaginary part of the refractive index of a medium.

$n_r$  Real part of the refractive index of a medium.

$Q$  Value at which the longitudinal wavenumbers of a Frozen Wave are centered.

$r_0$  Radius of the central spot for a Bessel beam.

$W$  Total energy per unit length.

$Z_{\text{max}}$  Longitudinal extent of the Bessel beam.
Chapter 1

Introduction

Diffraction lies at the heart of optical physics and plays an important role in the design of all optical systems. It is a phenomenon that is interconnected with the wave nature of light and typically occurs when a wave is transmitted through an aperture or encounters an opaque obstacle, leading to the wave broadening with propagation even in the absence of reflection or refraction. In this process, the wave may be altered in amplitude and/or phase and, hence, wavefront spreading takes place, as described by the Huygen’s principle [1]. In such case, each point of the wavefront acts as source of secondary waves that in turn create a new wavefront and so forth — leading to the spatial spreading of the beam. One common case of diffraction is encountered in the output of a laser which, although deemed ‘pencil-like’ in nature with low divergence, yet is subject to spatial broadening due to diffraction.

The typical parameter used for characterizing the spread of light beams is the Rayleigh range \( Z_R \). Considering a Gaussian beam profile, the Rayleigh range denotes the distance at which the Gaussian beam doubles its cross-sectional area and is given by \( Z_R = \frac{\pi W_0^2}{\lambda} \), where \( \lambda \) is the wavelength and \( W_0 \) is the beam waist size [1]. Interestingly, there exist a family of beams that travel for long propagation distances while encountering negligible diffraction effects. Such family of beams is usually referred-to as non-diffracting beams, shape preserving beams, or localized waves. Indeed, the notion of overcoming the fundamental limit of diffraction is very intriguing and attractive for numerous applications such as imaging, lithography, optical communications, and particle micro-manipulation. As such, shape preserving beams will be the main focus of this dissertation. A brief background on shape preserving beams in the more general context of localized waves is presented next.

1.1 Historical Background

In addition to plane waves, a class of waves that exhibits resistance to diffraction in free space (for a relatively long propagation distances) is the so called localized waves [2]. Such waves represent a valid solution to the Wave Equation which can render either wave packets (pulses) or monochromatic beams.
Shape preserving beams, a subclass of localized waves, have been theoretically predicted long time ago [3, 4]. In Ref. [3], Stratton obtained a monochromatic solution to the wave equation whose transverse shape was concentrated in the vicinity of its propagation axis, and represented by the Bessel function. He then established that Bessel beams maintain a non-variant transverse profile with propagation. Afterwards, Courant and Hilbert theoretically demonstrated how a wide class of equations — including the wave equation — can accept “non-distorted progressing waves” as valid solutions [4]. During that time, non-diffracting Bessel beams did not attract much attention, hindered by the fact that their theoretical formulation postulated a non-integrable solution with infinite energy, which is not practically feasible.

The possibility of generating practical Bessel beams (with finite energy) was not established until 1987, when Durnin presented the theoretical formulation [5] and the experimental demonstration [6] of non-diffracting Bessel beams generated from a finite aperture. In his seminal experiment, Durnin generated a Bessel beam by illuminating an annular slit in the back focal plane of a lens. The beam was generated at $\lambda = 632.8\text{ nm}$ with a central spot of $\Delta \rho_0 = 59\text{ }\mu\text{m}$ passing through a finite aperture of 3.5 mm. Back then, he demonstrated that such beam can propagate for a distance of 85 cm while maintaining an invariant transverse profile. Notably, such propagation distance observed in the Bessel beam is 28 times longer than its Gaussian counterpart which, at the same wavelength, would double its width after propagating for merely 3 cm [1]. Following this experiment, Turunen presented an alternative simple holographic method for the generation of Bessel beams confirming its diffraction-free behaviour [7] before another realization using axicons was reported [8]. While these earlier efforts were based on a scalar treatment of Bessel beams, a full vectorial formulation, satisfying Maxwell equations, was provided later [9, 10, 11].

The experimental realization of Bessel beams as diffraction-free waves invited many researchers to investigate them further. In Ref. [12], Macdonald et al. used the term “shadowing” to describe and establish the self-healing property of Bessel beams, demonstrating how the outer rings of the beam are capable of reproducing its center spot after being obstructed by an obstacle. Such peculiar behavior was then confirmed by Bouchal [13] and is understood to be the main contributing factor to the diffraction-resistance property of Bessel beams in which the outer rings of the beam constructively interfere while propagating to keep the central spot of the beam intact — thus preserving its shape [14].

In addition to describing monochromatic shape preserving beams, the term localized waves also refer to propagating wavepackets (pulses). With the space-time duality in mind [15], one can see that the temporal counterpart of diffraction is the well known temporal dispersion phenomenon which causes pulses to be distorted, often spreading in time, as a result of its spectral components traveling at different phase velocities. In analogy with non-diffracting beams, localized pulses that remain intact in time, thus immune to dispersion, have also been reported in the literature. For instance, a non-singular packet-like solution was introduced by Brittingham in [16] followed by a more general Gaussian laguerre [17] and Gaussian solutions [18] — altogether forming a valid set of localized packet-like solutions to the
homogeneous wave equation. Later on, a family of waveforms localized in both space and time, known as “X-waves” was introduced together with their corresponding finite aperture realization [19]. Not only are these waveforms localized in space and time, but they can also undergo superluminal propagation — an intriguing behavior that occurs without violating relativistic causality [20]. Such superluminal behavior in X-waves was first demonstrated by Mugnai in Ref. [21], few years after acoustic X-waves [22] and their optical counterpart [23] were reported.

More recently, with the advent of spatial light modulators and advanced laser beam shaping techniques, it became possible to engineer light into more complex structures — realizing localized waves that do not obey rectilinear propagation but rather travel along curved trajectories. Airy beams are one form of curved localized beams which are also referred to as self-accelerating beams. First introduced in the optical framework in 2007 [24, 25], Airy beams possess unusual features such as the ability to remain diffraction-free for relatively long distances while they tend to freely accelerate during propagation. Additionally, they share the interesting self-healing characteristics similar to non-diffracting beams [26]. While the formulation of Airy pulses was developed in the paraxial regime, non-paraxial beams with parabolic trajectories have also been reported [27] including Mathieu and Weber beams [28]. Interestingly, in analogy with their counterpart in space, Airy pulses are relatively immune to pulse spreading that typically arises from temporal dispersion, and for which the pulse envelope tends to either accelerate or decelerate [29, 30]. Importantly, it is possible to construct spatio-temporal wave packets that are both non-dispersive and non-diffractive as well, known as Airy bullets. Such class of wave packets can be realized by combining an Airy pulse with a 2D non-diffracting beam and can be useful in many applications [31, 32].

In parallel with investigating the fascinating science behind non-diffracting beams, many researchers also revealed a handful of their potential applications. For instance, improvements to the imaging properties of optical scanning systems using Bessel beams were reported as early as 1992 [33], whereas the robustness of non-diffracting beams under atmospheric turbulence was elaborated in [34], highlighting their usefulness in long range optical communications. Other early developments include using Bessel beams for second harmonic generation in a non-linear KDP crystal, satisfying the phase-matching condition at angles that are not usually suited for phase-matching, and postulating that a Bessel beam can be viewed as a light beam with tunable wavelength [35]. Afterwards, Bessel beams were applied as efficient source of stimulated Raman scattering in H₂ gas [36] and were also deployed in tunable distributed-feedback lasers [37]. Additionally, particle manipulation is another rich venue in which non-diffracting beams were extensively applied. Owing to their localized behavior, self-reconstruction, and long depth-of-field, Bessel beams were utilized in trapping micro particles at multiple planes along the beam’s axis for the first time [38, 39, 40], exhibiting a powerful capability that was not previously feasible using standard Gaussian beams. Furthermore, Bessel beams have made their path to atom guiding experiments and transporting Bose-Einstein condensates [41, 42, 43].
The introduction of the Laguerre-Gauss distributions in 1992 as beams that carry well-defined and stable orbital angular momentum (OAM) [44] ignited a new era of shape preserving beams. Unlike the spin angular momentum associated with circular polarization (which is bounded to a value of ±ℏ per photon), OAM has unlimited values of ℓℏ per photon where ℓ is referred to as the topological charge of the beam [44, 45]. As such, OAM beams (also known as vortex beams) were directly utilized in optical trapping and micro-manipulation motivated by their ability to transfer momentum into trapped objects.

The earliest demonstration of OAM transfer to a low-index particle was carried out using higher order Bessel beams by Garcés-Chávez et al. in [46] and later on in Ref. [47]. Moreover, the first observation of simultaneous transfer of spin and orbital angular momenta to a particle which is not positioned on the beam axis was also observed using the Bessel beam [48]. To date, non-diffracting beams in general and Bessel beams in particular have been extensively utilized in optical trapping and micro-manipulation [49, 50, 51, 52, 53].

In addition to optical trapping and micro-manipulation, the OAM of light opened many new venues in optical data communications. Notably, OAM states manifest as orthogonal degrees-of-freedom that can be coherently generated and detected and have thus been extensively deployed in encoding information; both classically and in the quantum regime [54, 55, 56, 57, 58, 59]. The list goes on for other applications of non-diffracting beams with OAM whether in imaging, materials processing, microscopy and so forth; creating an urgent need for advanced light manipulation techniques [60, 61, 62]. Besides OAM, the polarization state of shape preserving beams is another degree-of-freedom that has been widely exploited in materials processing [63], polarimetry [64], microscopy [65, 66] and optical communications [67, 68], to name but a few. Hence, the domain of localized waves, in general, and shape preserving beams, in particular, is wide and covers a plethora of applications. The objective of this thesis is to develop a new generation of optical beams that can meet the rising demand for Structured Light, as outlined next.

1.2 Thesis Objectives

Modern advances in laser beam shaping techniques has led to generating complex light structures; addressing existing challenges and opening new opportunities. The branch of optics that deals with the generation and application of custom light fields with structured intensity, polarization and phase, is known as Structured Light [69]. Extensive research efforts have been dedicated to tailor non-diffracting light structures in order to address emerging needs in areas such as imaging, materials processing, optical trapping, and free space optical communications as discussed in the previous section. In the majority of these efforts, when a shape preserving beam is exploited, the beam maintains its properties along its direction of propagation when traveling in a linear, homogeneous, isotropic, and transparent medium. Here, by beam “properties” we are explicitly referring to the beam’s intensity, orbital angular momentum, polarization state, and wavelength.
Very few attempts have been made to control the properties of non-diffracting optical beams along their axial direction, despite the potential and new possibilities this may merit. In Refs. [70, 71], Zamboni-Rached introduced a systematic method for controlling the longitudinal intensity profile of Bessel beams — a theory known as the “Frozen Wave”. The method relies on superposing multiple co-propagating Bessel beams with different wavenumbers (spatial frequencies), carefully weighted, to modulate the intensity profile of the envelope. In such case, the envelope of the beam is static (hence, the term “Frozen”) and can acquire any predetermined arbitrary intensity profile. Another versatile method was introduced by Čižmár and Dholakia in [72] together with its holographic implementation. The ability to engineer the intensity profile of non-diffracting beams opened the door for exploiting such beams in absorbing (lossy media) where conventional beams would suffer from exponential decay (attenuation) with propagation. Attenuation resistance can be achieved by engineering the beam’s envelope such that it follows a growing intensity profile, counteracting the loss profile of the absorbing medium, and thus extending the beam’s propagation range [73]. This approach has proven to tackle many challenges in microscopy by providing high contrast-to-noise ratios inside thick biological specimens [74]. Motivated by these earlier efforts that attempted to tailor the intensity profile of shape preserving beams, the objective of this thesis is three-fold as outlined next:

1.2.1 Engineering the Degrees-of-Freedom of Non-Diffracting Beams along the Axis of Propagation

One of the main objectives of this thesis is to present a systematic approach through which new classes of non-diffracting light beams can be designed, experimentally generated, and analyzed. We show how various degree-of-freedom of light can be engineered to follow a predetermined profile along the propagation direction of the beam. In particular, we report advanced classes of non-diffracting beams in which the following degrees-of-freedom can all be controlled along the beam’s axis:

1. Intensity Value.
2. Orbital Angular Momentum (OAM).
3. State of Polarization (SoP).
4. Wavelength.
5. Propagation Trajectory.

We refer to those beams as “Longitudinally Structured Light (LSL)” and we present the unified expression through which such degrees-of-freedom of light can be controlled (independently or concurrently) along the propagation direction. Longitudinal structuring of non-diffracting light structures can offer new degrees-of-freedom and unlock new possibilities in light science and its applications. For instance, shaping the intensity profile of light in lossy media can be useful for imaging inside absorbing
biological tissues or for data transmission in absorbing media — where conventional beams would suffer from an exponential decay and short propagation distances. On the other hand, longitudinal control of the topological charge (OAM) can offer new degrees-of-freedom in micro-manipulation [52, 53, 75], and can dramatically enhance communication channel capacities [76].

Additionally, tailoring the polarization state along the beam’s axis can be useful in handful of applications. For example, it can be exploited to spatially modulate the absorption profile of a polarization dependent optically pumped medium [64], or to spatially tailor the spectrum profile of quantum emitters (which are typically polarization dependent) [77]. Furthermore, beams with spatially varying polarization states can be deployed to control the shape and size of laser-machined structures by inducing polarization-dependent ablation effects [78]. Likewise, such developments can be carried out using an optical beam with longitudinally varying wavelength; where the response of optically pumped media [79], quantum emitters excitation [80], and material ablation [81] are all wavelength dependent. Moreover, engineering propagation trajectory of beams offers additional degrees-of-freedom in materials processing and optical trapping [82, 83].

1.2.2 Investigating the Physical Dynamics Associated with LSL

Along with generating new classes of Longitudinally Structured Light beams, this thesis also investigates several unusual physical phenomena; for instance, cases in which a vortex beam that carry OAM can change the sign and/or magnitude of its OAM, locally, with propagation. We investigate such behavior, and shed the light on many new intriguing physical mechanisms in which the OAM of a non-diffracting beam can be reversed and/or changed under unperturbed propagation in air. In this process we also provide quantitative interpretations on how these topological transitions and associated OAM evolution occur without violating any conservation laws. In doing so, we calculate the local and global OAM quantities, and show that indeed OAM can be varied locally while keeping the global OAM, across the entire beam’s cross section, conserved. Hence, highlighting the role of the outer rings of non-diffracting beams as energy and momentum reservoirs. Although not the main subject of this work, such a breakdown of local OAM conservation may find its analog in other fields such as plasma physics, fluid dynamics, and atmospheric physics.

1.2.3 Exploiting LSL in Refractive Index Sensing

In addition to generating new classes of non-diffracting beams, and investigating their underlying physical phenomena, we also exploit some of those developed classes of Structured Light and demonstrate how longitudinal control over the beam’s properties (in particular, its OAM and intensity) can offer new degrees-of-freedom that can tackle a twofold problem in refractive index sensing: i.e., providing high resolution measurements while covering a wide dynamic range for sensing applications.
Chapter 1. Introduction

1.3 Thesis Organization

This thesis is organized as follows:

In Chapter 2, we present an overview starting with the derivation of Bessel beams as a valid solution to the homogeneous wave equation. Then, by exploiting the imaging property of Bessel beams, we describe how a judicious superposition of Bessel beams can allow us to modulate the longitudinal intensity profile of their envelope, on demand, based on the well established Frozen Wave method.

Chapter 3 builds on the Frozen Wave method to present what is believed to be the first experimental demonstration of a Frozen Wave inside an absorbing fluid. Several interesting “attenuation-resistant” waveforms are generated in which the longitudinal intensity profile can be spatially modulated on demand while overcoming the propagation losses inside the absorbing fluid — a development that can address many challenges in imaging, materials processing, and optical trapping.

Chapter 4 establishes the methodology for creating non-diffracting vortex beams with topological charges that can be made to vary “at-will” with propagation. To do so, we modify the original Frozen Wave theory and incorporate higher-order Bessel modes, i.e. vortex beams that carry non-zero OAM. We start by presenting the theoretical methodology via which the sign and magnitude of the topological charge $\ell$ can be changed along the beam’s axis of propagation. Using modal decomposition and field reconstruction, we experimentally reconstruct the beam’s OAM density, Poynting vector, and phase profile. We then describe the physical mechanism via which topological transition occurs; reporting cases that manifest creation, movement, and annihilation of singularities across the beam — transforming its topology without disturbing the net topological charge. As such, several intriguing observations are reported wherein the OAM can vary locally along the $z$-direction while the global OAM remains conserved. Finally, we showcase two scenarios of rotating light structures that can reverse the direction of its rotation or change its rotational speed, on demand. This class of beams will particularly be useful in remote sensing, as will be discussed in details later on in Chapter 8.

Chapter 5 presents a new systematic platform to control the polarization state of non-diffracting Bessel beams as they propagate. This is achieved by a suitable superposition of Bessel beams, whose parameters are deliberately chosen based on the Frozen Wave method, introduced in Chapter 3. Using a holographic setup incorporating a spatial light modulator, several peculiar beam patterns are generated. For example, one in which the polarization state of a horizontally polarized beam evolves to radial polarization and is then changed to vertical polarization, while the beam intensity remains constant; and others in which both the polarization state and the longitudinal intensity profile are engineered along the beam’s axis. Furthermore, by exploiting the attenuation-resistant feature described in Chapter 3, the ability to control the polarization state while overcoming propagation losses inside absorbing fluids is illustrated experimentally. Finally, we demonstrate an advanced beam structure in which multiple degrees-of-freedom can be controlled simultaneously along the beam’s axis; in essence, creating an attenuation-resistant non-diffracting beam that changes both its polarization and topological charge...
Chapter 1. Introduction

with propagation, on demand.

Chapter 6 examines another degree-of-freedom of light: its wavelength. Notably, wavelength is a fundamental property of light that dictates its linear momentum and thus remains unchanged under unguided free space propagation. By exploiting the ability to modulate the intensity envelope of Bessel beams, via the Frozen Wave method, we theoretically and experimentally demonstrate non-diffracting beams in which the wavelength can be changed from one color to another along beam’s axis of propagation. Unlike standard chromatic aberration, longitudinal variation of the wavelength is achieved through controlled interference along the beam’s axis of propagation. We then combine this approach with the one presented in Chapter 4 to control both the wavelength and topological charge of the beam concurrently and at-will. Longitudinal control of both the wavelength and charge can offer many new degrees-of-freedom in materials processing, optical trapping, imaging, and dense data communications.

Chapter 7 describes, in theory and experiment, a class of non-diffracting and self-healing optical beams whose central spot can be designed to follow off-axis curved trajectories. The proposed method allows the generated beams to traverse a spiral path with the possibility of reversing its handedness and changing its longitudinal intensity pattern with propagation, on demand. The central spot can also be designed to evolve from a straight to a curved “snake-like” trajectory and vice versa — thus offering new degrees of control that can be useful in atomic guiding, optical trapping, and material processing.

In Chapter 8, a novel mechanism for Refractive Index (RI) sensing — exploiting OAM of Structured Light — is proposed and experimentally demonstrated. A superposition of higher order monochromatic co-propagating Bessel beams with equally-spaced longitudinal wavenumbers, arranged in a comb-like setting, is used to generate non-diffracting rotating light structures in which the rotation speed is sensitive to medium’s index of refraction. In principle, the sensitivity of this scheme can exceed $\sim 2700^\circ$/RIU with a resolution of $\sim 10^{-5}$ RI unit (RIU). Furthermore, it will be shown how the unbounded degrees of freedom of OAM can be deployed to achieve wide dynamic range by generating light structures that evolve to different modes based on the RI change — thus extending the dynamic range to cover RI values from 1 to over 2.9 RIU. We expect this method to open new directions in refractometry and remote sensing as it tackles one of the traditional trade-offs in RI sensing: that is achieving high resolution while covering a wide dynamic sensing range.

Chapter 9 highlights the main contributions and significance of this thesis in the context of what has been developed in the literature. It also shows that each longitudinally structured beam pattern presented in this thesis can be regarded as a special case of a more unified method, denoted as the Generalized Frozen Wave method. We present the general formula that governs all the generated light patterns discussed throughout the thesis, and show how every LSL pattern developed in each chapter could be interpreted as a special case of this general formula.

Lastly, Chapter 10 summarizes the concepts and developments presented throughout the thesis then sets the scene for future considerations and potential extensions of the present work. This includes
generating LSL under the non-linear absorption regime, manipulating Structured Light in the non-paraxial regime, investigating the theory of Frozen Waves in a full vectorial form, and proposing ideas to further linearize the response of the proposed sensing scheme, besides highlighting unexplored areas that can benefit from LSL in the single photon limit.
Chapter 2

Bessel Beams and the Frozen Wave Method

This chapter starts by introducing the mathematical derivation of Bessel functions, which represent valid solution to the Helmholtz equation, and considered among the most commonly used classes of non-diffracting beams. This is discussed in Section 2.1. Afterwards, the experimental generation of Bessel beams is discussed in Section 2.2. Finally, the ability to modulate the longitudinal intensity profile of the non-diffracting beam’s envelope via a superposition of co-propagating Bessel beams, based on the “Frozen Wave” method, is discussed in Section 2.3.

2.1 Non-Diffracting Propagation

Under ideal propagation conditions, the monochromatic spatially coherent non-diffracting beam propagating along the z-direction is interpreted as a mode-like field with complex amplitude and is expressed as

\[ U(x, y, z, t) = u(x, y)e^{i(\omega t - kz z)}, \quad (2.1) \]

where \( u(x, y) \) describes the transverse amplitude profile, whereas \( \omega \) and \( k_z \) are the angular frequency and the wavenumber, respectively. Accordingly, the slowly varying amplitude \( u(x, y) \) is independent of the \( z \)-coordinate and the intensity of the beam is thus propagation invariant. In order for such field to satisfy the homogeneous wave equation, its phasor representation, \( a(x, y, z) = u(x, y)e^{-ik_z z} \), must also satisfy the Helmholtz equation, such that

\[ [\nabla^2 + k^2]a(x, y, z) = 0. \quad (2.2) \]
Here, \( k = \omega/c \), where \( c \) is the speed of light in vacuum. Using the cylindrical coordinate system to comply with the rotationally symmetric solution, and deploying the separation of variables principle, the field is assumed to be of the form

\[
a(\rho, \phi, z) = R(\rho)\Phi(\phi)e^{-ik_z z}.
\]  

(2.3)

The function \( \Phi \) describes the azimuthal profile of the beam and typically acquires the following functional form

\[
\Phi(\phi) = e^{i\ell \phi}, \quad \ell = 0, 1, 2, \ldots
\]  

(2.4)

By substituting Eqs. (2.3) and (2.4) in Eq. (2.2), the differential equation for the radial function \( R(\rho) \) can be obtained:

\[
\frac{\partial^2 R(\rho)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R(\rho)}{\partial \rho} + k^2_R R(\rho) \left( \frac{1 - \ell^2}{k^2_R \rho^2} \right) = 0,
\]  

(2.5)

where \( k^2_R = k^2 - k^2_z \). Equation (2.5) is known as the Bessel differential equation. Its solution is given by the \( \ell \)-th order Bessel function of first kind \( J_\ell \) and the \( \ell \)-th order Neumann function of the first kind \( N_\ell \), such that \[84\]

\[
R_\ell(\rho) = c_1 J_\ell(k_\rho \rho) + c_2 N_\ell(k_\rho \rho).
\]  

(2.6)

Here, \( c_1 \) and \( c_2 \) are constants. Note that the Bessel function \( J_\ell \) is typically considered the valid solution, whereas the solution introduced by the Neumann function \( N_\ell \) is discarded because it introduces a singularity at the \( \rho = 0 \) point. With the expressions for \( R(\rho) \) and \( \Phi(\phi) \) in hand, Eq. (2.3) is then written as

\[
a(\rho, \phi, z) = J_\ell(k_\rho \rho)e^{i\ell \phi}e^{-ik_z z}.
\]  

(2.7)

One can then restore the temporal dependence and re-express the propagating Bessel beam in the following form:

\[
U(\rho, \phi, z, t) = J_\ell(k_\rho \rho)e^{i\ell \phi} \cos(\omega t - k_z z),
\]  

(2.8)

where \( k_\rho \) and \( k_z \) denote the transverse and longitudinal wavenumbers, respectively, and are connected via the consistency relation: \( k^2_\rho + k^2_z = k^2 \).

### 2.2 Bessel Beam Generation

A Bessel beam can be experimentally generated using multiple techniques. The most basic approach is to place an illuminated annular ring in the back focal plane of a lens, as first demonstrated by Durnin [6]. Figure 2.1 depicts the experimental setup used by Durnin in his seminal experiment. A circular ring with mean diameter \( d \) is located in the focal plane of a lens with radius \( R \) and is illuminated by a collimated Gaussian beam, thus creating a Bessel beam that extends for a distance \( Z_{\text{max}} \) on the other side of the
lens. In this case, each point on the ring acts as a point source that undergoes Fourier transformation by the lens into a plane-wave. In essence, a Bessel beam is generated as a result of superposing multiple plane-waves whose wavenumbers lie on the surface of a cone (See Fig. 2.1).

Figure 2.1: Schematic diagram showing a basic scheme for generating a Bessel beam. A collimated light of wavelength $\lambda$ illuminates a circular slit located in the focal plane of a lens. The mean diameter of the slit is $d$, the focal length of the lens is $f$, and the radius of the output aperture (here, the lens) is $R$. The distance $Z_{\text{max}}$ indicates the beginning of the geometrical shadow zone along the $z$-axis.

The Bessel beam created using the above setup will have few defining characteristics [14]: first, as stated earlier, it can be decomposed into a set of plane waves whose wavenumbers lie on the surface of a cone with the cone angle $\theta$; such that $\theta = \tan^{-1}\left(\frac{k_\rho}{k_z}\right) = \tan^{-1}\left(\frac{d}{2f}\right)$. Second, the transverse localization of the beam is dictated by $k_\rho$; where, from the zero properties of the Bessel function, the spot size of the Bessel beam, $r_0$, is given by $r_0 = \frac{2.405}{k_\rho}$. Third, because the Bessel beam is experimentally generated over a finite aperture\(^1\), it only extends for a finite distance, $Z_{\text{max}}$, proportional to the aperture size; such that $Z_{\text{max}} = \frac{R}{\tanh(\theta)}$. Fourth, owing to its inherent conical plane waves nature, a Bessel beam can reconstruct its central spot even if its propagation path is obstructed by an obstacle — a distinguishable property of all non-diffracting waves known as self-healing, first demonstrated for the Bessel beams in [12]. In this process, the central spot of the beam is able to reform after a minimum propagation distance of $z_{\text{min}} \approx \frac{\hat{W}^2}{2k_z}$ as depicted in Fig. 2.2, where $\hat{W}$ denotes the width of the obstacle.

While the annular slit technique is relatively simple to implement, it is not very efficient in terms of energy utilization of the incident Gaussian excitation. To mitigate this, more efficient Bessel beam generation techniques such as axicons or conical lenses [85, 8] have been developed. More recently, Bessel beams have been generated using other techniques; including meta-surfaces and holographic setups. The

\(^1\)Some researchers use the term “quasi-Bessel” to describe a Bessel beam generated from a finite aperture. Throughout this thesis, and for the sake of brevity, the term Bessel beam will be used instead of “quasi-Bessel” when referring to apertured Bessel beams.
latter approach typically incorporates programmable spatial light modulators (SLMs) to “print” the desired Bessel profile onto the incident Gaussian beam. Holographic generation takes advantage of the reconfigurability of SLMs; hence, it will be adopted in this work as discussed in more details shortly.

2.3 Modulating the Envelope of Bessel Beams

2.3.1 The Bessel Beam Profile

While the energy carried by zero-th order Bessel beams (with \( \ell = 0 \)) is mainly localized in their central spot and outer rings, higher order Bessel beams (with \( \ell \geq 1 \)) are characterized by a region of zero intensity in the center and they carry their energy over a circular ring. Such higher order beams possess
non-zero orbital angular momentum as will be discussed in details in Chapter 3. Figure 2.3 depicts the transverse intensity profiles of three Bessel beams with orders: $\ell = 0$, $\ell = 1$, and $\ell = 2$, as obtained from Eq. 2.7. Notice how the energy in the beam’s center gets distributed over a ring with larger diameter as a result of increasing the Bessel beam’s order, $\ell$.

Under ideal propagation conditions in a homogeneous, isotropic, transparent, and linear medium, the Bessel beam would maintain a propagation invariant intensity profile along its axis. Interestingly, it is possible to modulate the longitudinal intensity profile of Bessel beams by exploiting a property known as self-imaging, as discussed next.

### 2.3.2 Self-Imaging of Bessel Beams

When two Bessel beams with slightly shifted transverse wavenumbers ($k_\rho$) are coherently superimposed, the intensity profile of the resulting beam is no longer propagation invariant. In this case, the intensity level in the central spot varies, in a sinusoidal manner, as the beam propagates — owing to the beating among its different spatial frequencies (wavenumbers). Similarly, when $m$ Bessel beams with slightly different $k_\rho$ (and, hence, $k_z$) values are coherently superimposed, the resulting beam exhibits a $z$-dependent intensity profile as a result of the constructive and destructive interference among its Bessel modes. Under unguided propagation, the spatial profile of the beam will be repeated periodically along the beam’s axis. As such, the complex profile of the resultant beam can be described as [86]

$$a(\rho, \phi, z) = a(\rho, \phi, z + L).$$

(2.9)

Here, $L$ denotes the longitudinal periodic distance after which the same transverse spatial profile of the beam will be repeated. In this case, the beam’s intensity can be more localized in the longitudinal direction by incorporating larger number of co-propagating Bessel modes in the coherent superposition, whereas transverse localization is realized by including Bessel modes with large values of $k_\rho$.

### 2.3.3 The Frozen Wave Method

In Refs. [70, 71], Zamboni-Rached exploited the self-imaging property of Bessel beams to develop a systematic approach allowing the design and generation of non-diffracting beams; in which the intensity profile can be arbitrarily tailored along the beam’s axis of propagation — a method known in the literature as the Frozen Wave (FW) method. Based on this method, the envelope of a non-diffracting beam $\psi(\rho, \phi, z, t)$ can be arbitrarily modulated along the axis of propagation of the beam. This is achieved via a superposition of co-propagating Bessel modes at equal frequencies but with slightly shifted wavenumbers such that
Chapter 2. Bessel Beams and the Frozen Wave Method

\[
\psi(\rho, \phi, z, t) = e^{-i\omega t} \sum_{m=-N}^{N} A_m J_\ell(k_\rho^m \rho) e^{i\phi} e^{ik_z^m z}.
\] (2.10)

Equation (2.10) is an exact solution to the wave equation, given by Eq. (2.2), in which \(k_\rho^m\) and \(k_z^m\) denote the transverse and longitudinal wavenumbers of the \(m\)-th Bessel beam, respectively. The superposition of Eq. (2.10) allows us to control the longitudinal intensity profile of the waveform \(\psi(\rho, \phi, z, t)\) through a judicious choice of the coefficients \(A_m\); where \(A_m\) represent complex weighting factors for each Bessel beam in the superposition. For instance, consider a scenario in which the on-axis intensity profile (along \(\rho = 0\)) is designed to follow the pattern represented by an arbitrary function denoted by \(|F(z)|^2\) over a finite space interval, \(0 \leq z \leq L\), where \(L\) is the self-imaging period. In this case, the function \(F(z)\) is expanded using a Fourier series such that

\[
F(z) = \sum_{m=\infty}^{m=-\infty} B_m e^{(i \frac{2\pi}{L} m) z},
\] (2.11)

where

\[
B_m = \frac{1}{L} \int_0^L F(z) e^{-(i \frac{2\pi}{L} m) z} dz.
\] (2.12)

In order for the superposition of Eq. (2.10) to yield a waveform with on-axis intensity profile that follows the pattern given by \(|F(z)|^2\), the following condition should be satisfied

\[
\left| \sum_{m=-N}^{m=N} A_m e^{ik_z^m z} \right|^2 \approx |F(z)|^2, \quad \text{within } 0 \leq z \leq L.
\] (2.13)

By comparing Eqs. (2.13) and (2.11), one might chose to set \(k_z^m = \frac{2\pi}{L} m\) to realize a truncated Fourier series that approximately represents the pattern given by \(F(z)\). However, such choice for the longitudinal wavenumbers is accompanied with two main limitations. First, it yields negative values for \(k_z^m\) (when \(n < 0\)), which implies backward propagating components. Secondly, for \(L \gg \lambda\), the main terms of the series will correspond to very small values for \(k_z^m\), which result in very short depth of field of the generated waveforms. This sets a limit on how far the desired envelopes can be generated away from the source. Owing to these two limitations, the values of \(k_z^m\) are chosen according to [70, 71]

\[
k_z^m = Q + \frac{2\pi}{L} m.
\] (2.14)

Here, \(Q\) is a constant parameter that defines the central longitudinal wavenumber (\(k_z^{m=0}\)) and is chosen subject to the constraint \(0 \leq Q + \frac{2\pi}{L} m \leq \omega/c\). Accordingly, Eq. (2.10) can be re-written as

\[
\psi(\rho, \phi, z, t) = e^{-i\omega t} e^{iQz} \sum_{m=-N}^{N} A_m J_\ell(k_\rho^m \rho) e^{(i \frac{2\pi}{L} m) z} e^{i\phi}.
\] (2.15)
In the above, \( A_m \) represent complex weighting factors for each Bessel beam in the superposition and are obtained by solving

\[
A_m = \frac{1}{L} \int_0^L F(z) e^{-(i \frac{\pi}{2} m z)} dz. \tag{2.16}
\]

Here, the pattern \( F(z) \) (which we will refer to the “morphological function”) defines the desired longitudinal intensity profile to be encoded in the envelope of \( \psi(\rho = 0, \phi, z, t) \) over the finite longitudinal extent \( L \). In essence, the phases and amplitudes of \( A_m \) are engineered to shape the longitudinal intensity profile of \( \psi(\rho, \phi, z, t) \), via controlled interference. In principle, this method is equivalent to Fourier series in which arbitrary periodic waveforms are constructed from a discrete superposition of multiple frequency harmonics weighted by suitable Fourier coefficients. Similarly, here, the discrete superposition of suitably weighted spatial harmonics, \( J_\ell(k_m \rho)e^{i \ell \phi} e^{ik_m z} \), in Eq. (2.15), enables us to spatially modulate the envelope of the ensemble via controlled beating among its spatial frequencies. In such case, the beam’s envelope is static (Frozen) and can acquire any longitudinal profile — as determined by \( F(z) \).

Few examples of customized Frozen Waves have been demonstrated experimentally by Vieira et al. [87, 88, 89] under free propagation in air using a holographic setup. In Refs. [90, 91], Zamboni-Rached further developed the Frozen Wave method so that the generated beam can acquire predetermined intensity profiles, not only in air (lossless medium), but under propagation in absorbing media as well — without suffering from the undesired propagation losses. The basic concept was to modify the function \( F(z) \) such that it exhibits an exponentially growing behavior that could counteract the loss profile of the medium. Chapter 3 presents the first experimental demonstration of attenuation-resistant Frozen Waves inside an absorbing fluid. Afterwards, each chapter demonstrates how almost every degree-of-freedom of light (i.e, its orbital angular momentum, polarization, and wavelength) can be controlled along the beam’s axis by introducing new modifications to the Frozen Wave method. A unified approach through which all such degrees-of-freedom of light can be engineered along the propagation axis of the beam — denoted as the Generalized Frozen Wave method — will be presented in Chapter 9.
Chapter 3

Shaping the Longitudinal Intensity of Diffraction-Attenuation-Resistant Beams

This chapter presents what is believed to be the first experimental demonstration of a class of beams, called attenuation-resistant Frozen Waves (FWs). Such waveforms can maintain a predefined longitudinal intensity profile inside an absorbing fluid. The attenuation-resistance effect is achieved by shaping the longitudinal intensity profile of the waveform such that it acquires a growing intensity that counteracts the exponential loss profile of the absorbing medium. In addition to the compelling features of FWs presented here, namely the arbitrary longitudinal control of the intensity profile while overcoming propagation losses, FWs also possess non-diffracting and self-reconstructing behaviors, which are inherent properties of Bessel beams. As such, we envision that these beams can lead to many new advancements to the field of optical sciences and beam manipulation.

3.1 Motivation

The idea of creating non-diffracting beams that can overcome propagation losses has always been an intriguing one. The main objective is to offer a class of attenuation-resistant and diffraction-free beams that can exhibit longer propagation range inside absorbing media — where other beams would typically suffer from an exponential intensity decay. This becomes particularly useful in applications where light-matter interaction occurs in lossy fluids or absorbing biological tissues, for example. Few researchers have tried to mitigate the attenuation suffered by non-diffracting beams in absorbing media. This has been

Footnote: The experimental demonstration of diffraction-attenuation-resistant FW beams is the first contribution of this thesis and was first published in Refs. [92, 93, 94, 95] and adapted in this chapter.
at tempted by deploying an exponential intensity axicon (exicon) to generate constant intensity Bessel beams in an absorbing dye solution \[73\], or to generate loss-proof self-accelerating beams in the presence of two photon absorption \[96\]. More recently, shape preserving surface-plasmon polariton beams have been demonstrated in the presence of plasmon losses \[97\]. Although these efforts are promising steps to alleviate beam attenuation in lossy media, the ability to freely shape the longitudinal intensity profile of the beam (for example, turning it on and off with propagation distance) in an absorbing medium has not been demonstrated before this work. This is very beneficial for applications such as imaging, data communications, and material processing, and will be fulfilled here using attenuation-resistant FWs.

### 3.2 Concept

The generated non-diffracting and attenuation-resistant waveform, \(\psi(\rho, \phi, z, t)\), is designed to acquire a longitudinal intensity profile over a finite distance \(L\) in a linear, isotropic, and homogeneous medium with a complex index of refraction \(n = n_r + i n_i\). The beam is a result of adding \(2N+1\) equal frequency co-propagating Bessel beams of order \(\ell\) as given by Eq. (2.15), and re-written here for convenience

\[
\psi(\rho, \phi, z, t) = e^{-i\omega t} \sum_{m=-N}^{N} A_m J_{\ell}(k^m_\rho \rho) e^{ik^m_z z} e^{i\ell\phi}.
\]  

(3.1)

For the \(m^{th}\) Bessel beam of order \(\ell\) in the superposition, the transverse wavenumber \(k^m_\rho\) is related to the longitudinal wavenumber \(k^m_z\) according to the consistency relation \(k^m_\rho = \sqrt{k^2 - (k^m_z)^2}\); where \(k\) is the complex wavenumber in the medium; given by \(k = k_r + ik_i\). Here, the problem is formulated under the assumption that the beam is initially generated in a lossless material before penetrating the absorbing medium \[91\]. Additionally, the wavenumbers \(k^m_z\) are set to be equally separated in the k-space, in a comb-like setting, with a separation of \(2\pi/L\); where \(L\) is desired the longitudinal extent of the beam. The real part of \(k^m_z\) is chosen as \(Re\{k^m_z\} = Q + \frac{2\pi m}{L}\); such that \(Q\) is a constant parameter that defines the beam localization \[70\]. In essence, larger values for \(Q\) imply smaller values for \(k_\rho\) (via consistency equation) and thus ensures stronger paraxiality. If the transverse wavenumbers are restricted to real values (since the beam is initially created in a lossless medium), then it follows that the imaginary part of the longitudinal wavenumber is expressed as \(Im\{k^m_z\} = \frac{\omega^2 n_r n_i}{c^2 \Re\{k^m_\rho\}}\). The transverse wavenumbers are thus given by \[91\]

\[
k^m_\rho = \sqrt{(n_r^2 - n_i^2) \frac{\omega^2}{c^2} - (Q + \frac{2\pi m}{L})^2 + \left(\frac{\omega^2}{c^2} Q + \frac{2\pi m}{L}\right)^2}.
\]  

(3.2)

Moreover, the complex coefficients \(A_m\) for each Bessel beam in the superposition are obtained via

\[
A_m = \frac{1}{L} \int_0^L F(z) e^{-(i\frac{2\pi m}{L} - Im(k^m_\rho))z} dz.
\]  

(3.3)
Note that the effect of material loss is compensated here in Eq. (3.3) by including the exponential term $e^{im(k_0)z}$, which was not present in Eq. (2.15) for the case of ordinary FWs inside lossless media. This term allows the resulting beam to exhibit a growing intensity profile that equalizes the loss profile of the absorbing medium — thus extending the propagation range of the beam.

There are two possibilities for encoding the desired longitudinal pattern $F(z)$ onto the beam: First, the pattern can be encoded on the beam’s axis at $\rho = 0$ if only zero order Bessel beams (i.e. $\ell = 0$) are incorporated in Eq. (3.1). In this case, the radius of the central spot is given as $\Delta \rho_0 = \frac{2k_0}{\pi} \rho_0$; where $k_0$ is a function of the $Q$ parameter, according to Eq. (3.2). Alternatively, the pattern can be encoded on the surface of a cylinder with finite radius $\rho_\ell$, when higher order Bessel beams (i.e. $\ell \geq 1$) are superimposed in Eq. (3.1). In such a case, the cylinder radius is evaluated by solving $\frac{d}{d\rho} J_\ell(\rho\sqrt{\omega^2/c^2 - Q^2})|_{\rho=\rho_\ell} = 0$; which also depends on the localization parameter $Q$. In this chapter, only beams created from a superposition of zero-order Bessel modes ($\ell = 0$) are considered. As such, the desired longitudinal intensity profile is encoded in the central spot of the beam.

### 3.3 Experimental Setup

The experimental generation of attenuation-resistant beams represents the first contribution of this dissertation. To achieve this, a holographic setup that incorporates a programmable was used. The experimental procedure was as follows: first, the Bessel Beam superposition in Eq. (3.1) was computed and transformed into a 2D Computer Generated Hologram (CGH). The complex transmission function of the FW at the origin of propagation (i.e. $\psi(\rho, \phi, z = 0, t)$) was encoded into an amplitude hologram that was addressed to a transmissive SLM (Holoeye LC2012 Amplitude SLM). The hologram equation is given by

$$H(x, y) = \frac{1}{2}(\beta(x, y) + \alpha(x, y)\cos[\Theta(x, y) - 2\pi(u_0x + v_0y)])],$$

where, $\alpha(x, y)$ and $\Theta(x, y)$ represent the amplitude and phase of $\psi(\rho, \phi, z = 0, t)$, respectively, and $\beta(x, y)$ is a bias function chosen as a soft envelope for the amplitude $\alpha(x, y)$ according to $\beta(x, y) = [1+\alpha(x, y)^2]/2$ [98]. The pattern was interfered with a plane wave $\exp[2\pi i (u_0x + v_0y)]$. This shifts the encoded pattern off-axis (in the Fourier plane) to the spatial frequencies $(u_0, v_0)$; thus making it easier to filter out the shifted pattern from the undesired on-axis noise by simply using an iris. In our experiment, $u_0$ and $v_0$ were set to $1/(4\Delta x)$; where $\Delta x$ is the SLM pixel pitch ($\Delta x = 36 \mu m$).

In principle, our theoretical model yields periodic waveforms (with infinite power flux) [91]. As such, a finite truncated circular aperture is superimposed over the generated hologram. For the efficient generation of the desired waveform over a spatial range $L$, a sufficient condition for the aperture diameter $(D)$ is

$$D \geq 2L \sqrt{\left(\frac{k_n}{\text{Re}\{k_\ell^{m=-N}\}}\right)^2 - 1}.$$
Chapter 3. Shaping the intensity of diffraction-attenuation-resistant beams

Figure 3.1: The experimental setup used to generate and detect attenuation-resistant Frozen Waves. Figure reproduced with permission from Dorrah et al. [92] (@ 2016 The Optical Society).

The apertured hologram was addressed onto the SLM (Holoeye LC2012 SLM) which operates with maximum efficiency on linearly polarized incident light. Accordingly, a polarizer-analyzer combination was used and oriented at (0°) and (90°) with respect to the SLM axis, as depicted in Fig. 3.1. The SLM then encodes the amplitude CGH on an expanded and collimated 532 nm laser beam. The generated waveform was imaged using a 4-f optical system and an iris which picks the shifted first diffraction order while blocking the higher diffraction orders and the on-axis noise.

The generated waveform was then transmitted (at normal incidence) into an acrylic container filled with red dye solution with an absorption window at $\lambda = 532$ nm. The absorbing fluid was prepared by adding few drops of red food color (propylene glycol, citric acis, and sodium benzoate) to 10 L of water. As such, the index of refraction of the resultant lossy solution was equal to $n = 1.4 + i0.32 \times 10^{-6}$ at $\lambda = 532$ nm. The container was placed in the focal plane of the imaging system ($z = 0$ plane), as shown in Fig 3.1. The generated waveform was then recorded inside the absorbing fluid using a sliding CCD camera that scanned the beam profile at discrete steps of 1 cm along the longitudinal direction. Finally, the evolution of attenuation-resistant FWs was compared with ordinary FWs that are not compensated for propagation losses (i.e. whose complex coefficients $A_m$ are obtained using Eq. (2.15) instead of Eq. (3.3)).

Three different intensity profiles were generated: a) constant intensity profile, b) on-off-on-off intensity profile, and c) increasing intensity profile. Each pattern was generated within a range $L = 1$ m, whereas the parameter $Q$ was set to 0.99999565 $\times k_r$. This choice of $Q$ ensures highly paraxial beams with small values of $k^m_\rho$ that respect the SLM bandwidth. It also yields a maximum value of $N = 6$ which results in $2N + 1 = 13$ Bessel beams in the superposition of Eq. (3.1). Furthermore, the choice of $N = 6$ enforces positive values for the real part of the longitudinal wavenumber ($Re\{k^m_z\}$) and real values for the transverse wavenumbers ($k^m_\rho$). Here, the central spot radius is $\sim 70$ µm. In order to verify that the generated waveforms are immune to propagation loss, each attenuation-resistant FW pattern was compared with its corresponding waveform that is not compensated for losses; i.e. generated without the exponential term $e^{lm(k^0_\rho)z}$ in Eq. (3.3). These waveforms are denoted as ordinary FWs. The results of our experiment are presented next.
3.4 Results

3.4.1 Constant Intensity Profile

For the first pattern, a beam with a constant intensity profile \( F(z) \) was generated over a finite distance \( L \) such that

\[
F(z) = \begin{cases} 
1 & 10 \text{ cm} \leq z \leq 50 \text{ cm}, \\
0 & \text{elsewhere}. 
\end{cases} 
\]  

(3.6)

Figure 3.2: Simulated and measured intensity profiles of an ordinary FW (not compensated for losses) and the attenuation-resistant FW inside the lossy fluid; both FWs are designed with the constant intensity profile, \( F(z) \), given by Eq. (3.6). a) Simulated intensity profile of an ordinary FW. b) Measured intensity profile of the ordinary FW. c) Simulated intensity profile of an attenuation-resistant FW. d) Measured intensity profile of the attenuation-resistant FW. e) Intensity profile, \( |\psi|^2 \), sampled at \( \rho = 0 \). The input energy of the ordinary FW and attenuation resistant FW are equal. Figure reproduced with permission from Dorrah et al. [92] (© 2016 The Optical Society).
Figures 3.2(a) and (b) depict the simulated and measured longitudinal intensity profiles of the ordinary FW (not compensated for losses). The experimental data was produced by aggregating consecutive slices sampled from the transverse images captured by the CCD camera with discrete steps of 1 cm along the longitudinal direction, $z$. The rendered 2D longitudinal profile was then filtered to reduce the CCD camera background noise.

As seen in Figs. 3.2(a) and (b), instead of maintaining the desired constant intensity profile defined by $F(z)$ in Eq. (3.6), the central spot of the ordinary FW suffers an exponential decay with propagation, as expected. Compensating for the medium losses by introducing the factor $e^{lm(k_0^2)z}$ in Eq. (3.3) addresses the decay limitation and yields a FW with predefined intensity profile that can maintain its required power level even in the presence of losses. The simulated and measured intensity profiles of the attenuation-resistant FWs are illustrated in Figs. 3.2(c) and (d), respectively. As the figures indicate, the desired longitudinal profile is preserved at fixed intensity level in very good agreement with the simulation results. The slightly larger spot size in the experimental data can be attributed to the averaging process of the CCD camera images.

Realizing non-decaying intensity profile inside the absorbing solution requires increasing the amount of power in the lateral rings (side lobes) of the FW. These rings act as an energy reservoir that intensifies the beam central spot with propagation, thus compensating for the medium losses and satisfying the conservation of energy principle. The energy transfer from the side rings to the central spot counteracts the absorption effects and preserves the spot at the desired intensity level. The reported attenuation-resistant behavior can thus be interpreted as an intensified self-healing process [96]. In the absence of propagation losses, such attenuation-resistant beams will exhibit an exponential growth in their longitudinal intensity profile.

The axial intensity was sampled at $\rho = 0$, as depicted in Fig. 3.2(e). The experimental measurements are in very good agreement with the simulation results which fully characterizes the generated beam propagation. The sampled intensity profile was compared with the case of ordinary FW, which clearly exhibits exponential decay inside the lossy fluid. To perform this comparison, the ordinary FW was generated in such a way that its transverse profile carries the same total power as the compensated FW at the $z = 0$ plane. This ensures fair comparison among the beams but also led to the high intensity level observed in the central spot of ordinary FW at shorter distances. Clearly, the ordinary FW, even with the same total input energy, does not preserve the desired intensity level for the entire propagation distance. This suggests that maintaining the axial intensity level using attenuation-resistant FWs is a result of careful redistribution of the energy with propagation as opposed to merely increasing the input flux.
3.4.2 Multi-Foci Intensity Pattern

In the second scenario, we demonstrate more complex control over the longitudinal intensity profile in the presence of losses by considering the following multi-foci intensity pattern.

![Simulated and measured intensities of an ordinary FW and the attenuation-resistant FW inside the lossy fluid](image)

Figure 3.3: Simulated and measured intensities of an ordinary FW and the attenuation-resistant FW inside the lossy fluid; the FWs are designed with the on-off-on-off intensity profile, $F(z)$, given by Eq. (3.7). a) Simulated intensity profile of an ordinary FW. b) Measured intensity of the ordinary FW. c) Simulated intensity profile of an attenuation-resistant FW. d) Measured Intensity profile of the attenuation-resistant FW. e) Measured transverse profile of attenuation-resistant FW at $z = 20, 30, 40,$ and 55 cm. f) Intensity profile, $|\psi|^2$, sampled at $\rho = 0$. The input energy of the ordinary FW and attenuation-resistant FW are equal. Figure reproduced with permission from Dorrah et al. [92].
Chapter 3. Shaping the intensity of diffraction-attenuation-resistant beams

\[ F(z) = \begin{cases} 
1 & 10 \text{ cm} \leq z < 25 \text{ cm}, \\
0 & 25 \text{ cm} \leq z < 35 \text{ cm}, \\
1 & 35 \text{ cm} \leq z \leq 50 \text{ cm}, \\
0 & \text{elsewhere}. 
\end{cases} \] (3.7)

The multi-foci intensity pattern defined by Eq. (3.7) is encountered in material processing applications and can be utilized in selective targeting of body tissues over a desired region, which is a common practice in imaging and treatments. The simulated and experimental ordinary FW patterns are illustrated in Figs. 3.3(a) and (b), showing a severe decay in the second peak of the beam. This effect is clearly alleviated in the simulated and measured attenuation-resistant FW depicted in Figs. 3.3 (c) and (d). In all cases, the measured waveforms show a very good agreement with the simulation model.

Furthermore, the evolution of the beam center spot was recorded at four different distances \( z = 20, 30, 40, \) and \( 55 \) cm, as depicted in Fig. 3.3(e). It is observed that, over the region where the desired pattern carries zero intensity \( (25 \text{ cm} \leq z \leq 35 \text{ cm}) \), a counter self-healing effect takes place where energy flows away from the intense spot and is redistributed over a ring of larger radius before it is focused again to form the second peak.

The axial intensity, sampled at \( \rho = 0 \), is also shown in Fig. 3.3(f). It is noticed that the ordinary FW suffers from an exponential decay (pronounced by comparing the intensity of the second peak with the first peak of the red curves). This is clearly mitigated in the attenuation-resistant FW (blue curves), which also demonstrate a good agreement with the desired pattern.

Similar to the case of Fig. 3.2(e), here we ensured that the total power in the transverse profile for the ordinary and attenuation-resistant FWs are the same at the \( z = 0 \) plane. In other words, realizing a predetermined intensity pattern in a lossy medium is an act of careful self-healing of attenuation-resistant FWs as opposed to merely increasing the input power.

3.4.3 Growing Intensity Profile

Lastly, we demonstrate an attenuation-resistant FW with an increasing intensity profile along the beam axis. For this pattern, the function \( F(z) \) is defined as

\[ F_{3}(z) = \begin{cases} 
\sqrt{5}z & 10 \text{ cm} \leq z \leq 50 \text{ cm}, \\
0 & \text{elsewhere}. 
\end{cases} \] (3.8)

As such, the generated waveform should exhibit a quadratic increase of intensity along its axis of propagation. The simulated and measured ordinary FWs are depicted in Figs. 3.4(a) and (b), respectively, and the corresponding attenuation-resistant patterns are shown in Figs. 3.4(c) and (d). The simulated and measured longitudinal patterns are in a very good agreement.
Figure 3.4: Simulated and measured intensities of the ordinary FW and the attenuation-resistant FW inside the lossy fluid; the FWs are designed with the increasing intensity profile, $F(z)$, given by Eq. (3.8). a) Simulated intensity profile of an ordinary FW. b) Measured intensity of the ordinary FW. c) Simulated intensity profile of an attenuation-resistant FW. d) Measured intensity profile of the attenuation-resistant FW. e) Intensity profile sampled at $\rho = 0$. The input energy of the ordinary FW and attenuation-resistant FW are equal. Figure reproduced with permission from Dorrah et al. [92] (© 2016 The Optical Society).

In addition, the axial intensity profile is sampled at $\rho = 0$ and plotted in Fig. 3.4(e). It is observed that for the ordinary FW the beam fails to exhibit the desired quadratic increase in its axial intensity with propagation. At longer propagation distances ($35 \text{ cm} \leq z \leq 50$), the exponential decay of the beam inside the fluid exceeds the quadratic increase in its axial intensity and thus the increase in the axial intensity is not pronounced. On the other hand, for the attenuation-resistant FW, the energy is redistributed and stored in the outer rings of the beam. These rings act as an energy supply that continuously intensify the beam center spot with propagation. Hence, the quadratic increase in the axial intensity becomes more pronounced. This can also be understood in the context of intensified self-healing
of Bessel beams [96]. It should be noted that, similar to the previous cases, the input energy for the ordinary and attenuation-resistant FWs is the same.

3.5 Summary

This chapter presented what is believe to be the first experimental demonstration of a class of beams, called attenuation-resistant Frozen Waves (FWs), that can maintain a predefined longitudinal intensity profile inside an absorbing fluid. The attenuation-resistance feature shown here is a result of shaping the longitudinal intensity profile of the waveform such that it acquires a growing intensity that counteracts the exponential loss profile of the absorbing medium. In addition to the compelling features of FWs presented here, namely the arbitrary longitudinal control of the intensity profile while overcoming propagation losses, FWs also posses non-diffracting and self-reconstructing behaviors, which are inherent properties of Bessel beams.
Chapter 4

Evolution of Orbital Angular Momentum in Longitudinally Structured Light

Light beams with an azimuthal phase dependency of $e^{i\ell \phi}$, known as vortex beams, have helical phase fronts and thus carry orbital angular momentum (OAM), a strictly conserved quantity with propagation. In this chapter, we describe how structured vortex Bessel beams can be engineered along the propagation direction, demonstrating unusual scenarios in which the OAM value can vary locally in both sign and magnitude along the beam’s axis, in a controlled manner, under free-space propagation. To reveal the underlying mechanisms of this phenomenon, we perform full modal decomposition and reconstruction of the generated beams to describe the evolution of their intrinsic OAM and topological charge with propagation. It will be shown that topological transition and the associated variation in local OAM rely on the creation, movement, and annihilation of local vortex charges without disturbing the global net charge of the beam, thus conserving the global OAM while varying it locally. Finally, we showcase two scenarios of rotating light structures that can reverse the sense of rotation or vary their rotational speed, on demand. This class of beams will particularly be useful in remote sensing, as will be discussed in details in Chapter 8. The results shown here may be perceived as an experimental demonstration of the Hilbert Hotel paradox, while advancing our understanding of topological deformations in general\(^1\).

\(^1\)Controlling the topological charge of non-diffracting beams with propagation and investigating the underlying dynamics are two main contributions of this thesis that were first published in Refs. [99, 100, 101] and adapted in this chapter.
4.1 Overview

Vortex beams refer to a class of Structured Light beams [69] characterized by azimuthal phase dependency $\sim e^{i\ell \phi}$, where $\ell$ is known as the topological charge of the beam [44, 45, 60]. Such beams possess $\ell$ intertwined helical phase-fronts with an on-axis phase singularity and carry orbital angular momentum (OAM) value of $\ell \hbar$ per photon. The handedness and order of the helical phase twist are determined by the sign and magnitude of $\ell$, respectively. OAM differs fundamentally from spin angular momentum (SAM) associated with circular polarization. A striking difference between the two momenta is manifested in the range of their allowable values: where SAM is limited to $\pm \hbar$ per photon, the OAM can acquire unbounded value of $\ell \hbar$ per photon, thus dramatically exceeding the value of SAM. Light’s OAM has been utilized in many applications including optical trapping, materials processing, and imaging, and has been extensively reviewed to date [50, 54, 61, 62]. In particular, the unbounded and orthogonal OAM states of light have been extensively deployed in data communications as means of encoding information; both classically and in the quantum regime [54, 55, 56, 57, 58, 59].

Examples of vortex beams that carry OAM include Laguerre-Gauss and higher order Bessel beams [44, 60]. Figure 4.1 depicts the helical wavefront, transverse intensity, and transverse phase profiles of vortex beams with different topological charges. It is observed that vortex beams with larger topological charge values exhibit larger amount of twist in their phase and carry their energy over a ring with larger radius.

In principle, when optical vortices propagate in a homogeneous isotropic transparent medium, both their spin and orbital angular momenta are conserved [102, 103, 104, 105]. In other words, light’s OAM is manifested as a strictly conserved quantity, signified by a quantized topological charge $\ell$, and cannot be altered under free unperturbed propagation. In very special cases, however, non-trivial topological deformations have been deliberately realized; originally, by interfering vortex modes with Gaussian beams [106], then by realizing charge flipping induced in a non-linear medium [107, 108], and in noncanonical vortices generated by an astigmatic optical setup [109, 110]. More recently, non-diffracting optical vortices with longitudinally varying topological charge have been observed in air [111, 99, 112, 113], thus opening new opportunities in venues like optical trapping [52, 53, 75], dense data communications [76], and remote sensing [114]. In all these developments, a fundamental question on how OAM conservation is seemingly broken, without violating any laws of physics, naturally arises. However, a satisfactory answer that resolves such paradoxical behavior, observed in recently developed classes of Structured Light, has not been provided yet. In addition, the physical dynamics associated with the boundaries at which the light beam undergoes its topological transition has scantily been investigated.

In this chapter, we examine the evolution of OAM in longitudinally (quasi 3D) structured vortex beams — where the topological charge exhibits unusual, yet controlled, non-trivial transitions with

---

2 Exploiting OAM of light in remote sensing is one of the main contributions of this thesis, first published in Ref. [114], and will be discussed in details in Chapter 8.
propagation. We start by establishing a general methodology for designing and generating such beams, and we consider two case studies in which the sign and magnitude of \( \ell \) are changed with propagation. We then perform full modal decomposition and reconstruction of those beams using the Bessel bases. This approach allows us to: a) gain insights into the interplay between the intermodal phases within the beam and its evolution, hence, understanding the underlying mechanism of its topological deformation, b) it allows us to quantitatively measure the OAM density in addition to the local and global values of OAM (and topological charge) of the reconstructed beam as it propagates. As such, we reveal the general mechanism that governs the topological transitions along the beam’s axis. We discuss how this mechanism manifests as a practical realization of transfinite mathematics, exhibiting a striking analogy with what is known as the Hilbert’s hotel paradox [115, 116, 117, 118]. We then demonstrate how the OAM, despite its local variation, is always conserved globally — thus providing a quantitative interpretation addressing the OAM conservation paradox in 3D Structured Light.

Figure 4.1: Schematic diagram illustrating the helical wavefront, transverse intensity and phase profiles of vortex modes with different topological charges, \( \ell \) (© 2011 E. Karimi).
4.2 Controlling the Topological Charge of Vortex Beams along the Propagation Direction

4.2.1 Concept

Controlling the topological charge of structured vortex beams is realized here by superimposing multiple vortex modes, $\psi_\ell$, of topological charge $\ell$. Further, each vortex mode itself consists of a superposition of multiple co-propagating beams carrying the same charge $\ell$ but with different spatial frequencies. Expanding on the formulation given by the Frozen Wave method, discussed in Chapter 3, the vortex beam with varying charge $\ell$ is expressed as:

$$U(\rho, \phi, z, t) = \sum_{\ell = -\infty}^{\infty} \psi_\ell(e^{-i\omega t} \sum_{\ell = -\infty}^{\infty} \sum_{m = -N}^{N} A_{\ell, m} J_{\ell}(k_{\rho, m}) e^{i(\ell\phi + k_{\ell, m}z)}).$$

(4.1)

Each OAM state $\psi_\ell$ is composed of $2N + 1$ Bessel beams of order $\ell$. Additionally, for the $m^{th}$ Bessel beam in each OAM state, the transverse wavenumber $k_{\rho, m}$ is related to the longitudinal wavenumber $k_{z, m}$ via $k_{\rho, m} = \sqrt{k^2 - (k_{z, m})^2}$. The real part of the longitudinal wavenumber is $\text{Re}(k_{z, m}) = Q_\ell + \frac{2\pi m}{L}$, where $Q_\ell$ is a constant that defines the transverse localization of the beam. Further, the complex coefficients $A_{\ell, m}$ represent weighting factors for the Bessel beams in the superposition such that

$$A_{\ell, m} = \frac{1}{L} \int_0^L F_\ell(z) e^{-(i\frac{2\pi m}{L})z} dz,$$

(4.2)

where the morphological function $F_\ell(z)$ defines the desired longitudinal intensity profile of each OAM state ($\psi_\ell$) over a finite distance $L$.

By properly adjusting $F_\ell(z)$, the contributions of each OAM state $\psi_\ell$, along the beam axis, can be determined at-will. As such, specific OAM state(s) with desired topological charge ($\ell$) can be selected to effectively contribute to the beam’s center over finite space interval while other states are readily switched-off by being dispersed over larger space in the outer rings of the beam, as seen in Refs. [70, 71, 87, 88] and in Chapter 3. The effective topological charge ($\ell$) within the beam’s center can thus be controlled along the propagation direction. Longitudinal control of $\ell$ can yield waveforms whose phase-fronts change their helicity with propagation. Figure 4.2(a) schematically illustrates an idealistic view of phase-front that reverses its helicity (transitioning from $\ell = 1$ to $\ell = -1$) and doubles its number of helices (from $\ell = -1$ to $\ell = -2$) with propagation.

This vortex beam can be realized via superposition of OAM states ($U = \psi_1 + \psi_{-1} + \psi_{-2}$) over length

---

3Summation over the topological charge values is an important extension to the Frozen Wave method and is one of the main contributions of this thesis, first published in Ref. [99, 100]
Chapter 4. Evolution of Orbital Angular Momentum in 3D Structured Light

Figure 4.2: Wavefront reversing its handedness and increasing its topological order with propagation. a) An idealistic schematic view of the helical phase-front of a beam whose topological charge takes on the values $\ell = 1, -1, \text{ and } -2$ over three consecutive space intervals. The phase-front reverses its twist from $\ell = 1$ to $\ell = -1$, and increases the number of its helices from $\ell = -1$ to $\ell = -2$. b) Simulated evolution of the transverse phase and intensity profiles of a beam with 3 OAM states ($U = \psi_1 + \psi_{-1} + \psi_{-2}$). The local phase reverses its sense of rotation (marked by the arrow) and increases its order of twist with propagation. The insets depict the intensity profile: the beam radius increases upon the transition $\ell = -1$ to $\ell = -2$, implying an increase in the order of the Bessel modes. Figure adapted from Dorrah et al. [99] (© 2016 American Physical Society).

$L = 1 \text{ m}$, by incorporating the morphological function $F_\ell(z)$ defined as

$$F_\ell(z) = \begin{cases} 
F_1 = 1 & 5 \text{ cm} \leq z \leq 35 \text{ cm}, \\
F_{-1} = 1 & 35 \text{ cm} \leq z \leq 65 \text{ cm}, \\
F_{-2} = 1 & 65 \text{ cm} \leq z \leq 95 \text{ cm}, \\
F_\ell = 0 |_{\ell \neq \ell} & \text{elsewhere.}
\end{cases}$$

(4.3)

A numeric evaluation of the phase and intensity profiles of $U$, at three different propagation distances ($z = 26 \text{ cm}$, $z = 49 \text{ cm}$, and $z = 84 \text{ cm}$), is given in Fig. 4.1(b). It is observed that the effective charge $\ell$ (and, hence, the intrinsic OAM [119]) undergo predetermined (and non-trivial) transition with propagation — a curious behavior that seemingly breaks OAM conservation. To reveal the underlying mechanisms that govern such topological transition, in the following, we experimentally generate and analyze few scenarios in which the vortex beams change their topological charge with propagation.
4.2.2 Experimental Setup

Figure 4.3: A holographic setup to create, digitally propagate and then detect 3D structured vortex beams. Here, BE refers to the beam expander, Pol is the polarizer, and BS is the beam splitter. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

The longitudinally structured vortex beams were generated using digital holograms realized via programmable spatial lights modulators (SLMs). Figure 4.3 illustrates the experimental setup used for beam generation and detection. First, a linearly polarized He-Ne ($\lambda = 632.8$ nm) Gaussian mode was expanded, collimated, and imaged onto SLM-1 (LCOS reflective phase SLM with 1920x1080 resolution and 8 $\mu$m pixel pitch). Next, following the modulation scheme described in Refs. [120, 121], the desired pattern given by Eq. (4.1) was encoded into a 2D hologram that was displayed on SLM-1. In addition, a 2D grating function was encoded on the hologram to spatially separate the generated signal in the $k$-space from the on-axis carrier. As such, the reflected beam from SLM-1 was imaged and spatially filtered using a 4$f$ system incorporating an iris to remove unwanted diffraction orders. A CCD camera (CCD-1) was then used to monitor the transverse intensity profile of the resulting beam with propagation. Finally the detection module, comprised of SLM-2 and a 2-f system, was used to decompose the propagating beam. More details on the modal decomposition and reconstruction can be found in Appendix A.

In the following section, we showcase two scenarios in which the sign and magnitude of $\ell$ vary with propagation. We then describe the evolution of the aforementioned physical quantities with propagation and show how the OAM conservation is always satisfied.

4.3 Generating and Measuring Longitudinally Structured Vortex Beams with Varying OAM and Charge

In this section, we experimentally demonstrate two different light patterns constructed from a superposition of the vortex modes $\psi_\ell$. Each mode $\psi_\ell$ is composed of 5 Bessel beams. The longitudinal wavenumbers $k_{\ell,m}^z$ were centered at $0.999995 \times k_0$ ($k_0 = 2\pi/\lambda_0$), and were equally spaced with separations of $4\pi$ in the k-space. This choice of $k_{\ell,m}^z$ yield Bessel beams whose transverse wavenumbers...
are sufficiently separated to ensure low cross-talk among the Bessel modes (<-10 dB). This becomes particularly useful for fulfilling the orthogonality requirement in the process of modal decomposition and reconstruction as discussed in the previous section (additional details can be found in Appendix A). To this end, we generated and fully reconstructed two structured beams in which the sign and magnitude of the topological charge were made to vary with propagation. We start by discussing the first pattern as described below.

### 4.3.1 Pattern 1: Reversing the Sign of the Topological Charge

In the first experimental scenario, the propagating field \(U(\rho, \phi, z, t)\) flips its topological charge from \(\ell = 2\) to \(\ell = -2\) as it propagates. In this case, \(U(\rho, \phi, z, t) = \psi_2 + \psi_{-2}\). The morphological function \(F_\ell(z)\) associated with each vortex mode \(\psi_\ell\) was defined as

\[
F_\ell(z) = \begin{cases} 
F_2 = 1 & 0 \text{ cm} \leq z \leq 14 \text{ cm}, \\
F_{-2} = 1 & 14 \text{ cm} \leq z \leq 28 \text{ cm}, \\
F_2 = F_{-2} = 0 & \text{elsewhere}. 
\end{cases}
\]  

(4.4)

According to Eq. (4.4), the beam is designed to possess a topological charge \(\ell = 2\) over the region \((0 \text{ cm} \leq z \leq 14 \text{ cm})\), and then undergoes topological inversion at \(z = 14 \text{ cm}\), at which the charge is reversed to become \(\ell = -2\) for the remaining distance \((14 \text{ cm} \leq z \leq 28 \text{ cm})\). The complex amplitude hologram for this beam has been computed based on Eq. (4.1) and the procedure described in Ref. [120], then displayed on SLM-1. The corresponding intensity profiles, measured by CCD-1, are depicted in Fig. 4.4. The longitudinal intensity profile in Fig. 4.4(a) has been rendered by aggregating 1D slices obtained from the transverse CCD images taken over 30 cm (with 1 cm steps) in the \(z\) direction. The rendered profile is in good agreement with the simulated results shown in the inset. The transverse profiles are displayed in Fig. 4.4(b) at various propagation distances along the \(z\) direction. Note that there exist a region in space, at \(z = 14\) cm, where the vortex mode collapse from a closed ring into a petal-like structure. This breakdown is a signature of topological inversion as will be shown later.

### Modal Decomposition for Pattern 1

The intensity measurements shown in Fig. 4.4, however, are not sufficient to examine key properties of the generated beam such as its OAM and topological charge. To access these quantities, we performed full modal decomposition and reconstruction of \(U(\rho, \phi, z, t)\) into its Bessel basis. The measured amplitudes of the Bessel beams correspond to the absolute values \(|A_{\ell,m}|\), whereas the measured intermodal phases accounts for both the phases associated with \(A_{\ell,m}\) in addition to the phases accumulated with propagation. More specifically, the intermodal phases describe the differences among the phases of \(A_{\ell,m}e^{ik\ell mz}\), where phase accumulation is different for each Bessel beam in the superposition.
Chapter 4. Evolution of Orbital Angular Momentum in 3D Structured Light

Figure 4.4: Measured and simulated intensity of the beam reversing its charge. a) Measured longitudinal intensity profile (the inset depicts the simulated intensity profile). b) Measured and simulated transverse intensity profiles obtained at propagation distances: $z = 0$ cm, $z = 7$ cm, $z = 14$ cm, $z = 21$ cm, and $z = 28$ cm. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

The measured amplitudes and intermodal phases among the Bessel beams are shown in Fig. 4.5 in comparison with the theoretical predictions. Here, we show the coefficients at five distances: $z = 0$ cm, $z = 7$ cm, $z = 14$ cm, $z = 21$ cm, and $z = 28$ cm, chosen as representative samples for the beam evolution. In principle, the amplitudes of $A_{\ell,m}$ remain constant throughout the beam’s propagation (as long as it is not perturbed), as seen from Fig. 4.5(a). On the other hand, the intermodal phases vary with propagation and play a critical role in shaping the longitudinal beam profile. For instance, at $z = 0$ cm, the Bessel beams within $\psi_{-2}$ are out-of-phase and, hence, destructively interfere and do not contribute to the beam’s intensity at the center. As the beam propagates, the intermodal phases evolve such that, at $z = 7$ cm, the Bessel beams within $\psi_2$ become in-phase and, hence, interfere constructively in the beam’s center while, at the same $z$ position, the Bessel beams within $\psi_{-2}$ are still out-of-phase and destructively interfere.

Destructive interference in the vortex mode $\psi_{-2}$ implies that its energy is dispersed in the outer rings of the beam. In this case, the topological charge in the beam’s center is predominately $\ell = 2$ as a result of the contributions of $\psi_2$, as will be verified shortly. This state of intermodal phases is reversed later on. For example, at $z = 21$ cm, the Bessel beams associated with $\psi_{-2}$ become in-phase whereas those associated with $\psi_2$ become out-of-phase. Hence, the beam’s topology is reversed. Evidently, there exist
Mechanism of Topological Charge Inversion

To investigate the topological inversion more closely, we compare the measured Poynting vector (inferred from modal reconstruction) to the theoretically predicted one. Recall that the Poynting vector points in the direction perpendicular to the wavefront [44] and is given by the following expression for scalar
beams [122]
\[
S = \frac{\epsilon_0 \omega}{4} \left[ i(U \nabla U^* - U^* \nabla U) + 2k_0|U|^2 \hat{z} \right].
\] (4.5)

Here, \( \epsilon_0 \) is the free space permittivity and \( \omega \) is the angular frequency. In vortex modes, the Poynting vector typically follows a spiraling path with propagation (perpendicular to the rotating helical wavefront) and, hence, it has non-zero transverse components. Figure 4.6(a) shows the transverse components of the Poynting vector of the reconstructed beam at various planes along its propagation direction, and compared with the simulated results. At \( z = 7 \) cm, it is observed that the energy of the vortex beam predominately flows in the clockwise direction. This behavior is then disrupted at \( z = 14 \) cm, after which the energy reverses its flow to the counter-clockwise direction (as seen at \( z = 21 \) cm). Topological inversion is confirmed by looking at the reconstructed phase profiles depicted in Fig. 4.6(b), from which it is evident that the phase inverts its helicity from \( \ell = 2 \) to \( \ell = -2 \) as the beam propagates.

Reversal of the topological charge hinges on several intriguing dynamics. First, the vortex beam collapses from a closed ring (with charge \( \ell = 2 \)) to a petal-like structure. This transformation is a consequence of overlapping vortex modes with opposite helicities (\( \psi_2 \) and \( \psi_{-2} \)) over the same space region. Second, the phase-front gradually loses its helicity until it becomes unfolded (flat) at \( z = 14 \) cm. This unwrapping of the phase-front is associated with the formation of chains of smaller vortices with alternating signs located along the binary phase dislocations, as marked by ‘+’ and ‘-’ signs. Note that chains of vortices with alternating signs are typically a signature of fractional (non-integer) vortex beams [123, 124, 125], and are experienced here during the topological inversion. Third, the vortices with ‘-’ sign approach the beam center, guided along the path of phase dislocation, and coalesce into one large vortex that replaces the \( \ell = 2 \) vortex, thus reversing its topology. Simultaneously, the \( \ell = 2 \) vortex is divided into smaller vortices that exit from the beam center towards its outer ring. Next, the phase-front gradually acquires a helical nature but in the opposite sense (\( \ell = -2 \)). Finally, after all the charges have been judiciously transported to and from the beam’s center (guided on the path of phase dislocations), the ‘+’ charges annihilate and the petal-like structure closes again into a ring, this time with charge \( \ell = -2 \).

The OAM density in the \( z \)-direction for this beam is obtained from [122]
\[
\jmath_z = \left( \rho \hat{\rho} \times \frac{S}{c^2} \right)_z.
\] (4.6)

Figure 4.6(c) illustrates the distribution of OAM density in the beam center as the beam propagates. The OAM density evolves from positive values at \( z = 7 \) cm (associated with \( \ell = 2 \)), to acquire negative values with same magnitude and opposite sign at \( z = 21 \) cm — as expected from a vortex beam with \( \ell = -2 \). Interestingly, there exist an overlap region midway, at \( z = 14 \) cm, where the OAM density is dispersed over concentric rings with alternating signs in agreement with Poynting vector and phase pictures discussed above.
Figure 4.6: Theoretical and experimental evolution of the Poynting vector, phase, and OAM density of our 3D Structured Light reversing its charge. All plots reveal the transverse plane at various propagation distances, \( z \). a) Transverse components of the Poynting vector. The white arrows trace the direction of energy flow and the ‘+’ and ‘-’ signs denote the polarity of the local vortices. The boxed regions represent enlarged sections of the image. b) Reconstructed and simulated phase profile. The arrows depict the sense of helicity in the phase and their number denote the topological charge. c) OAM density profiles reconstructed at different propagation distances. The insets depict the simulated profiles. The OAM density evolves from positive to negative distribution as the beam propagates. Figure reproduced from Dorrah \textit{et al.} [101] (© 2018 American Physical Society).
Conservation of OAM and Charge

Thus far we have described the mechanism governing the topological inversion from $\ell = 2$ to $\ell = -2$ in our 3D structured beam. The transition occurs as a result of judicious creation, movement, and annihilation of phase singularities as the beam propagates. The evolution of the topological charge is further illustrated in Fig. 4.7 which depicts the 3D path traversed by the charges in the vicinity of the beam’s center as it undergoes the topological transition. The reconstructed paths have been obtained by tracking the location of phase singularities near the beam’s center.

Figure 4.7: Movement of charges in the vicinity of the beam’s center. The solid lines depict the experimentally reconstructed paths whereas the markers represent the theoretical paths. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

The plot emphasizes the role of the outer rings, acting as a reservoir that injects new charges into the beam’s center, when needed, to satisfy the desired OAM profile dictated by Eq. (4.4). Notice how the negative charges (in turquoise markers) are injected into the beam’s center while the positive charges exit from the center into the outer rings of the beam to realize the topological inversion at $z = 14$ cm. A fundamental question on how the OAM and charge are conserved, in such case, naturally arises. To provide a quantitative answer, we computed the OAM and effective topological charge under two aperture sizes. The total angular momentum $J_z$ per unit length is evaluated by integrating the OAM density over the transverse plane of the beam according to

$$J_z = \int \int j_z \hat{z} \cdot dA. \quad (4.7)$$

In addition, the effective topological charge of the beam is inferred from the ratio between its OAM $J_z$
and Energy $W$ per unit length such that [126]

$$\frac{J_z}{W} = \int \int_R (\rho \hat{\rho} \times \langle E \times B \rangle)_z \cdot dA = \frac{\ell_{\text{eff}}}{\omega}.$$  

(4.8)

Figure 4.8: Evolution of the OAM and topological charge considering two aperture diameters: $D = 300 \mu m$, and $D = 5 \text{ mm}$. a) Local and global OAM of the beam obtained along its direction of propagation. b) Local and global effective topological charge evaluated along the beam’s axis. It is noticed that the local charge evolves from $\ell = 2$ to $\ell = -2$ when observed over the smaller aperture while the global charge remains zero at all planes along the $z$-direction. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

The OAM and effective topological charge are depicted in Fig 4.8(a) and (b), respectively. The experimental results are compared with the simulations and are in very good agreement. We note that these quantities have been obtained over two different aperture diameters: a) $D = 300 \mu m$, and b) $D = 5 \text{ mm}$. When computed over the smaller aperture, the OAM exhibits a transition from positive to negative quantities along the propagation direction, and it reaches an inflection point characterized by zero-OAM in between. Evidently, the OAM conservation is seemingly broken over this section of the beam. We refer to this quantity as the local OAM of the beam. Interestingly, when the OAM is evaluated across the entire cross section of the beam ($D = 5 \text{ mm}$) it maintains the same value regardless of the propagation distance $z$. The global OAM is always zero in this case. These observations quantitatively establish that while the OAM may vary locally, the global OAM is always conserved.

Similar behavior is observed when evaluating the effective topological charge $\ell_{\text{eff}}$ under two aperture sizes, as shown in Fig. 4.8(b). It is noticed that the local charge evolves from $\ell_{\text{eff}} = 2$ to $\ell_{\text{eff}} = -2$ when observed over the smaller aperture while the global $\ell_{\text{eff}}$ remains zero at all planes along the $z$ direction. Therefore, topological inversion occurs without violating neither the OAM nor the charge conservation. While the number of charges within a sub volume of the beam may vary, the net charge across the entire volume remains in balance. In essence, charges are created in pairs across the beam and topological inversion relies on the movement of some of those charges towards the beam’s center leaving the opposite charges in the outer rings of the beam. To complement these findings, next, we present another scenario in which the magnitude of the topological charge is designed to vary with propagation.
4.3.2 Pattern 2: Changing the Magnitude of the Topological Charge

In the second case study, the propagating field $U(\rho, \phi, z, t)$ was designed to change the magnitude of its topological charge from $\ell = 1$ to $\ell = 3$ as it propagates. For this scenario, the propagating waveform is expressed as $U(\rho, \phi, z, t) = \psi_1 + \psi_3$. The morphological function $F_\ell(z)$ associated with each vortex mode $\psi_\ell$ is chosen as

$$
F_\ell(z) \begin{cases} 
F_1 = 1 & 0 \text{ cm} \leq z \leq 11 \text{ cm}, \\
F_3 = 1.15 & 11 \text{ cm} \leq z \leq 23 \text{ cm}, \\
F_1 = F_3 = 0 & \text{ elsewhere}. 
\end{cases}
$$

(4.9)

![Figure 4.9](image)

Figure 4.9: Measured and simulated intensity of the beam varying its topological charge. a) Measured longitudinal intensity profile (the inset depicts the simulated intensity profile). b) Measured and simulated transverse intensity profiles obtained at propagation distances: $z = 3$ cm, $z = 6$ cm, $z = 11$ cm, $z = 16$ cm, and $z = 19$ cm. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

Note that a value slightly larger than one was assigned to $F_3$ over the interval $(11 \text{ cm} \leq z \leq 23 \text{ cm})$ to ensure that $\psi_1$ and $\psi_3$ are generated at equal intensity levels, i.e. to compensate for the energy mismatch between their Bessel modes.

Based on Eq. (4.9), the beam is designed to have a topological charge $\ell = 1$ over the interval $(0 \text{ cm} \leq z \leq 11 \text{ cm})$, and then experiences a transition in its topological charge at $z = 11$ cm, from $\ell = 1$ to
\( \ell = 3 \). The beam then maintains this charge \( (\ell = 3) \) for the remaining distance \((11 \text{ cm} \leq z \leq 23 \text{ cm})\) before it switches-off. Figure 4.9 shows the intensity profiles of the generated beam. The longitudinal intensity profile has been rendered by aggregating 1D slices obtained from the transverse CCD images taken over 30 cm (with 1 cm steps) in the \( z \) direction, and is in good agreement with the simulated results shown in the inset. The transverse profiles are shown in Fig. 4.9(b) at different planes along the direction of propagation. Note how the vortex beam diameter is increased at \( z = 19 \text{ cm} \) — a signature of increasing the topological charge. In addition, similar to the case of topological inversion, here, there exist a region in space (at \( z = 11 \text{ cm} \)) where the closed ring of the vortex undergoes shape deformation. This deformation is a characteristic of topological transition. It can be attributed to the spatial redistribution incurred to the phase singularities, associated with varying the topological charge.

In essence, the intensity profile is deformed judiciously to allow certain number of charges to enter to (or exit from) the beam’s center to satisfy Eq. (4.9), as will be shown.

### Modal Decomposition for Pattern 2

Similar to the first generated pattern, to access key quantities of the second generated beam such as its Poynting vector and OAM, full modal decomposition and reconstruction of the field into its Bessel basis was performed along the propagation direction. Modal decomposition also provides useful insights into the dynamics of topological transition of the 3D structured beam from \( \ell = 1 \) to \( \ell = 3 \).

Figure 4.10 depicts the measured amplitudes and intermodal phases among the Bessel beams of \( U(\rho, \phi, z, t) \) in comparison with the theoretical predictions. The modal coefficients are displayed at five distances: \( z = 3 \text{ cm}, z = 6 \text{ cm}, z = 11 \text{ cm}, z = 16 \text{ cm}, \text{ and } z = 19 \text{ cm} \), selected as illustrative samples for the 3D structured beam’s evolution.

Evidently, the amplitudes \( |A_{\ell,m}| \) remain constant throughout the beam propagation, whereas the intermodal phases (proportional to \( \angle A_{\ell,m}e^{ik\ell mz} \)) vary with propagation, thus shaping the longitudinal beam profile. The frame at \( z = 3 \text{ cm} \) in Fig. 4.10 captures the intermodal phases within \( \psi_1 \) as it progresses to become in-phase later on at \( z = 6 \text{ cm} \). The Bessel beams of \( \psi_3 \), on the other hand, are out-of-phase at this position. They destructively interfere and their energies are dispersed into the outer rings of the beam. Hence, they do not contribute to the beam’s center and the topological charge thereby becomes predominately \( \ell = 1 \). The opposite picture is seen later on at \( z = 16 \text{ cm} \) and \( z = 19 \text{ cm} \); where the Bessel beams associated with \( \psi_3 \) progress to become in-phase and those associated with \( \psi_1 \) become out-of-phase. Hence, the topological charge \( \ell \) evolves from \( \ell = 1 \) to \( \ell = 3 \). Certainly, there is an overlap region, which exists at \( z = 11 \text{ cm} \), and where the contributions of both \( \psi_1 \) and \( \psi_3 \) are present in the beam’s center, and where the topological transition takes place.
Mechanism of Topological Charge Transition

To investigate the mechanism by which the vortex beam undergoes topological charge transition, we obtained the Poynting vector of the reconstructed field. Figure 4.11(a) shows the transverse components of the Poynting vector of the reconstructed beam at multiple planes along the beam’s axis. At the plane $z = 5$ cm, the energy circulates in the clock-wise direction, over a small ring, where the charge is supposedly $\ell = 1$. Eventually, at $z = 17$ cm, the charge has evolved from $\ell = 1$ to $\ell = 3$, signified by the energy circulation over a larger ring. Such topological transition is also confirmed by looking at the reconstructed phase profiles depicted in Fig. 4.11(b); from which it is clear that the number of helical phase-fronts is increased from 1 to 3, while maintaining the same sense of helicity, as the beam propagates.
Figure 4.11: Theoretical and experimental evolution of the Poynting vector, phase, and OAM density of our 3D Structured Light varying its charge. All plots reveal the transverse plane at various propagation distances, $z$. a) Transverse components of the Poynting vector. The white arrows trace the direction of energy flow and the ‘+’ and ‘−’ signs denote the polarity of the local vortices. The boxed areas represent enlarged sections of the image. b) Reconstructed and simulated phase profile. The arrows depict the sense of helicity in the phase and their number denote the topological charge. c) OAM density profiles reconstructed at different propagation distances. The insets depict the simulated profiles. The OAM density triples its magnitude as the beam propagates. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).
Evolution of the topological charge from $\ell = 1$ to $\ell = 3$ relies on several interesting dynamics that are reminiscent to those associated with topological inversion. First, the closed ring forming the vortex beam (with $\ell = 1$) gradually splits into two sections. This deformation is accompanied by the formation of two smaller vortices around the beam’s center, signified by the ‘+’ sign in Fig. 4.11(a). It is also associated with the creation of two new singularities in the phase-front, which gradually acquire an azimuthal phase gradient as the beam propagates. Second, the newly formed vortices (charges) approach the beam’s center and come into the vicinity of the original vortex. Third, as the three vortices (charges) come closer, the divided ring gradually merges again, but this time into a larger diameter, consistent with increasing its charge. Simultaneously, the phase-front evolves into three intertwined helices — a signature of acquiring a charge $\ell = 3$. Interestingly, the intertwined helices possess three distinguished singularities connected via branch cuts as opposed to sharing one singularity.

Shape deformation observed here is a generic behavior and can be regarded as a signature of topological transition. In any structured (non-trivial) transition from $\ell_1$ to $\ell_2$, the beam undergoes shape deformation, thus creating $|\ell_2 - \ell_1|$ channels of zero intensity allowing to transport charges to (from) the beam’s center. This observation holds true in the case of topological inversion as well. These results advance prior work on single component beams vortex dynamics by Gouy phase [127], confirming the prediction by the authors of the anticipated rich dynamics of superposition fields.

The OAM density of the reconstructed beam has been obtained via Eq. (4.6), and is depicted in Fig. 4.11(c) at three planes along the $z$ direction. Note how the OAM density acquires larger values, three times its initial value, and gets redistributed over larger diameter as the charge evolves from $\ell = 1$ to $\ell = 3$. A discussion on OAM and charge conservation is presented next.

### Conservation of OAM and Charge

The transition from $\ell = 1$ to $\ell = 3$ in our structured vortex beam occurs as a result of controlled creation and movement of two phase singularities in the vicinity of the beam’s center as it propagates.

Figure 4.12 illustrates the 3D path traversed by the charges near the beam’s center as it undergoes the topological transition. The charges created in the outer rings of the beam are transported into the beam’s center at the prescribed distance, $z = 11$ cm, to satisfy the desired OAM profile dictated by Eq. (4.9).

To examine OAM conservation, we evaluated the OAM of the beam, given by Eq. (4.7) over two aperture diameters: a) $D = 400 \mu m$, and b) $D = 5$ mm. Additionally, the effective topological charge $\ell_{\text{eff}}$ was computed from Eq. (4.8) considering the same two aperture sizes.

The OAM and $\ell_{\text{eff}}$ are depicted in Fig. 4.13(a) and (b), respectively. The experimental results are compared with the simulations and are in very good agreement. In the limit of small aperture size (excluding the outer rings of the beam), the OAM exhibits a transition into larger values as it propagates. Therefore, the OAM conservation is seemingly broken locally, i.e. over this finite section.
Chapter 4. Evolution of Orbital Angular Momentum in 3D Structured Light

Figure 4.12: Movement of charges in the vicinity of the beam’s center. The solid lines depict the experimentally reconstructed paths whereas the markers represent the theoretical paths. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

Figure 4.13: Evolution of the OAM and topological charge considering two aperture diameters: \( D = 400 \) µm, and \( D = 5 \) mm. a) Local and global OAM of the beam obtained along its direction of propagation. b) Local and global effective topological charge evaluated along the beam’s axis. It is noticed that the local charge evolves from \( \ell = 1 \) to \( \ell = 3 \) when observed over the smaller aperture while the global charge remains fixed at all planes along the \( z \)-direction. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).

of the beam. However, when the OAM is evaluated across the entire cross section of the beam (\( D = 5 \) mm), it maintains a fixed value regardless of the propagation distance \( z \). The \textit{global} OAM maintains a value that lies in between the minimum and maximum local OAM values, and is always conserved in this case. Similar to the case of topological inversion, here, while the OAM varies locally, the global
OAM is always conserved. A similar picture is also observed in the effective topological charge $\ell_{\text{eff}}$, as shown in Figure 4.13(b). While the local charge evolves from $\ell_{\text{eff}} = 1$ to approach $\ell_{\text{eff}} = 3$, the global $\ell_{\text{eff}}$ remains fixed at $\ell_{\text{eff}} = 2.15$ at all planes along the $z$ direction. Note that $\ell_{\text{eff}}$ is slightly larger than 2 (the average of $\ell = 1$ and $\ell = 3$) due to the asymmetric weightings assigned to $\psi_1$ and $\psi_3$ in Eq. (4.9).

These interpretations are general and prove that while the total number of charges entering/exiting a finite volume within the beam can vary, the total number of charges carried by the entire system is always in equilibrium, provided that it is a closed system. In this case, local topological transitions can occur as a result of spatial redistribution of the charges and the local OAM densities without altering the respective global quantities. While the OAM density ($j_z$) and the total OAM per unit length ($J_z$) can vary along the beam’s axis when observed over a finite open aperture, it is worth noting that $j_z$ together with its associated flux will always satisfy the continuity equation when observed over a closed surface (enclosing a volume).

4.3.3 Discussion

Conservation of OAM and charge manifests in a manner that can be regarded as a practical realization of the Hilbert’s Hotel paradox [116, 117, 118]. The paradox, attributed to David Hilbert in 1924, postulates a hotel with infinitely countable number of occupied rooms with no vacancies. Nevertheless, such a hotel can still accommodate an infinite number of new guests at any given time by merely shifting each guest to its neighbor higher-numbered room, thus creating a seemingly infinite number of vacancies. Similarly, here, the accommodation of new positive charges (new guests) in the beam’s center is associated with the formation of negative charges (vacancies) in the outer rings of the beam (and vice versa). In this case, the outer rings act as a reservoir that dynamically compensates for the controlled imbalance in both the charges and OAM within the beam’s center. The underlying mechanism of charge conservation thus exhibits a fascinating analogy with the seemingly abstract concept of Hilbert’s hotel.

Thus far, we have demonstrated control over the topological charge and OAM along the propagation direction of longitudinally structured vortex modes. In the following, we extend our approach in order to control the sense of rotation of petal-like light structures carrying OAM. Such control over rotating petal-like structures is particularly useful in refractive index sensing, as will be discussed in Chapter 8.

4.4 Controlling the Topological Charge in Rotating Petal-Like Structures

4.4.1 Overview

When two or more waveforms $\Psi_{\ell}^1$ and $\Psi_{\ell}^2$, with opposite signs of $\ell$, are coherently superimposed, the helicity in the phase-front maps to an intensity modulation in the transverse beam profile as a result
of introducing singularities in the azimuthal direction — often creating a petal-like structure [128, 129, 130, 131, 132, 133]. Such superposition, in our case, can be expressed as

$$U(\rho, \phi, z, t) = \Psi_1^1 + \Psi_2^2 = \sum_{\ell=-\infty}^{\infty} \psi_1^\ell + \sum_{\ell=-\infty}^{\infty} \psi_2^\ell. \quad (4.10)$$

Further, when the waveforms, \(\Psi_1^1\) and \(\Psi_2^2\), have slightly shifted longitudinal wavenumbers the petal-like light structures starts to rotate along the beam’s axis of propagation. These rotating structures can carry non-zero OAM [134] or zero global OAM [122, 135, 136]; whereas in some cases the beam structures can exhibit accelerated rotation [137]. The transverse beam profile of the rotating light structure depends on the topological charge and the definition of the morphological function \(F^\ell_v(z)\) associated with the waveforms \(\Psi_1^1\) and \(\Psi_2^2\). By controlling these parameters, the transverse intensity of the beam can be made to evolve into various profiles along the beam’s propagation direction.

Several topologies of rotating beam structures are illustrated in Fig. 4.14 by tuning \(\ell^u\) and \(F^v_\ell(z)\); as indicated on the figure. In such a case, the angular orientation of the beam as it propagates along the \(z\)-direction is given by

$$\Phi_{\ell_1, \ell_2} \propto \frac{z(\Delta Q)}{|\ell_1| + |\ell_2|}, \quad (4.11)$$

where \(\Delta Q\) is the shift between the central longitudinal wavenumbers (\(Q^1\) and \(Q^2\)) associated with the OAM modes within \(\Psi_1^1\) and \(\Psi_2^2\), respectively.

In this section, we extend our longitudinal control over the OAM and charge to create rotating light structures that can change their sense of rotation and topological charge (number of petals) with
propagation, on-demand. Such level of control offers new degrees-of-freedom that can be exploited in sensing applications as will be discussed in details in Chapter 8. For each of the two generated patterns, a value of $0.9999958 \times k_0$ was assigned to $Q_1^1$ (for all the OAM states in $\Psi_1$). This yields maximum value of $N = 7$ in the superposition of Eq. (4.1). The parameter $Q_2^2, \ell_2$, associated with FW states of $\Psi_2$, was set to $0.999993 \times k$ resulting in $N = 13$. Each OAM state $\psi_v^{\ell}$ thus consists of $2N+1=15$ equal frequency Bessel beams of order $\ell_v$ with equally spaced longitudinal wavenumbers. The longitudinal wavenumbers in $\Psi_1$ and $\Psi_2$ are centered around slightly shifted values $Q_1^1$ and $Q_2^2$. For this particular experiment, we used a green laser source at $\lambda = 532$ nm, at which $\Delta k_{\ell,m}^z = \Delta Q = 33$ m$^{-1}$.

### 4.4.2 Pattern 1: Reversing the Sense of Rotation

In the first pattern, $\Psi^1$ and $\Psi^2$ consist of two OAM states; $\Psi^1 = \psi_{-2}^1 + \psi_{-2}^1$ and $\Psi^2 = \psi_2^2 + \psi_{-2}^2$. The morphological function is given by

$$
F_v^\ell(z) = \begin{cases} 
F_{1-2}^2 = F_{2-2}^2 = 1 & 5 \text{ cm} \leq z \leq 35 \text{ cm}, \\
F_2^2 = F_{-2}^2 = 1 & 35 \text{ cm} \leq z \leq 75 \text{ cm}, \\
F_{1-2}^2 = F_{2-2}^2 = 1 & 75 \text{ cm} \leq z \leq 90 \text{ cm}, \\
F_v^\ell = 0 & \forall \forall \ell \quad \text{elsewhere.}
\end{cases}
$$

(4.12)

According to $F_v^\ell(z)$, the generated beam should possess four revolving petals that invert their sense of rotation twice: at $z = 35$ cm and again at $z = 75$ cm. The evolution of the transverse profile of the rotating beam is recorded via CCD camera over a range of 1 m as shown in Fig. 4.15(a). The small arrows are superimposed on the recorded images to track the local sense of rotation of the beam. As suggested by Eq. (4.11), the petals exhibit counter clockwise rotation before inverting their sense of rotation as depicted in the middle row of Fig. 4.15(a). Afterwards, at $z = 75$ cm, the petals recover their initial rotation direction. The angular orientation of the rotating petals is plotted in Fig. 4.15(b). The curve inverts its slope twice over the range of $35 \text{ cm} \leq z \leq 75$ and $75 \text{ cm} \leq z \leq 90 \text{ cm}$, signifying a reverse in the sign of the topological charge. Moreover, the rotating petals exhibit an angular velocity $\frac{\partial \Phi}{\partial z} \sim 0.0825$ rad/cm, in agreement with Eq. (4.11). This angular velocity, depicted in Fig. 4.15(c), shows acceleration and deceleration associated with reversing rotation.

As demonstrated earlier in Sec. 4.3.1, the observed charge flipping is realized while conserving the total momentum of the field; when some of the OAM states are selected to switch-off — with the help of morphological function $F(z)$ — the energy of these states does not disappear but becomes diffused over larger space (outer rings of the beam). These outer rings act, not only as an energy reservoir, but also as an OAM reservoir and can be restored at further propagation distance to the central ring (depending on $F(z)$). For instance, when the petals are revolving in the CW direction, the outer rings carry OAM in the opposite (CCW) direction but distributed over a larger space. This momentum is then transferred
to the central ring when the scheme is reversed. Hence, the global OAM of the beam is preserved in full agreement with the discussion in Sec. 4.3.1.

Figure 4.15: Evolution of the rotating light structure as it reverses its sense of rotation. a) The four petals rotate in a CCW direction over the range of 5 cm ≤ z ≤ 35 cm. Inversion in the sense of rotation is observed in the middle row over the range of 35 cm ≤ z ≤ 75 cm before the beam center retains its initial sense of rotation within the range of 75 cm ≤ z ≤ 90 cm. b) Measured angular orientation of the beam (red circles) compared with theoretical prediction of Eq. (4.11) (black dashed lines). Clearly, the petals encounter linear rotation. The inversion of slope implies reversal of the sense of rotation. c) Angular velocity of the petals (blue circles) compared with theory (black dashed lines). Figure reproduced from Dorrah et al. [99] (© 2016 American Physical Society).

4.4.3 Pattern 2: Changing the Order of Phase-Twist

In the previous scenario, we have demonstrated the possibility of inverting the sign of the effective topological charge as the beam propagates. Here, we show the ability to control the sign and absolute value of the effective topological charge along the beam’s axis.

In this scenario, Ψ^1 and Ψ^2 both possess four OAM states such that Ψ^1 = ψ^1_{-1} + ψ^1_{-2} + ψ^1_{0} + ψ^1_{3} and
\[ \Psi^2 = \psi^2_2 + \psi^2_3 + \psi^2_0 + \psi^2_{-4} \]. The morphological function is chosen as

\[
F^\ell_z(z) = \begin{cases} 
F^1_1 = F^2_2 = 1 & 5 \text{ cm} \leq z \leq 35 \text{ cm}, \\
F^1_2 = F^2_3 = 1 & 35 \text{ cm} \leq z \leq 55 \text{ cm}, \\
F^1_0 = F^2_4 = 0 & 55 \text{ cm} \leq z \leq 70 \text{ cm}, \\
F^3_1 = F^4_2 = 1 & 70 \text{ cm} \leq z \leq 95 \text{ cm}, \\
F^\ell \equiv 0 & \text{ elsewhere.}
\end{cases}
\] (4.13)

Figure 4.16: Intensity pattern of the petal structure as it changes its phase twist. a) Three petals rotate with a CCW sense of rotation over the range of 5 cm \( \leq z \leq 35 \) cm before it evolves to five petals, indicating an increase in the local phase twist. The petals then carry zero intensity over the range of 55 cm \( \leq z \leq 70 \) cm before they evolve in the form of seven petals rotating in the CW direction. b) Measured angular orientation of the rotating beam (red circles) in comparison with theory (black dashed line) calculated from Eq. (4.11). The slope signifies the rate of rotation (angular velocity) and its sign indicates the sense of rotation. c) Measured (blue circles) and simulated (black dashed line) angular velocity of the petals, also calculated from Eq. (4.11), with propagation distance. Figure reproduced from Dorrah et al. [99] (© 2016 American Physical Society).

From Eq. (4.13), the beam carries three petals rotating with linear speed in the counterclockwise direction over the range of 5 cm \( \leq z \leq 35 \) cm. The beam then evolves into five petals rotating in the
same direction for $35 \, \text{cm} \leq z \leq 55$ before petals carry zero intensity in the range of $55 \, \text{cm} \leq z \leq 70$ cm. The beam then evolves into a rotating structure comprised of seven petals with an opposite sense of rotation (clockwise direction), signifying both an increase and inversion in its azimuthal phase twist. As discussed in Sec. 4.3.2, increasing the topological charge here is associated with introducing new singularities in the azimuthal phase which translate to a discontinuity in the intensity profile.

The evolution of intensity distribution associated with $U = \Psi_1 + \Psi_2$ is depicted in Fig. 4.16(a). The angular orientation and velocity of the rotating petals are plotted in Figs. 4.16 (b) and (c), respectively; showing good agreement with theoretical predictions. The rotating beam structure initially exhibits larger angular velocity (when it carries three petals) as compared to the case when it evolves into five and seven petals, in agreement with Eq. (4.11). The radial extent of the beam increases when its effective topological charge increases, as dictated by Helmholtz wave equation.

4.5 Summary

In summary, we presented a systematic approach — expressed in closed-form analytic expressions — to design, create, and characterize the longitudinally structured vortex beams using a simple all-optical holographic setup. This significantly expands some earlier efforts that relied on complicated setups incorporating non-linear media to achieve non-trivial topological inversion. Furthermore, we have employed modal decomposition in pseudo-orthonormal function space to realize a holistic quantitative diagnostic tool for our Structured Light, determining the intensity, phase, wavefront, Poynting vector and OAM density. In this regard, we were able to monitor the movement, creation, and annihilation of optical vortices, which accounts for the local variation of the OAM density within regions of our quasi 3D field. As such we are able to offer a complete interpretation of the OAM dynamical evolution; illustrating how the local OAM can vary along the beam’s axis without violating the global OAM conservation. In addition, we established that at any structured transition from $\ell_1$ to $\ell_2$, the beam undergoes deliberate shape deformation to create exactly $|\ell_2 - \ell_1|$ channels of zero intensity allowing to judiciously transport charges to (from) the beam’s center. This is a generic signature that we emphasize here for the first time and is a step towards establishing other governing laws in advanced singular optics.

Further, we extended our approach to include rotating OAM modes that can change the sense of their rotation and order of phase-twist with propagation. These rotating modes will become particularly useful for remote sensing as we discuss in details in Chapter 8. Interestingly, our approach to control the OAM and charge along the beam’s axis, presented here, can also be combined with the formulation presented in Chapter 3 to create attenuation-resistant vortex beams that may find many applications in absorbing media encountered in micro-manipulation, imaging, remote sensing...etc. This could be achieved by engineering the function $F(z)$ such that it carries the inverse of the medium loss profile [92]. Finally, although here we focused on the orbital component of the angular momentum by considering
linearly polarized light, the photon’s spin, associated with circular polarization, also contributes to the total angular momentum [104]. Longitudinal control over the polarization degree-of-freedom of light will be the subject of Chapter 5.
Chapter 5

Controlling the Polarization State and Intensity of Diffraction-Attenuation-Resistant Beams along their Propagation Direction

This chapter demonstrates a class of non-diffracting beams in which the state of polarization (SoP) can be controlled along the propagation direction of the beam. The beams are composed of a superposition of equal frequency co-propagating Bessel beams with different transverse and longitudinal wavenumbers, and weighted by suitable complex coefficients derived from closed-form analytic expressions. The desired polarization states (i.e., linear, radial, azimuthal and so on) are decomposed into their orthogonal components which are then independently encoded onto different sets of Bessel beams. Through constructive (and destructive) interference, specific SoPs can be designed to switch on (and off) along the propagation direction; effectively altering the longitudinal SoP and intensity of the beam. Five different case studies are presented showing the ability of the structured waveform to evolve from one polarization state to another as it propagates. It will also be shown how this level of control over polarization can be applied to independently control other degrees of freedom of light, namely its intensity and topological charge, even in the presence of propagation losses. This development can be of great interest to optical tweezers, atom guiding, material processing, microscopy, and optical communications.\(^1\)

\(^1\)The results presented here are among the main contributions of this thesis and were first published in Refs. [138, 139, 140] and adapted in this chapter.
5.1 Overview

Polarization is an important degree-of-freedom of light: it describes the geometrical orientation of the field oscillations and is also associated with the photon’s spin [141]. The state of polarization (SoP) of optical beams has been exploited in many areas, such as material processing [63], polarimetry [64], microscopy [65, 66] and optical communications [67, 68]. Moreover, the ability to control the SoP of optical beams along the propagation direction is very intriguing and can offer new degrees-of-freedom in our existing light manipulation tool set. In material processing, for instance, this can be utilized to control the shape and size of laser-machined structures by inducing a polarization-dependent ablation effect along the structure [78], which can be deployed in waveguide writing or to create periodic structures. In addition, spatially varying SoP can be used to modulate the absorption profile of polarization-dependent optically pumped medium [64], or to tailor the spectrum profile of quantum emitters over a given volume [77].

Few studies have demonstrated non-diffracting optical beams in which the SoP can vary continuously along the propagation direction [111, 142, 143, 144, 145]. Despite showing interesting beam patterns with effective control over the longitudinal SoP profile, these earlier studies either did not provide a systematic recipe to design the desired beams [111, 142, 143, 144, 145] or did not allow special polarization states — such as azimuthal and radial — to be generated [144]. In this chapter, the Frozen Wave (FW) method will be utilized to experimentally show several interesting beam patterns where the SoP and intensity can both be controlled along the propagation direction of the beam. We start by generating linearly polarized beams in which the SoP is altered from vertical to horizontal and back to vertical, while maintaining the same intensity level. Then we present scenarios in which the intensity level can be shaped while the SoP is simultaneously altered from vertical to horizontal polarization. Afterwards, more complicated polarization states will be generated; for instance, a case in which a linearly polarized beam in the horizontal direction becomes radially polarized and eventually evolves to linear polarization in the vertical direction. Finally, we demonstrate complex beam structures in which multiple degrees of freedom can be controlled simultaneously along the beam’s axis; in essence, creating an attenuation-resistant and non-diffracting beam that can change both its polarization state and topological charge with propagation, independently and on-demand.

5.2 Concept

Longitudinal control of the SoP and intensity is achieved via a judicious superposition of co-propagating monochromatic non-diffracting Bessel beams with different transverse and longitudinal wavenumbers, following the formulation given by the Frozen Waves (FWs) method, discussed in Chapter 3 and
A scalar FW is expressed as

$$\psi_\ell (\rho, \phi, z, t) = e^{-i\omega t} \sum_{m=-N}^{N} A_{\ell,m} J_\ell (k_\rho^m \rho) e^{i k_\rho^m \rho} e^{i k_z^m z}. \quad (5.1)$$

Here, $\ell$ is the order of the Bessel beam in the superposition, and $k_\rho$ and $k_z$ denote the transverse and longitudinal wavenumbers, respectively. The longitudinal wavenumbers, $k_z^m$, are equally spaced around a constant parameter $Q$ such that: $k_z^m = Q + \frac{2\pi m}{L}$, where $L$ is the desired longitudinal extent of the beam. The transverse wavenumbers are given by the constraint relation $k_\rho^m = \sqrt{k_0^2 - (k_z^m)^2}$, where $k_0 = \omega n/c$. The interference among the Bessel Beams with different wavenumbers leads to a beating in the envelope of the resulting waveform along the $z$-direction. By carefully choosing the parameters of each Bessel Beam, it is possible to tailor this interference so that the intensity of the envelope follows a predefined profile along the beam’s axis, denoted as $|F_\ell (z)|^2$. The coefficients $A_{\ell,m}$ are calculated via $A_{\ell,m} = \frac{1}{L} \int_{-L}^{L} F_\ell (z) e^{i m z} e^{-i k_\rho^m \rho} dz$, where the term $e^{i m z}$ compensates for the propagation losses if the beam is designed to propagate in an absorbing medium as discussed in Chapter 3 and Refs. [90, 92].

By superposing multiple FWs with different SoPs, it is possible to control the effective SoP of the resulting waveform, $\vec{E}(\rho, \phi, z, t)$, along its axis. For example, consider a scenario in which $\vec{E}(\rho, \phi, z, t) = \hat{x} \psi_1^1 + \hat{y} \psi_2^1$; where, just for simplicity, the beam is considered in the paraxial regime ($k_z \gg k_\rho$), so that the $E_z$ component can be neglected. If we define $F_\ell^1 (z) = 1$ for $z_1 \leq z < z_2$ (and zero otherwise) for the FW $\psi_1^1$, and $F_\ell^2 (z) = 1$ for $z_2 \leq z < z_3$ (and zero otherwise) for the FW $\psi_2^1$, it follows that the center of the resulting waveform $\vec{E}(\rho, \phi, z, t)$ will be predominantly composed of $\hat{x} \psi_1^1$ over the propagation range from $z_1$ to $z_2$. Then, over the range $z_2 \leq z < z_3$, $\vec{E}(\rho, \phi, z, t)$ will be dominated by the contributions of $\hat{y} \psi_2^1$. Accordingly, via deliberate interference, the effective SoP of the beam center will vary along the beam’s axis, evolving from $x$-polarized to $y$-polarized in this case. Further, if linearly polarized FWs with orthogonal SoPs are combined within the same space regions, circular and elliptical polarizations can be obtained. Also, special polarizations, such as azimuthal and radial, can be realized.

In the electromagnetic case, a scalar FW can be assigned to the desired transverse electric field component, with the minor constraint that for azimuthal and radial polarizations the FW has azimuthal symmetry and, as a consequence, can only be of order $\ell = 1$ [146]. The longitudinal electric field component is calculated using Gauss’s law (for source-free homogeneous media, it is $\nabla \cdot \vec{E} = 0$) and the magnetic field is then obtained from Faraday’s law ($\vec{B} = -\frac{1}{c} \nabla \times \vec{E}$). An azimuthally polarized FW, for example, has the form $\vec{E} = E_\phi \hat{\phi}$ with

$$E_\phi = e^{-i\omega t} \sum_{m=-N}^{N} A_m J_1 (k_\rho^m \rho) e^{i k_z^m z}, \quad (5.2)$$

A full vectorial treatment of FWs is provided in more details in Ref. [146].

In the case of radial and azimuthal polarizations, the FWs are of order $\ell = 1$ and do not have the $e^{i\ell \phi}$ term [146].
and has no other electric field components [146]. A radially polarized FW, on the other hand, is written as \( \vec{E} = E_\rho \hat{\rho} + E_z \hat{z} \) with [146]

\[
E_\rho = e^{-i\omega t} \sum_{m=-N}^{N} A_m J_1(k_m \rho) e^{ik_m \rho},
\]

\[
E_z = e^{-i\omega t} \sum_{m=-M}^{M} k_m^2 A_m J_0(k_m \rho) e^{ik_m \rho}.
\]

In addition, a linearly polarized FW of order \( \ell \) in the \( \hat{x} \) or \( \hat{y} \) direction has the form [146]

\[
\vec{E} = E_\perp (\hat{x} \hat{y}) + E_z \hat{z},
\]

\[
E_\perp = e^{-i\omega t} \sum_{m=-N}^{N} A_m J_\ell(k_m \rho) e^{i\ell \phi} e^{ik_m \rho},
\]

\[
E_z = e^{-i\omega t} \sum_{m=-M}^{M} A_m e^{ik_m \rho} e^{i\ell \phi} \left[ \left( \frac{\sin \phi}{\rho k_m^2} \frac{\ell}{2} \right) J_\ell + \left( \frac{\cos \phi}{\rho k_m^2} \frac{\ell+1}{2} \right) J_{\ell+1} \right],
\]

where the brackets notation is used to distinguish the terms corresponding to each direction and \( J_\ell \equiv J_\ell(k_m \rho) \). Any polarization state in the Poincaré sphere can be obtained by combining orthogonal linearly polarized FWs with suitable complex amplitudes \( a_x \) and \( a_y \), that is, \( \vec{E} = (a_x \hat{x} + a_y \hat{y}) E_\perp + E_z \hat{z} \) where \( E_z \) is a combination of results of the type (5.7). Hence, in the general case, it is possible to write the expression for the \( x \) and \( y \) components of the resulting waveform in the following compact form

\[
E_u(\rho, \phi, z, t)|_{u=x,y} = e^{-i\omega t} \sum_{v=1}^{M} G_v^u(\phi) \sum_{m=-N}^{N} A_{v,m} \ell J_\ell(k_m \rho) e^{ik_m \rho},
\]

where \( M \) is the number of superposed FWs and \( G_v^u(\phi)|_{u=x,y} \) is a morphological function that defines the azimuthal dependence for the FW over each subregion \( v \) and \( u \) denotes the \( x \) and \( y \) components of the FW. As such, \( G_v^u(\phi) \) is \( e^{i\ell \phi} \) for linear polarization (due to the structure of linearly-polarized FWs) and \( \cos(\phi) \) or \( \sin(\phi) \) when decomposing radial and azimuthal polarizations into their \( x \)- and \( y \)-polarized components (also, for these polarizations, \( \ell \) has to be equal to 1 [146]), as stated above. Our proposed method thus relies on combining horizontally and vertically polarized Bessel modes, then modulating their longitudinal intensity, to control the effective SoP of the resultant beam throughout propagation, as demonstrated next.

### 5.3 Experimental Procedure

Several non-diffracting beam patterns were experimentally generated showing control over SoP along the propagation direction both in air and inside a lossy fluid. The experimental setup is depicted in Fig. 5.1. The desired transverse electric field at the initial plane \( \vec{E}_\perp(\rho, \phi, z = 0, t) \) is decomposed into two
Figure 5.1: Experimental setup used to demonstrate longitudinal control of the polarization state and intensity. An amplitude-only transmissive spatial light modulator is addressed by a computer generated hologram that is designed at 532 nm wavelength. Two collimated beams with linear and orthogonal polarization states are incident on the SLM screen, where each beam is encoded on one half of it. The output patterns are combined and then filtered using a 4-f system. The combined beam is then detected by a CCD camera along the beam axis. For the case of beam propagation inside a lossy fluid, the CCD camera is immersed inside the container while covered in a waterproof case. A polarizer is used to analyze the polarization state at the plane of the CCD camera. Here, BE is the beam expander, BS refers to the beam splitter, BC is the beam combiner, M refers to the mirrors, HWP is the half wave plate, Pol refers to the polarizers, and SLM refers to the spatial light modulator. Figure reproduced from Dorrah et al. [139] (© 2018 SPIE).

components with linear and orthogonal SoP along $x$ and $y$ directions. Each component, $E_x(\rho, \phi, z = 0, t)$ and $E_y(\rho, \phi, z = 0, t)$, is calculated and transformed into a 2D computer generated hologram (CGH) that is mapped onto one part of the SLM (HOLOEYE LC2012 Amplitude SLM). This SLM consists of a twisted nematic liquid crystal transmissive screen that operates in the amplitude-only mode. The SLM screen is divided into two adjacent sections, each being addressed independently by $E_x$ and $E_y$. A 532 nm collimated laser beam with linear polarization along the vertical axis $y$ is split into two beams via the beam splitter “BS1”. The first beam, which is now $y$-polarized, passes through the right side of the SLM. The second beam passes through a half wave plate (HWP) that is rotated at an angle of 45° with respect to the vertical axis. This rotates the polarization state of the beam by an angle of 90° (i.e. rotating it to the orthogonal direction $x$). Such beam, now $x$-polarized, then passes through the left side of the SLM. Due to the twisted nematic nature of the SLM, the output patterns encoded from the CGHs on the right and left sides of the SLM then pass through polarizers oriented at 90° and 0° with respect to the vertical axis, respectively. This helps to clean the encoded patterns from unwanted polarizations and generate high contrast images.

The patterns $E_u(\rho, \phi, z = 0, t)|_{u=x,y}$ are encoded using an amplitude mask. The hologram equation is given by

$$H(x, y) = \frac{1}{2} \{ \beta(x, y) + \alpha(x, y) \cos[\Theta(x, y) - 2\pi(u_0x + v_0y)] \},$$

(5.9)
where \( \alpha(x, y) \) and \( \Theta(x, y) \) are the amplitude and phase of \( E_u(\rho, \phi, z = 0, t) \), respectively. \( \beta(x, y) \) is a bias function chosen as a soft envelope for the amplitude \( \alpha(x, y) \) and is given by \( \beta(x, y) = \frac{1 + \alpha(x, y)^2}{2} \) [98]. The two output beams, with orthogonal polarization states, are then combined via the beam splitter “BS2”, resulting in the vector waveform \( \vec{E}(\rho, \phi, z, t) \). The resulting beam is then imaged and filtered using a 4-\( f \) system (with \( f = 20 \text{ cm} \)) and an iris. For efficient filtering, a plane wave exp\( [2\pi i(u_0 x + v_0 y)] \) was superposed to the computer generated hologram. This shifts the encoded pattern off-axis to the spatial frequencies \( (u_0, v_0) \) in the Fourier plane; thus making it easier to filter out the shifted pattern from the undesired on-axis noise by using an iris. The parameters \( u_0 \) and \( v_0 \) were set to \( 3/(16\Delta x) \), where \( \Delta x \) is the SLM pixel pitch (36 \( \mu \)m). The filtered pattern is then imaged back at the focus of “Lens2”, referred to as the \( z' = 0 \) plane. This plane maps to an actual propagation distance of 20 cm along the beam axis, as this is the approximate distance it propagates after the SLM and before it is imaged by the 4-\( f \) system. The evolution of the generated waveform is then recorded using a CCD camera along the beam axis either in air or inside a lossy fluid. At each plane of detection, the polarization state of the measured beam is analyzed by a polarizer oriented at 0°, 90°, 45°, and 135°, with respect to the vertical axis \( y \), to verify the change in the SoP with propagation.

### 5.4 Experimental Results

Several beam patterns have been experimentally generated with \( N = 5 \). Accordingly, each of the \( x \)- and \( y \)-polarized components \( E_x \) and \( E_y \) of the resulting waveform consists of \( 2N + 1 = 11 \) equal-frequency Bessel beams of order \( \ell = 1 \) with equally spaced longitudinal wavenumbers. The parameter \( Q \) was set to values of 0.9999995 \( \times k_0 \) and 0.999997 \( \times k_0 \) for the FWs generated in air and the lossy fluid, respectively. This choice of \( Q \) ensures highly paraxial beams with small values of \( k_{\rho, m} \). As such, the generated Bessel beams possess transverse wavenumbers that are compatible with the spatial bandwidth of the available SLM. In all cases, the longitudinal extent of the beam was set as \( L = 1 \text{ m} \).

#### 5.4.1 Pattern 1: Longitudinal Control Over the Linear Polarization State

In the first pattern, the SoP will be controlled while keeping the intensity at a constant level along the beam’s axis. In this case, zero order Bessel Beams \( (\ell = 0) \) are considered in the superposition of Eq. (5.8), which implies no azimuthal \( (\phi) \) dependence. As an example, the beam was designed such that it is \( x \)-polarized initially and then becomes \( y \)-polarized before it retains its initial \( x \)-polarization. As such, the beam is constructed by superposing two FWs with orthogonal polarizations, the morphological functions are defined as \( G_1^x(\phi) = 0 \), \( G_1^y(\phi) = 1 \), \( G_2^x(\phi) = 1 \), and \( G_2^y(\phi) = 0 \), and the respective functions
$F_{\ell=0,v}(z)$ are set as

$$
F_{\ell=0,v}(z) =
\begin{cases}
F_{\ell,1}(z) = 1, & F_{\ell,2}(z) = 0, \quad \text{for } 0 \text{ cm} \leq z' < 10 \text{ cm}, \\
F_{\ell,1}(z) = 0, & F_{\ell,2}(z) = 1, \quad \text{for } 10 \text{ cm} \leq z' < 30 \text{ cm}, \\
F_{\ell,1}(z) = 1, & F_{\ell,2}(z) = 0, \quad \text{for } 30 \text{ cm} \leq z' < 40 \text{ cm}, \\
F_{\ell,1}(z) = 0, & F_{\ell,2}(z) = 0, \quad \text{elsewhere}
\end{cases}
$$

(5.10)

where the resulting polarization is indicated under each interval.

Figure 5.2: Normalized intensity profiles of the measured beam for the first example. The top row indicates the analyzer angle used before the CCD camera to verify the change in SoP with propagation and the leftmost column indicates the desired SoP. The angles are measured with respect to the vertical axis $y$. The beam is linearly polarized in the $y$-direction over the range of $0 \text{ cm} \leq z' < 10 \text{ cm}$, before the SoP is rotated to become $x$-polarized, as observed at $z' = 20 \text{ cm}$. Then, the SoP goes back to its initial $y$-polarization, as seen at $z' = 35 \text{ cm}$. Finally, the beam center is switched-off, as shown at $z' = 43 \text{ cm}$. In all the SoPs, the longitudinal intensity profile is kept constant along the beam axis, as seen in the first column (“No Analyzer”). Figure reproduced from Dorrah et al. [139] (© 2018 SPIE).

The evolution of the transverse profile of the resulting beam, recorded with the CCD camera at different propagation distances, is shown in Fig. 5.2. The arrows in the left column indicate the predefined SoP of the beam, whereas the arrows in the top row depict the analyzer angle. The beam exhibits linear SoP in the $y$-direction at $z' = 5 \text{ cm}$. This is evident by looking at the transverse intensity profile, which
is at maximum level when the analyzer is oriented along the \( y \)-direction (0° with respect to the vertical axis \( y \)) and becomes negligible when the analyzer is in the orthogonal \( x \)-direction (90° with respect to the vertical axis \( y \)). Also, the intensity reaches approximately half the maximum level in between. The \( y \)-polarization is dominant because only the \( E_y \) component is significant near the beam center by the virtue of the choices for \( F_{\ell,v}(z) \). Afterwards, at the distance of 20 cm, the SoP is changed to \( x \)-polarized. Then, at \( z' = 35 \) cm, the beam recovers its initial SoP state and becomes \( y \)-polarized again. Finally, the beam center is switched-off, as observed at \( z' = 43 \) cm. This is done by assigning \( F_{\ell,v}(z) = 0 \) for all FWs after the previous interval. Notice that the peak intensity is kept constant during propagation, as seen in the “No Analyzer” column.

In this example, the SoP of the beam was rotated while fixing the intensity level along the beam’s axis. In the next example, another scenario will be presented wherein both the SoP and the intensity level are modified with propagation.

### 5.4.2 Pattern 2: Simultaneous Control Over Polarization and Intensity Level

In the second example, we show the possibility of controlling both the SoP and the intensity along the beam axis. In this scenario, \( E_x \) and \( E_y \) are designed to exhibit different intensity levels by using two linearly-polarized FWs with orthogonal polarizations. We still consider zero order Bessel Beams and we set \( N = 5 \). The morphological functions are \( G_x = 0, G_y^1 = 1, G_x^2 = 1, G_y^2 = 0 \), and the corresponding functions \( F_{\ell=0,v}(z) \) are chosen as

\[
F_{\ell=0,v}(z) = \begin{cases} 
F_{\ell,1}(z) = 1, & \text{for } 0 \text{ cm} \leq z' < 10 \text{ cm}, \\
F_{\ell,1}(z) = 0, & \text{for } 10 \text{ cm} \leq z' < 25 \text{ cm}, \\
F_{\ell,1}(z) = 0, & \text{for } 25 \text{ cm} \leq z' < 35 \text{ cm}, \\
F_{\ell,1}(z) = 0, & \text{elsewhere}.
\end{cases}
\] (5.11)

where the resulting polarization is again indicated below each interval.

Fig. 5.3 depicts the evolution of the transverse profile of the resulting beam. It exhibits linear SoP in the \( y \)-direction at \( z' = 5 \) cm, which is evident from the transverse intensity profile that records maximum level when the analyzer is oriented along the \( y \)-direction and negligible intensity along the orthogonal direction. The beam center is then switched-off by assigning \( F_{\ell,v}(z) = 0 \) to both \( E_x \) and \( E_y \), as shown at \( z' = 17 \) cm. As such, the central spot vanishes because the energy is stored away from the center of the beam into the outer rings via an inverse self-healing process. Afterwards, the beam center is switched-on again at double the intensity level with an SoP in the \( x \)-direction, as shown at \( z' = 30 \) cm. Finally, the beam’s center is switched-off, as observed at \( z' = 42 \) cm. Intensity modulation of the envelope along
the $z$-direction is realized as a result of the intensity beating (interference) among the Bessel modes with different spatial frequencies and complex weights. Note that this approach is versatile and can be applied to generate beams with more complicated polarization states, as shown next.

![Figure 5.3: Normalized intensity profiles of the measured beam for the second example. The top row indicates the analyzer angle used before the CCD camera to verify the change in SoP with propagation and the leftmost column indicates the expected SoP. The angles are measured with respect to the vertical axis $y$. The beam is linearly polarized in the $y$-direction over the range $0 \text{ cm} \leq z' < 10 \text{ cm}$ before it is switched-off for the interval $10 \text{ cm} \leq z' < 25 \text{ cm}$. Then, it is made to possess a linear polarization in the $x$-direction with double the original intensity level over the range $25 \text{ cm} \leq z' < 35 \text{ cm}$, after which the beam is switched-off again. Figure reproduced from Dorrah et al. [139] (© 2018 SPIE).](image)

### 5.4.3 Pattern 3: Evolving from Linear to Radial Polarization

For this pattern, the beam is designed to change its SoP from linear to radial and then back to linear, while maintaining the same intensity level. The morphological functions are set as $G_x^1(\phi) = e^{i\phi}$, $G_y^1(\phi) = 0$, $G_x^2(\phi) = \cos(\phi)$, $G_y^2(\phi) = \sin(\phi)$, $G_x^3(\phi) = 0$, $G_y^3(\phi) = e^{i\phi}$ and the associated functions $F_{\ell=1,v}(z)$ are
given by

\[
F_{\ell,v}(z) = \begin{cases} 
F_{\ell,1}(z) = 1, & F_{\ell,2}(z) = 0, & F_{\ell,3}(z) = 0 \\
for \ 0 \text{ cm} \leq z' < 10 \text{ cm}, & (x\text{-polarization}) \\
F_{\ell,1}(z) = 0, & F_{\ell,2}(z) = 1, & F_{\ell,3}(z) = 0 \\
for \ 10 \text{ cm} \leq z' < 25 \text{ cm}, & (radial \text{ polarization}) \\
F_{\ell,1}(z) = 0, & F_{\ell,2}(z) = 0, & F_{\ell,3}(z) = 1 \\
for \ 25 \text{ cm} \leq z' < 35 \text{ cm}, & (y\text{-polarization}) \\
F_{\ell,1}(z) = 0, & F_{\ell,2}(z) = 0, & F_{\ell,3}(z) = 0 \\
elsewhere.
\end{cases}
\] (5.12)

where the resulting polarization is indicated under each interval.

![Normalized intensity profiles of the measured beam for the third example.](image)

**Figure 5.4:** Normalized intensity profiles of the measured beam for the third example. The top row indicates the analyzer angle used before the CCD camera to verify the change in SoP with propagation and the leftmost column indicates the desired SoP. The angles are measured with respect to the vertical axis y. The beam is linearly polarized in the x-direction over the range of 0 cm ≤ z' < 10 cm, before the SoP evolves to radial polarization, which is observed at z' = 17 cm. Then, the SoP is changed to linear polarization in the y-direction, as seen at z' = 28 cm. Finally, the beam center is switched-off, as shown at z' = 38 cm. In all the SoPs, the longitudinal intensity profile is kept constant along the beam axis, as seen in the first column (“No Analyzer”). Figure reproduced from Dorrah et al. [139] (© 2018 SPIE).

The evolution of the transverse profile of the resulting beam is shown in Fig. 5.4. The beam initially exhibits linear SoP in the x-direction at z' = 5 cm. This is evident by looking at the transverse intensity profile, which is at maximum level when the analyzer is oriented along the x-direction and is negligible when the analyzer is in the orthogonal direction (along the y-direction). Also, it reaches approximately half the intensity level in between. Afterwards, at the distance of 17 cm, the SoP is altered and the
beam becomes radially polarized. This is achieved by designing the waveform $\vec{E}$ to have equal in-phase contributions from $E_x$ and $E_y$. We note that a phase bias (retardation) has been added to the path of $E_y$ to compensate for any phase difference with respect to $E_x$, thus ensuring radial polarization. The radial polarization is verified by looking at the intensity profile, which now has the form of two petals recording maximum intensity level along the analyzer axis and zero intensity in the orthogonal direction.

Then, at $z' = 28 \text{ cm}$, the SoP is $y$-polarized, as shown by the results in the third row, which represent the complementary scenario of the first row ($z' = 5 \text{ cm}$). Finally, the beam center is switched-off, as observed at $z' = 38 \text{ cm}$. Notice that the peak intensity is kept constant during propagation, as seen in the “No Analyzer” column.

It can also be observed from Fig. 5.4 that the intensity of the outer rings changes with propagation. This is a typical behavior that results from interfering multiple Bessel beams with different (transverse) wavenumbers [92]. The outer rings observed at $z' = 5$ and $17 \text{ cm}$ get focused throughout propagation and are responsible for constructing the $y$-polarized inner ring at $z' = 28 \text{ cm}$. When the inner ring is designed to switch off, for example at the plane $z' = 38 \text{ cm}$, the energy in the inner ring is redistributed to the outer rings again. At this same position (and at the plane $z' = 28 \text{ cm}$ as well), the outer rings that were responsible for the $x$-polarized part of the beam cannot be observed, as they already interfered previously and are spread outside of the observation window.

Thus far, all the presented results were for SoP control of non-diffracting beams propagating in air (lossless medium). In order to demonstrate the attenuation-resistance property of the proposed beams, next, we present a scenario in which the beam changes its SoP from x-polarized to y-polarized changes with propagation inside a lossy fluid, thus overcoming attenuation.

### 5.4.4 Pattern 4: Attenuation-Resistant Beam with Longitudinally Varying SoP

In this example, a beam varying its SoP along its axis is transmitted inside a lossy fluid while maintaining its intensity level — thus counteracting the exponential decay associated with propagation losses. The fluid was prepared by adding few drops of propylene glycol, citric acid and sodium benzoate to 4L of water. The resulting fluid has a complex index of refraction $n = 1.4 + i1.2 \times 10^{-6}$ at 532 nm. Here, $E_x$ and $E_y$ are engineered to maintain the same intensity level while overcoming propagation losses. In this case, attenuation-resistance is achieved by designing the longitudinal intensity profile of the beam, $F(z)$, such that it carries the inverse loss profile of the medium, as demonstrated in Chapter 3. For this case, higher order Bessel beams were adopted in the superposition of Eq. (5.8) — with $\ell = 1$. The morphological functions are give by $G_1^x = e^{i\phi}$, $G_1^y = 0$, $G_2^x = 0$, $G_2^y = e^{i\phi}$ and $F_{\ell=1,v}(z)$ is defined as
According to these parameters, the generated beam is $x$-polarized over the range $0 \text{ cm} \leq z' < 15 \text{ cm}$, since it mainly has the contribution of $E_x$ in the beam center. Then, in the interval $15 \text{ cm} \leq z' < 30 \text{ cm}$, the polarization is rotated such that the beam becomes $y$-polarized. This is done while maintaining the same intensity level with propagation inside the lossy fluid. The evolution of the transverse profile of the resulting beam recorded with the CCD camera at different propagation distances is shown in Fig. 5.5.

\[
F_{\ell,1}(z) = 1, \quad F_{\ell,2}(z) = 0 \quad \text{for } 0 \text{ cm} \leq z' < 15 \text{ cm}, \quad \text{(x-polarization)}
\]

\[
F_{\ell,1}(z) = 0, \quad F_{\ell,2}(z) = 1 \quad \text{for } 15 \text{ cm} \leq z' < 30 \text{ cm}, \quad \text{(y-polarization)}
\]

\[
F_{\ell,1}(z) = 0, \quad F_{\ell,2}(z) = 0 \quad \text{elsewhere.}
\]  

(5.13)

The generated beam denoted as attenuation-resistant FW is depicted in the second column of Fig. 5.5 with no analyzer. The beam is also compared with the ordinary FW (not compensated for losses), as shown in the first column. It is observed that the intensity of the ordinary FW suffers from exponential decay with propagation. This is clearly mitigated in the second column. Such attenuation-resistant property is a result of intensified self-healing process in which the outer rings of the beam act as an energy reservoir that intensifies the central ring continuously with propagation. For example, in Fig. 5.5,
the outer rings of the beam at \( z' \) for the attenuation-resistant beam is clearly pronounced (compared to the case of ordinary FW). These rings are \( y \)-polarized as observed from third column under the “0° Analyzer” recordings. The energy in these rings focuses to intensify the central part of the beam with propagation.

Figure 5.6: Normalized intensity profiles of the measured beam for the fourth example (generated in a lossy fluid) taken at three consecutive planes at the SoP transition: \( z' = 14 \), 15, and 16 cm, with different analyzer angles. The figure depicts the transition of the beam from \( x \)-polarization to \( y \)-polarization. Figure reproduced from Ref. [138] (© 2018 American Physical Society).

In order to investigate the transition of the beam from \( x \)-polarization to \( y \)-polarized, the beam evolution has been recorded around the \( z' = 15 \) cm plane. The results are depicted in Fig. 5.6. It is observed that the \( x \)-polarized component of the beam washes away and diffuses to the outer rings of the beam with propagation, as observed from column labeled “90° Analyzer”. Meanwhile, the \( y \)-polarized component gets focused and more significant with propagation, as observed from the column no labeled “0° Analyzer”.

A powerful advantage of our technique is that it can be combined with the approach presented in Chapter 4 to control the topological charge of the beam in addition to its SoP. This becomes particularly useful in applications that require full manipulation over the angular momentum of light in terms of its spin and orbital momenta. An attenuation-resistant and non-diffracting beam that can vary its SoP and topological charge is shown next.

### 5.4.5 Pattern 5: Attenuation-Resistant Beam with Longitudinally Varying SoP and Charge

In this final example, an attenuation-resistant beam that can vary both its SoP and topological charge with with propagation is transmitted into a lossy fluid with complex refractive index \( n = 1.4 + i \times 4 \times 10^{-6} \). In this case, the Bessel beams associated with \( x \)-polarization are of order \( \ell = 1 \) whereas those with \( y \)-
polarization are of order $\ell = 2$. The morphological functions set as $G_1^x = e^{i\phi}$, $G_1^y = 0$, $G_2^x = 0$, $G_2^y = e^{i2\phi}$ and $F_{\ell,v}(z)$ is given by

$$F_{\ell,v}(z) = \begin{cases} 
F_{\ell=1,v=1}(z) = 1, & F_{\ell=2,v=2}(z) = 0 \quad \text{for } 0 \text{ cm} \leq z' < 15 \text{ cm,} \\
F_{\ell=1,v=1}(z) = 0, & F_{\ell=2,v=2}(z) = 1 \quad \text{for } 15 \text{ cm} \leq z' < 30 \text{ cm,} \\
F_{\ell,v=1}(z) = 0, & F_{\ell,v=2} = 0 \quad \text{elsewhere.} 
\end{cases}$$ (5.14)

Figure 5.7: Normalized intensity profiles of the measured beam. The top row indicates the analyzer angle used before the CCD camera to verify the change in SoP with propagation and the leftmost column indicates the expected SoP. The angles are measured with respect to the vertical axis $y$. The beam is linearly polarized in the $x$-direction over the range $0 \text{ cm} \leq z' < 15 \text{ cm}$. Then, it is made to possess a linear polarization in the $y$-direction at the same intensity level over the range $15 \text{ cm} \leq z' < 30 \text{ cm}$, thus overcoming propagation losses. Figure reproduced from Dorrah et al. [140] (© 2018 The Optical Society).

Figure 5.7(b) shows the measured transverse intensity profiles of the beam inside the fluid. The top row indicates the analyzer’s angle before the camera (with respect to the $y$-axis). The desired SoP is depicted in the leftmost column. The Attenuation-Resistant FW is shown in the third to seventh columns (marked by the green frame) and is compared with the “Ordinary FW” (second column). Here, “Ordinary FW” refers to a waveform composed of the same superposition of Bessel Beams, but not compensated for losses. Note that the Attenuation-Resistant FW initially carries more energy in its outer rings than the “Ordinary FW”. This results in intensified self-healing that compensates for the attenuation encountered in the lossy fluid. Over the range $0 \text{ cm} \leq z < 15 \text{ cm}$, the beam is linearly polarized in the $x$-direction and carries a charge $\ell = 1$. However, for $15 \text{ cm} \leq z < 30 \text{ cm}$, the beam
becomes \( y \)-polarized, while the topological charge is simultaneously increased from \( \ell = 1 \) to \( \ell = 2 \), which is verified by the observed increase in the beam’s diameter. Finally, we note that other interesting SoPs, such as circular, elliptical, and azimuthal can be realized following the systematic approach provided in this chapter.

5.5 Summary

In this chapter it has been shown, both theoretically and experimentally, how a superposition of co-propagating Bessel modes allows the arbitrary and simultaneous control of the polarization state, intensity, and topological charge of the resulting beam as it propagates. Our technique provides a systematic and versatile method to engineer the longitudinal characteristics of a field in lossless and lossy media. This can address many challenges in applications that benefit from the manipulation of diffraction-attenuation-resistant beams. In particular, our approach to control the state of polarization with propagation can be useful for material processing, polarimetry, microscopy and optical communications, to name a few.
Chapter 6

Engineering the Wavelength and Topological Charge of Non-Diffracting Beams along their Axis of Propagation

Wavelength is a fundamental property of light that dictates its linear momentum and thus remains unchanged under unguided free space propagation. Similarly, light’s orbital angular momentum is another strictly conserved property determined by light’s quantized topological charge. In this chapter, using a simple holographic setup, we report on a new class of non-diffracting optical beams in which the wavelength (linear momentum) and the topological charge (orbital angular momentum) can both be changed, independently, along the beam’s axis while propagating in air. Longitudinal control of both the wavelength and charge can offer new degrees-of-freedom in materials processing, optical trapping, imaging, and dense data communications.

6.1 Overview

Bessel beams are one example of monochromatic Localized Waves, which are also self-healing; i.e. the beam can reconstruct its center spot after being scattered by an obstacle [14]. Owing to these interesting non-diffracting and self-healing behaviors, Bessel beams have been widely utilized in optical trapping, materials processing, and imaging [14, 61, 147]. Further, higher order Bessel beams possess an azimuthal phase dependency of the form $e^{i\ell\phi}$, where $\ell$ is a quantized quantity referred-to as the topological charge of the beam [44, 54]. Such beams possess $\ell$ intertwined helices in their phasefront and carry orbital
angular momentum (OAM) — a conserved degree-of-freedom that has been widely deployed in a variety of fields as optical communications and micro-manipulation [55, 60, 50].

In general, Bessel beams are monochromatic, and thus maintain a fixed wavelength (\( \lambda \)) while propagating in air. Since \( \lambda \) is a fundamental property of light that dictates its linear momentum according to \( P = h/\lambda \) (\( P \) being the linear momentum per photon and \( h \) the Planck’s constant) [45], it follows that monochromatic Bessel beams carry fixed linear momentum along their propagation direction. Similarly, higher order Bessel beams with azimuthal phase dependency of \( e^{i\ell\phi} \) maintain a quantized topological charge \( \ell \) and carry fixed OAM value of \( \ell h/2\pi \) per photon, a strictly conserved quantity with propagation\(^1\).

In this chapter we present, for the first time to the best of our knowledge, a class of non-diffracting Bessel beams in which two degrees-of-freedom of light, namely its wavelength and topological charge, can be changed independently (“at-will”) along the beam’s axis of propagation, when traveling in air and without incorporating any non-linear media. The ability to generate multi-chromatic Bessel beams, in which the wavelength (linear momentum) can be controlled along the beam’s axis, can unlock many possibilities. For instance, beams with spatially varying wavelengths can be deployed to control the excitation properties of quantum emitters, which are wavelength-dependent [80]. It can also be utilized to shape the spatial profile of optically pumped media [79], or to control the shape and size of laser-machined structures by inducing wavelength-dependent ablation effects [81]. On the other hand, longitudinal control of the topological charge (orbital angular momentum) can offer new degrees-of-freedom in micro-manipulation [53, 75] and remote sensing [114], and can dramatically enhance communication channel capacities [76]. We thus envision that simultaneous longitudinal control of the beam’s wavelength and topological charge can lead to new advances in the field of optical sciences and applications of light.

### 6.2 Concept

#### 6.2.1 Theory

Controlling the wavelength and topological charge of non-diffracting Bessel beams along their axis of propagation relies on spatially multiplexing co-propagating Frozen Wave (FW) modes \((\psi_{\lambda,\ell})\) in the longitudinal direction. Each mode \((\psi_{\lambda,\ell})\) carries a distinct wavelength (\( \lambda \)) and charge (\( \ell \)) and consists of a judicious superposition of co-propagating Bessel beams with different transverse and longitudinal wavenumbers denoted by \( k_{p,\lambda,\ell,m} \) and \( k_{z,\lambda,\ell,m} \), respectively. The resulting multi-chromatic waveform is thus expressed as

\[
U(\rho, \phi, z, t) = \sum_{\lambda=\lambda_1}^{\lambda_M} \sum_{\ell=\ell_1}^{\ell_P} \psi_{\lambda,\ell}(\rho, \phi, z, t),
\]  

\(6.1\)

\(^1\)In Chapter 4, we illustrated how the OAM can change locally over finite sections of the beam. Here, such development will be extended to generate a non-diffracting beam that can change both its topological charge and wavelength, independently, with propagation.
where,
\[
\psi_{\lambda,\ell}(\rho, \phi, z, t) = e^{-i\frac{2\pi c}{\lambda} t} \sum_{m=-N}^{N} A_{\lambda,\ell,m} J_\ell(k_{\rho,\ell,m} \rho) e^{i\ell \phi} e^{i k_{z,\ell,m} z}.
\]

(6.2)

Here, the formulation is presented in the scalar and paraxial regime only for simplicity [99]. The parameters \( A_{\lambda,\ell,m} \) represent complex weighting factors for each Bessel mode in the superposition. By carefully engineering the phases and amplitudes of \( A_{\lambda,\ell,m} \) it becomes possible to modulate the envelope of the resulting waveform (\( \psi_{\lambda,\ell} \)) such that it follows a predefined intensity profile defined by \(|F_{\lambda,\ell}(z)|^2\).

To accomplish this, the coefficients \( A_{\lambda,\ell,m} \) are calculated from
\[
A_{\lambda,\ell,m} = \frac{1}{L} \int_0^L F_{\lambda,\ell}(z) e^{-i\frac{2\pi m}{L} z} dz,
\]
where \( L \) is the beam’s longitudinal extent. Longitudinal intensity modulation of the envelope is the result of controlled beating among the co-propagating Bessel modes carrying different spatial frequencies \( k_{\rho,\ell,m} \) and \( k_{z,\ell,m} \), where the longitudinal wavenumbers \( k_{z,\ell,m} \) are centered around a constant value (\( Q_{\lambda,\ell} \)) and are equally spaced, in a comb-like setting, such that:
\[
k_{z,\ell,m} = Q_{\lambda,\ell} + \frac{2\pi m}{L}.
\]

By superposing multiple monochromatic waveforms \( \psi_{\lambda,\ell} \) centered at different wavelengths \( \lambda \) and carrying different topological charges \( \ell \), according to Eq. (6.1), while properly engineering the longitudinal intensity profile of each waveform via \( F_{\lambda,\ell}(z) \) using Eq. (6.3), it is possible to independently control the effective wavelength and topological charge of the resulting beam along its propagation direction. In essence, with suitable definition of \( F_{\lambda,\ell}(z) \), only some chosen waveforms \( \psi_{\lambda,\ell} \) (with specific \( \lambda \) and \( \ell \)) will “switch-on” (possess high intensity) over a predetermined spatial region, thus contributing to the beam’s intensity at the central spot. Simultaneously, with the help of \( F_{\lambda,\ell}(z) \) as well, the contributions of other waveforms carrying different wavelengths and topological charges can “switch-off” (possess zero intensity) over the same spatial region. Such configuration can then be changed at other locations along the beam’s axis. Accordingly, the wavelength of the resulting beam \( U(\rho, \phi, z, t) \) can be observed to evolve, on demand, from \( \lambda_1 \) to \( \lambda_M \) and the topological charge can independently exhibit any desired transition from \( \ell_1 \) to \( \ell_P \) along the beam’s axis.

### 6.2.2 Experimental Setup

Figure 6.1 depicts the experimental setup used to generate the non-diffracting beams with varying wavelength and topological charge. Two laser sources at 635 nm and 532 nm are expanded, collimated, and then encoded onto two adjacent sections of a transmissive spatial light modulator (Holoeye LC2012 Amplitude SLM), via two computer generated holograms. The holograms were encoded using amplitude modulation mode following the method described in Ref. [98]. Due to its twisted nematic nature, the SLM is sandwiched in a polarizer-analyzer setup to achieve optimum contrast. The output patterns \( \psi_{\lambda=635 \text{ nm},\ell} \) and \( \psi_{\lambda=532 \text{ nm},\ell} \) are then combined into a single beam and filtered using a 4-f system.
incorporating an iris, as shown in the figure. The transverse profile of the resultant beam \( U(\rho, \phi, z, t) \) is then detected via a color CCD camera moving along the \( z \)-direction at steps of 1 cm. In the following section we present the results of three generated multi-chromatic Bessel beam patterns.

![Figure 6.1: The holographic setup used to generate beams with varying wavelength and topological charge. Here, BE: beam expander, BC: beam combiner, M: mirrors, Pol: polarizers, and SLM: spatial light modulator.](image)

6.3 Experimental Results

Four different multi-chromatic light patterns have been generated. Such light structures were constructed from a superposition of co-propagating modes \( \psi_{\lambda,\ell} \). In our experiments, we have considered two wavelengths; \( \lambda = 532 \text{ nm} \) and \( \lambda = 635 \text{ nm} \), where each mode \( \psi_{\lambda,\ell} \) is composed of 13 Bessel beams (i.e. \( N = 6 \) and \( m \) goes from \(-6\) to \(6\)). The longitudinal wavenumbers associated with \( \psi_{\lambda=635 \text{ nm},\ell} \) (denoted by \( k_{z=635 \text{ nm},\ell,m} \)) were centered at \( Q_{\lambda=635 \text{ nm},\ell} = 0.999995 \times k_0 \) (\( k_0 = 2\pi/\lambda \)) and were equally spaced with steps of \( 2\pi \) in the k-space, in a comb-like setting. Similarly, the spatial frequencies \( k_{z=532 \text{ nm},m} \) were centered around \( Q_{\lambda=532 \text{ nm},\ell} = 0.9999965 \times k_0 \). This choice of \( k_{z,\ell,m} \) ensures that the resulting multi-chromatic beam maintains the same spot size along the beam’s axis regardless of the wavelength. It also ensures highly paraxial beams with small values of \( k_{\rho,\ell,m} \) that respect the SLM bandwidth. The first generated light pattern is described next.

6.3.1 Pattern 1: Multi-Chromatic Beam with Two Foci

In the first experimental scenario, the central spot of the propagating beam \( U(\rho, \phi, z, t) \) changes its color from green to red and is designed to switch-off midway with propagation, thus creating distinct multi-foci with different colors in space. In this case \( U(\rho, \phi, z, t) = \psi_{\lambda=532 \text{ nm},0} + \psi_{\lambda=635 \text{ nm},0} \). To realize
such profile, the morphological function $F_{\lambda,\ell}(z)$ associated with each mode $\psi_{\lambda,\ell}$ was defined as

$$
F_{\lambda,\ell}(z) = \begin{cases} 
F_{\lambda=532 \text{ nm}, \ell=0} = 1, F_{\lambda=635 \text{ nm}, \ell=0} = 0 & (12 \text{ cm} \leq z \leq 27 \text{ cm}), \\
F_{\lambda=532 \text{ nm}, \ell=0} = F_{\lambda=635 \text{ nm}, \ell=0} = 0 & (27 \text{ cm} \leq z \leq 42 \text{ cm}), \\
F_{\lambda=532 \text{ nm}, \ell=0} = 0, F_{\lambda=635 \text{ nm}, \ell=0} = 1 & (42 \text{ cm} \leq z \leq 57 \text{ cm}), \\
F_{\forall \lambda, \forall \ell} = 0 & \text{elsewhere.}
\end{cases}
$$

(6.4)

Figure 6.2: Measured intensity profile of the generated multi-foci color beam. a) Experimentally reconstructed longitudinal intensity profile. The inset depicts the simulated profile. b) Transverse intensity profiles detected by the CCD at $z = 19$ cm, $z = 34$ cm, and $z = 49$ cm.

Substituting $F_{\lambda,\ell}(z)$ in Eq. (6.3) yields the desired complex coefficients $A_{\lambda,\ell,m}$ which, when inserted in Eq. (6.2), gives the desired transmission functions to be encoded in the computer holograms. The longitudinal intensity profile of the resulting multi-foci beam is depicted in Fig. 6.2(a). This plot has been rendered by aggregating 1-D strips obtained from the transverse profiles taken by the CCD camera, and is in very good agreement with the simulated profile (in the inset). The transverse beam’s profiles are displayed in Fig. 6.2(b) at three propagation distances along the $z$-direction; $z = 19$ cm, $z = 34$ cm, and $z = 49$ cm, chosen as three representative samples of each region of the beam. As the figure shows, the generated beam has two foci with distinct wavelengths (red and green) along the $z$-direction following the profile dictated by Eq. (6.4). In such case, the beam changes its wavelength with propagation while maintaining its topological charge at $\ell = 0$. 
To investigate the underlying principles governing the evolution of our multi-foci beam, we examined the amplitudes and intermodal phases associated with the complex coefficients $A_{\lambda,\ell,m}$. Recall that each Bessel beam in the superposition of Eq. (6.2) is assigned an initial amplitude and phase via $A_{\lambda,\ell,m}$. As the co-propagating Bessel beams travel along the $z$-direction, they accumulate additional phases according to $\angle[A_{\lambda,\ell,m} e^{ik_{\lambda,m}z}]$ (\(\angle\) is the phase angle); where the accumulated phases are non-uniform and depend on the spatial frequencies of the Bessel beams. Figure 6.3 depicts the amplitudes and intermodal phases among the Bessel beams as they propagate. The amplitudes of the Bessel beams within the modes $\psi_{\lambda=532 \text{ nm}, \ell=0}$ and $\psi_{\lambda=635 \text{ nm}, \ell=0}$ remain fixed along the beam’s axis as shown in Fig. 6.3(a). On the other hand, the intermodal phases, $A_{\lambda,\ell,m}$, evolve with propagation to help shape the envelope (intensity profile) of the resultant beam. This is depicted in Figs. 6.3(b-d) which displays the intermodal phases at three propagation distances; $z = 19.5$ cm, $z = 34.5$ cm, and $z = 49.5$ cm.

It is observed that at $z = 19.5$ cm the Bessel modes within $\psi_{\lambda=532 \text{ nm}, \ell=0}$ are all in-phase and, hence, constructively interfere to create the first focus in green color. At the same $z$-location, the Bessel modes within $\psi_{\lambda=635 \text{ nm}, \ell=0}$ are out-of-phase and thereby destructively interfere. As such, the contributions of $\psi_{\lambda=635 \text{ nm}, \ell=0}$ are dispersed over a large transverse area in the outer rings of the beam. The dual picture is then seen at $z = 49.5$ cm, where the contributions of $\psi_{\lambda=635 \text{ nm}, \ell=0}$ are now pronounced in
the beam’s center, creating the red focus, whereas the contributions of $\psi_{\lambda=532 \text{ nm}, \ell=0}$ are dispersed in the outer rings. Evidently, there exist a region midway at $z = 34.5$ cm, where the contributions of both $\psi_{\lambda=532 \text{ nm}, \ell=0}$ and $\psi_{\lambda=635 \text{ nm}, \ell=0}$ are switched-off, in agreement with the definition of Eq. (6.4). In the next subsection we demonstrate another non-diffracting beam whose central spot exhibits an alternating wavelength transition without switching-off its intensity.

### 6.3.2 Pattern 2: Multi-Chromatic Beam with Alternating Color

In the second experimental pattern, $U(\rho, \phi, z, t)$ consists of a superposition of 0-th order Bessel beams ($\ell = 0$), where the beam’s center evolves from red color to green multiple times. In this case, the morphological function $F_{\lambda,\ell}(z)$ is defined as

$$F_{\lambda,\ell}(z) = \begin{cases} F_{\lambda=635 \text{ nm}, \ell=0} = 1 & (10 \text{ cm} \leq z \leq 28 \text{ cm}) \text{ and } (32 \text{ cm} \leq z \leq 50 \text{ cm}), \\ F_{\lambda=532 \text{ nm}, \ell=0} = 1 & (22 \text{ cm} \leq z \leq 40 \text{ cm}) \text{ and } (44 \text{ cm} \leq z \leq 60 \text{ cm}), \\ F_{\lambda,\ell} = 0 & \text{elsewhere.} \end{cases} \quad (6.5)$$

Consequently, the contributions of $\psi_{\lambda=635 \text{ nm}, \ell=0}$ (red) are “switched-on” over the regions $10 \text{ cm} \leq z \leq 28 \text{ cm}$ and $32 \text{ cm} \leq z \leq 50 \text{ cm}$, and “switched-off” elsewhere. In contrast, the contributions of $\psi_{\lambda=532 \text{ nm}, \ell=0}$ (green) are “switched-on” over the regions, $22 \text{ cm} \leq z \leq 40 \text{ cm}$ and $44 \text{ cm} \leq z \leq 62 \text{ cm}$, and are “switched-off” elsewhere, in an interlaced manner with $\psi_{\lambda=635 \text{ nm}, \ell=0}$. Figure 6.4(a) shows the measured longitudinal intensity profile of the combined beam compared to the simulated profile (the inset). It is observed that the central spot evolves from red to green, twice, while propagating along the $z$-direction, in agreement with Eq. (6.5). This behavior can be viewed as spatial longitudinal wavelength division multiplexing.

Figure 6.4(b) displays the transverse intensity profiles taken at different planes along the propagation direction. It is confirmed that the center spot changes its color as the beam propagates. Interestingly, there exist transitional regions (for instance, $z = 24$ and 47 cm) where the central spot is observed to possess a yellow color as a result of the overlap between red and green waveforms.

Therefore, by modulating the envelope via controlled intensity beating, we are able to change the wavelength of the resulting beam along its axis. We note that when the contributions of one waveform ($\psi_{\lambda,\ell}$) is designed to switch-off, the energy in the center is dispersed into the outer rings of the beam over larger transverse area. The outer rings thus act as a reservoir of energy and linear momentum that can feed the central spot again at further distances. Such energy and momentum exchange between the central spot and outer rings allows us to control/multiplex the wavelength with propagation without violating the conservation of energy or momentum. In particular, the linear momentum is conserved globally across the entire cross section of the beam (including its outer rings); a detailed analysis of this can be found in Chapter 4 and Ref. [101].
Thus far we have presented scenarios in which the wavelength of non-diffracting Bessel beams is engineered along the propagation direction while maintaining the topological charge at a fixed value $\ell = 0$. Our approach is general and can be applied to vortex Bessel beams (with $\ell > 1$), such that both the wavelength and topological charge ($\ell$) of the beam’s central spot can be changed along its axis of propagation.

### 6.3.3 Pattern 3: Multi-Chromatic Vortex Beam with Varying Charge $\ell$

In the third example, we will demonstrate a multi-chromatic vortex beam that, besides alternating its wavelength (twice), also undergoes a controlled topological transition from $\ell = 1$ to $\ell = 2$. Recall that an increase in the topological charge of a vortex beam is associated with an increase in the number of helices in its phase-front and increase in the radial extent of the beam. In essence, the vortex beam composed of OAM modes $\psi_{\lambda,\ell=1}$ $\psi_{\lambda,\ell=2}$. To create the beam with a transition from $\ell = 1$ to $\ell = 2$, the
The morphological function was defined as follows $F_{\lambda, \ell}(z)$

$$F_{\lambda, \ell}(z) =
\begin{cases}
F_{\lambda=635 \text{ nm}, \ell=1} = 1, & F_{\lambda=532 \text{ nm}, \ell=1} = 0 & (0 \text{ cm} \leq z \leq 18 \text{ cm}), \\
F_{\lambda=635 \text{ nm}, \ell=0} = 0, & F_{\lambda=532 \text{ nm}, \ell=1} = 1 & (12 \text{ cm} \leq z \leq 30 \text{ cm}), \\
F_{\lambda=635 \text{ nm}, \ell=2} = 1, & F_{\lambda=532 \text{ nm}, \ell=2} = 0 & (22 \text{ cm} \leq z \leq 40 \text{ cm}), \\
F_{\lambda=635 \text{ nm}, \ell=0} = 0, & F_{\lambda=532 \text{ nm}, \ell=2} = 1 & (34 \text{ cm} \leq z \leq 50 \text{ cm}), \\
F_{\forall \lambda, \forall \ell} = 0 & \text{elsewhere}.
\end{cases}
$$

Figure 6.5: Transverse intensity profiles of the multi-chromatic vortex beam captured at different planes along the propagation direction, $z$.

Figure 6.5 depicts the transverse intensity profiles of the multi-chromatic vortex beam recorded at different propagation distances along $z$. It is observed that, not only does the beam change its color from red to green (twice), but it also exhibits a transition in its topological charge from $\ell = 1$ to $\ell = 2$ as the beam propagates. This is confirmed by observing the increase in the radial extent of the beam as it propagates along the $z$-plane. Notice how the energy is exchanged between the center of the beam and its outer rings to control the $\lambda$ and $\ell$ with propagation. The role of the outer rings in conserving the energy and momentum is described in details in Chapter 4 and Ref. [101].
This approach is versatile and can be used to design and create rotating petal-like structures that changes their topology with propagation, as described next.

6.3.4 Pattern 4: Multi-Chromatic Petal-Like Beam with Varying Charge

In the final example, we show a multi-chromatic vortex beam that changes its wavelength twice and also undergoes a controlled topological transition (again from $\ell = 1$ to $\ell = 2$). However, in order to experimentally observe the change in the topological charge $\ell$ directly on the intensity profile, we created waveforms that carry opposite signs of $\ell$, hence carrying phase-fronts of opposite helicities. This allows the azimuthal phase dependency ($e^{i\ell\phi}$) to be directly mapped onto the intensity profile (in the form of intensity petals) as a result of introducing phase singularities in the azimuthal direction — as discussed in Chapter 4. In essence, a beam composed of $\psi_{\lambda,\ell=1}$ and $\psi_{\lambda,\ell=-1}$ would possess two petals, whereas a beam composed of $\psi_{\lambda,\ell=2}$ and $\psi_{\lambda,\ell=-2}$ would possess four petals and so forth [99].

To generate such beam, the morphological function $F_{\lambda,\ell}(z)$ was defined as follows

$$F_{\lambda,\ell}(z) = \begin{cases} 
F_{\lambda=635 \text{ nm}, \ell=1} = F_{\lambda=635 \text{ nm}, \ell=-1} = 1 & (0 \text{ cm} \leq z \leq 18 \text{ cm}), \\
F_{\lambda=532 \text{ nm}, \ell=1} = F_{\lambda=532 \text{ nm}, \ell=-1} = 1 & (12 \text{ cm} \leq z \leq 30 \text{ cm}), \\
F_{\lambda=635 \text{ nm}, \ell=2} = F_{\lambda=635 \text{ nm}, \ell=-2} = 1 & (22 \text{ cm} \leq z \leq 40 \text{ cm}), \\
F_{\lambda=532 \text{ nm}, \ell=2} = F_{\lambda=532 \text{ nm}, \ell=-2} = 1 & (34 \text{ cm} \leq z \leq 50 \text{ cm}), \\
F_{\psi_{\lambda,\ell}} = 0 & \text{elsewhere.}
\end{cases}$$

$$\text{(6.7)}$$

![Figure 6.6](image)

Figure 6.6: Transverse intensity profiles of the multi-chromatic petal-like beam captured at different planes along the propagation direction, $z$.

Figure 6.6 depicts the transverse intensity profiles of the multi-chromatic vortex beam recorded at different propagation distances along $z$. It is observed that, not only does the beam change its color from red to green (twice), but it also exhibits a transition in its topological charge from $\ell = 1$ to $\ell = 2$ as the beam propagates. This is confirmed by tracking the number of petals at each $z$-plane. Notice how the energy is exchanged between the center of the beam and its outer rings to control the $\lambda$ and $\ell$ with propagation.
Finally, it is important to note that the approach adopted here is qualitatively different from earlier efforts demonstrating achromatic (broadband) Bessel modes [148]. Our approach significantly expands on these previous efforts to engineer the spatial profile of the beam’s wavelength and charge. A powerful advantage of this scheme is that longitudinal control over $\lambda$ and $\ell$ occurs independently and without relying on any nonlinear effects. Finally, our method provides control over the longitudinal profile of each wavelength component of the multi-chromatic beam via judicious interference of Bessel beams, which is much more versatile and fundamentally different from the on-axis wavelength variation observed in standard chromatic aberration. We thus envision our method to open many new possibilities in Structured Light and its applications.

### 6.4 Summary

This chapter presented the experimental demonstration of a new class of non-diffracting and multi-chromatic beams in which the wavelength and topological charge can both be modified independently along the beam’s direction of propagation. The proposed method utilizes a simple holographic setup without incorporating any nonlinear effects. Such degrees of control set the foundation for manipulating the linear and orbital angular momentum of light locally as it propagates, thus offering new opportunities in micro-manipulation, dense data communications, and materials processing.
Chapter 7

Curved Frozen Waves Following Arbitrary Spiral and Snake-Like Trajectories in Air

In this chapter we present, in theory and experiment, a class of non-diffracting and self-healing optical beams whose central spot can be designed to follow off-axis curved trajectories. The intensity of the generated waveform can traverse a spiral path with the possibility of reversing its handedness and changing its longitudinal intensity pattern with propagation, on demand. The central spot can also be designed to evolve from a straight to a curved snake-like trajectory and vice versa. Such flexible level of control can be useful for many applications in atomic guiding, optical trapping, and material processing.

7.1 Overview

In general, a Bessel beam propagates in rectilinear path (straight-line) along its propagation axis. Other classes of non-diffracting beams, namely Airy and parabolic beams, are self-accelerating and can propagate over curved trajectories [24, 25, 151], thus offering additional flexibility for many applications including particle manipulation [82, 83], bending surface plasmons [152], and imaging [153]. In addition to Airy beams, non-paraxial self-accelerating beams that propagate in curved trajectories have been demonstrated in Refs. [154, 155] and in [28, 156]. The latter are known as the Mathieu and Weber beams.

However, only few studies have attempted to generate Bessel-like beams whose intensity follows curved (off-axis) trajectories [157] such as snake-like [158] and/or spiral paths [159, 160, 161, 162].

1Generating Frozen Waves over curved trajectories is among the main contributions of this thesis — first published in Refs. [149, 150] and adapted in this chapter.
When such a behavior is sought, the generated beams exhibit a periodic off-axis profile over the entire beam range; with no ability to simultaneously control the longitudinal behavior of Bessel beams while propagating along a curved trajectory (for example, changing the sense of spiraling, turning the beam intensity on and off, evolving from a straight to curved path at-will).

In this chapter, we demonstrate a class of non-diffracting and self-healing curved beams, noted here as Curved Frozen Waves (CFW), whose central intensity spot can be designed to exhibit a non-periodic off-axis spiraling or snake-like behavior over a predefined space region. The generated beams — which arise from a double superposition of Bessel beams — can exhibit a spiraling behavior in which the beam’s sense of rotation can be reversed and the longitudinal intensity profile can be controlled, on demand. Furthermore, the beam’s propagation can be made to evolve from straight to curved snake-like, and then retaining its original straight path, at-will. Such control takes place while conserving the energy and global momentum of the beam. These capabilities can be beneficial for many applications in optical trapping and material processing.

7.2 Concept

The proposed CFW builds on a generalization of the theory of Frozen Waves (FWs) discussed in Chapter 2 and in Refs. [70, 71]. The original formulation of the FWs dealt with a superposition of equal frequency co-propagating Bessel beams (of the same order \( \ell \), often \( \ell = 0 \)) with different transverse and longitudinal wavenumbers. This yields waveforms with static intensity envelopes that can maintain a predefined longitudinal profile along straight lines, only. While FWs were originally demonstrated in air [87, 88, 89], the possibility of generating such beams inside an arbitrary absorbing medium, overcoming propagation losses, has been presented theoretically [90] and recently demonstrated experimentally — as reported in Chapter 3 and in Ref. [92]. The CFW (\( \Psi \)), i.e., the superposition of the original FW (\( \psi_\ell \)) over the topological charges \( \ell \), is given by

\[
\Psi(\rho, \phi, z, t) = \sum_{\ell=-\infty}^{\infty} \psi_\ell = e^{-i\omega t} \sum_{\ell=-\infty}^{\infty} \sum_{m=-N}^{N} A_{\ell,m} J_\ell(k_{\rho,\ell,m} \rho) e^{i\ell \phi} e^{ik_{z,\ell,m} z},
\]

where each FW \( \psi_\ell \) is composed of a superposition of \( 2N+1 \) equal frequency Bessel beams of order \( \ell \). For the \( n^{th} \) Bessel beam in each FW, the transverse and longitudinal wavenumbers are denoted by \( k_{\rho,\ell,m} \) and \( k_{z,\ell,m} \), respectively. The longitudinal wavenumbers \( k_{z,\ell,m} \) are chosen such that \( k_{z,\ell,m} = Q_\ell + \frac{2\pi m}{L} \); where \( Q_\ell \) is a constant parameter that defines the beam transverse localization whereas \( L \) is a predefined distance along which the CFW displays the desired pattern. The transverse wavenumbers are thus given by the consistency relation \( k_{\rho,\ell,m} = \sqrt{k^2 - (k_{z,\ell,m})^2} \) with \( k = \omega n/c \) (\( n \), the index of refraction, is equal to 1 for air).

An important characteristic of the FWs and CFWs is their ability to carry a predefined longitudinal intensity profile given by the morphological function \( F_\ell(z) \). The complex coefficients \( A_{\ell,m} \) in Eq. (7.1)
represent different weighting factors for each Bessel beam in the superposition such that the interference of the co-propagating Bessel beams yields the desired longitudinal pattern \( F_\ell (z) \); thus allowing control over the beam’s longitudinal intensity with propagation. The coefficients \( A_{\ell,m} \) are obtained via

\[
A_{\ell,m} = \frac{1}{L} \int_0^L F_\ell (z) e^{-(i \frac{2\pi}{L} m) z} dz.
\]

In general, a FW composed of only zero-order Bessel beams \((\psi_0)\) has a uniform phase front, and carries the desired intensity profile \( F_0 (z) \) in its central spot on axis, while propagating along a straight line. On the other hand, a CFW composed of higher order Bessel beams \((\ell \geq 1)\) possesses a helical phase front and carries its intensity profile over a cylindrical surface and has a non-zero Orbital Angular Momentum (OAM).

While FWs only propagate along straight paths, CFWs — via judicious superposition of FWs — provide better control over both transverse and longitudinal intensity patterns of the resultant beam. For example, it is possible to generate a CFW beam with central spot that follows curved (off-axis) 3D spiraling path, or it experiences a lateral shift while propagating in a snake-like manner, as will be demonstrated in this chapter. Moreover, the longitudinal control provided via CFWs allows us to reverse the sense of rotation of the spiraling beam, turn the central spot intensity off and then on again with propagation, make the beam spot evolve from a straight to a curved path and vice versa, on demand. These degrees of control have not been previously shown and can open many new possibilities for material processing, optical trapping, and micromanipulation [52, 53].

The spiraling and snake-like beams were both generated over a distance \( L = 1 \) m. Such CFW beams consists of FWs \( \psi_0 \) and \( \psi_\ell \), such that \( \psi_0 \) itself is composed of zero order Bessel beams, whereas \( \psi_\ell \) is comprised of first order Bessel beams.

### 7.3 Experimental Setup

The CFWs were generated using a programmable spatial light modulator (SLM) model Holoeye LC2012 Amplitude SLM. For both generated patterns, the values of parameter \( Q_\ell \) were chosen as \( 0.9999958 \times k_0 \) for \( \psi_0 \) and \( 0.999993 \times k_0 \) for \( \psi_1 \) and \( \psi_{-1} \) (\( k_0 \) is the wavenumber in vacuum). This choice of \( Q_\ell \) ensures highly paraxial beams with small values of \( k_{\rho,m}^\ell \). Moreover, each FW mode \( \psi_\ell \) is composed of 15 Bessel beams with slightly shifted and equally spaced \( k_{z,m}^\ell \) that are centered about \( Q_\ell \). The choice of these parameters enforces positive values for the real part of the longitudinal wavenumber \( \text{Re}\{k_{z,m}^\ell\} \) while respecting the SLM’s bandwidth.

The experimental procedure was as follows: first, the CFW pattern \( \Psi = \psi_0 + \psi_1 + \psi_{-1} \) was computed and transformed into a 2D Computer Generated Hologram (CGH). An amplitude mask was adopted to express the complex transmission function of the CFW at the origin of propagation (i.e. \( \Psi(\rho,\phi,z = 0,t) \)). The hologram equation is given by

\[
H(x,y) = \frac{1}{2} \{ \beta(x,y) + \alpha(x,y) \cos[\Theta(x,y) - 2\pi(u_0 x + v_0 y)] \},
\]

where \( \alpha(x,y) \) and \( \Theta(x,y) \) represent the amplitude and phase of \( \Psi(\rho,\phi,z = 0,t) \), respectively, and \( \beta(x,y) \) is a
bias function chosen as a soft envelope for the amplitude $\alpha(x, y)$ according to $\beta(x, y) = [1 + \alpha^2(x, y)]/2$ [98]. The pattern was interfered with a plane wave $\exp[2\pi i(u_0x + v_0y)]$. This shifts the encoded pattern off-axis (in the Fourier plane) to the spatial frequencies $(u_0, v_0)$; thus making it easier to filter out the desired pattern from the undesired on-axis noise by using an iris. In our experiment, $u_0$ and $v_0$ were set to $1/(4\Delta x)$; where $\Delta x$ is the SLM pixel pitch ($\Delta x = 36 \mu m$).

The CGH was addressed to an amplitude SLM which encoded the transmission function on a 532 nm collimated laser beam. The resulting pattern was imaged and filtered using a 4-f optical system as shown in Fig. 7.1. The evolution of the generated waveform along the longitudinal direction was then recorded using a sliding CCD camera with 1 cm step resolution.

### 7.4 Experimental Results

#### 7.4.1 Pattern 1: Spiraling Beam Reversing its Helicity

In the first example we present a CFW that follows a spiraling path. The waveform was generated by performing a superposition between $\psi_0$, $\psi_1$, and $\psi_{-1}$ using the morphological function, $F_\ell(z)$

$$
F_\ell(z) = \begin{cases} 
F_0 = 1, & F_{-1} = 0.7, & F_1 = 0 & 5 \text{ cm} \leq z \leq 40 \text{ cm}, \\
F_0 = F_{-1} = F_1 = 0 & 40 \text{ cm} \leq z \leq 55 \text{ cm}, \\
F_0 = 1, & F_{-1} = 0, & F_1 = 0.7 & 55 \text{ cm} \leq z \leq 85 \text{ cm}, \\
F_0 = F_{-1} = F_1 = 0 & \text{ elsewhere}. 
\end{cases} 
$$

(7.2)

According to Eq. (7.2), over the range $5 \text{ cm} \leq z \leq 40 \text{ cm}$, $\Psi$ is comprised of FW $\psi_0$ (zero order Bessel beams with $\ell = 0$) and FW $\psi_{-1}$ (first order Bessel beams with $\ell = -1$). This yields a CFW that propagates along a spiral trajectory with the CW sense of rotation. Contributions of the FWs are then switched-off over the range $40 \text{ cm} \leq z \leq 55 \text{ cm}$. When contributions of a FW are switched-off (by assigning a zero value to $F_\ell(z)$), the energy of the selected FW flows away from the central spot and is dispersed over a larger space in the outer rings of the beam. Afterwards, within the interval $55 \text{ cm} \leq z \leq 85 \text{ cm}$, contributions of the FW modes $\psi_0$ and $\psi_1$ (FW of order $\ell = 1$) come into effect. As such,
Chapter 7. Frozen Waves following arbitrary spiral and snake-like trajectories

(a) Transverse Profiles - Simulation

(b) Transverse Profiles - Measured

Figure 7.2: Simulated and measured intensity patterns of the spiral beam. a) Simulated transverse profiles. b) Measured transverse profiles. The arrows depict the direction of rotation of the central spot. The central spot follows a spiral path in the CW direction with a periodicity Λ = 19 cm over the range 5 cm ≤ z ≤ 40. It is turned off in the range of 40 cm ≤ z ≤ 55 cm. The beam follows a spiral path but with an opposite sense of rotation (CCW) in the region 55 cm ≤ z ≤ 85 cm. c-e) Transverse intensity profiles sampled at z = 14, 47, and 72 cm, respectively. The intensities are normalized with respect to the beam at z = 14 cm. Figure reproduced from Dorrah et al. [149] (© 2017 American Institute of Physics).
the dispersed energy is retrieved from the outer rings of the beam and focused again in the center so
that the central spot retains its spiraling behavior, but with the opposite sense of rotation (CCW). The
periodicity of the spiral trajectory is given by \( \Lambda = 2\pi/(Q_0 - Q_1) = 19 \text{ cm} \). Finally, for the interval \( z \geq 85 \text{ cm} \), all the morphological functions \( F_\ell \) are set to zero, thus switching the beam off.

The simulated and measured beam’s intensity profiles are depicted in Figs. 7.2(a) and (b), respec-
tively. The simulated figures were obtained by evaluating Eq. (7.1) in MATLAB. It is observed that
the central spot of the beam exhibits off-axis rotation in the CW direction over the range \( 5 \text{ cm} \leq z \leq 40 \text{ cm} \). The intensity of the central spot is then turned off for \( 40 \text{ cm} \leq z \leq 55 \text{ cm} \) (within which the morphological function \( F_\ell(z) = 0 \)). Finally, the central spot is switched-on again, but this time it rotates in the opposite direction (CCW) with propagation.

The beam’s transverse intensity profile is sampled at three locations along the propagation direction
at \( z = 14, 47, \) and \( 72 \text{ cm} \) as depicted in Figs. 7.2(c-e), respectively. It is observed that the center of mass
of the beam is shifted about the beam’s axis with propagation where the beam acquires an asymmetric
intensity profile. When the intensity of the beam’s center spot is designed to switch-off, as shown at
\( z = 47 \text{ cm} \) for instance, the energy is dispersed over a larger volume within the outer rings as shown
in Fig. 7.2(d). These outer rings act as an energy reservoir that reconstruct the center spot again, on
demand, at a longer distance and thus conserves the global energy of the beam.

The spiral trajectory of the beam’s central spot, obtained directly from CCD camera measurements,
is illustrated in Fig. 7.3. The upper limit on the spiraling radius of the CFW center spot can be estimated
by solving for \( \rho_\ell \) in \( \frac{\partial}{\partial \rho} J_\ell(\rho_\ell r, \theta, 0)\big|_{\rho = \rho_\ell} = 0 \). This gives the mean radius of the first order Bessel beams in

---

Figure 7.3: The 3D trajectory of the central spot of the spiraling beam obtained from the CCD camera measurements. Figure reproduced from Dorrah et al. [149] ( © 2017 American Institute of Physics).
The superposition of Eq. (7.1), along which the central spot follows a spiral path.

Furthermore, the reversal in the sense of spiraling is pronounced in Fig. 7.3. It should be noted that such longitudinal control over the beam intensity and topological charge occurs while respecting the conservation of energy and momentum of CFW. The generated spiraling beam is a result of superposition of FW modes \( \psi_0 + \psi_1 + \psi_{-1} \), where each FW mode carries 15 Bessel beams with the same topological charge \( \ell \) but with different wave numbers and weighting coefficients \( A_{\ell,m} \). Via the morphological function given in Eq. (7.2), the Bessel beams within the FW modes \( \psi_{-1} \) and \( \psi_0 \) are made to constructively interfere for 5 cm \( \leq z \leq 40 \) cm, whereas, the contribution of the Bessel beams within FW \( \psi_1 \) is made to destructively interfere over the same space region and is dispersed over the side rings of the beam. Similarly, the Bessel beams within \( \psi_0 \) and \( \psi_1 \) constructively interfere over 55 cm \( \leq z \leq 85 \) cm while the contribution of \( \psi_{-1} \) destructively interfere, which reverses the beam’s sense of spiraling.

In order to effectively achieve such longitudinal control over the beam’s intensity profile and topological charge over a given propagation distance \( L \), the diameter of the hologram aperture should be larger than a threshold size. This ensures the effective contribution of the side rings of the beam with propagation. Performing longitudinal control over the beam profile for longer distance \( L \) requires a hologram with larger aperture size. In our experiment, a sufficient condition on the aperture size (diameter) of the computer generated hologram is given by

\[
D \geq 2L \sqrt{\left( \frac{k}{k_{\ell=-1}} \right)^2 - 1}.
\]

Our approach is flexible and allows us to generate other interesting patterns along the beam’s direction of propagation. For instance, by simultaneously incorporating both FWs, with \( \ell = 1 \) and \( -1 \), within the same space interval, it is possible to realize a waveform that follows a snake-like path, as shown next.

### 7.4.2 Pattern 2: Snake-Like Beam Evolving from On-Axis to Off-Axis Propagation

Snake-like CFWs can be generated, still from a superposition of FW modes \( \psi_0, \psi_1, \) and \( \psi_{-1} \), while using the following morphological function \( F_\ell(z) \).

\[
F_\ell(z) = \begin{cases} 
F_0 = 1.2, F_{-1} = F_1 = 0 & 5 \text{ cm} \leq z \leq 25 \text{ cm}, \\
F_0 = 1, F_{-1} = F_1 = 0.6 & 25 \text{ cm} \leq z \leq 65 \text{ cm}, \\
F_0 = 1.2, F_{-1} = F_1 = 0 & 65 \text{ cm} \leq z \leq 85 \text{ cm}, \\
F_0 = F_{-1} = F_1 = 0 & \text{ elsewhere.}
\end{cases}
\]  

(7.3)

Here, the FW \( \psi_0 \) is composed of zero order Bessel beams, whereas \( \psi_1 \) and \( \psi_{-1} \) are comprised of the first order Bessel beams with opposite helicity. As a result of defining \( F_\ell(z) \) according to Eq. 7.3, the contributions of \( \psi_1 \) and \( \psi_{-1} \) are only pronounced over the range 25 cm \( \leq z \leq 65 \) cm and switched-off elsewhere. Meanwhile, the contribution of \( \psi_0 \) is in effect along the entire propagation range 5 cm \( \leq z \leq 85 \) cm. As such, the resultant CFW is effectively composed of only zero order Bessel beams over the
Figure 7.4: Simulated and measured intensity patterns of the snake-like beam in air. a) Simulated beam intensities at various transverse planes. b) Measured beam intensities at various transverse planes. The arrows depict the direction of transverse motion of the central spot. c) Simulated longitudinal intensity profile. d) Measured longitudinal intensity profile. The central spot propagates on-axis over the range $5 \text{ cm} \leq z \leq 25 \text{ cm}$ before it follows snake-like behavior over $25 \text{ cm} \leq z \leq 65 \text{ cm}$ with periodicity $\Lambda = 19 \text{ cm}$. Afterwards, the spot retains its on-axis path for $65 \text{ cm} \leq z \leq 85 \text{ cm}$. Transverse intensity profiles sampled at $z = 14, 38, 49,$ and $75 \text{ cm}$, respectively. The intensities are normalized with respect to the beam at $z = 38 \text{ cm}$. Figure reproduced from Dorrah et al. [149] (© 2017 American Institute of Physics).
space regions $5 \text{ cm} \leq z \leq 25 \text{ cm}$ and $65 \text{ cm} \leq z \leq 85 \text{ cm}$. Within these two intervals, the central spot of the beam exhibits a straight on-axis propagation as expected from the zero order Bessel beam. More interestingly, over the space region $25 \text{ cm} \leq z \leq 65 \text{ cm}$ (when $\psi_1$ and $\psi_{-1}$ contributions are switched-on), the central spot of the beam follows an off-axis snake-like propagation with periodicity $\Lambda = 19 \text{ cm}$.

Figures 7.4(a) and (b) depict the simulated and measured transverse intensity profiles of the generated pattern along the propagation direction, $z$. The on-axis central spot is shifted from the beam axis to follow a snake-like propagation path before it recovers its original on-axis behavior, consistent with $F_\ell(z)$ and above discussion. The simulated and measured longitudinal intensity profiles of the snake-like CFW are shown in Figs. 7.4(a) and (b), respectively, and are in good agreement. Furthermore, the beam’s transverse intensity profile is sampled at four locations along the propagation direction at $z = 14, 38, 49$ and $75 \text{ cm}$ as depicted in Figs. 7.4(e-h), respectively. It is observed that the beam evolve from a symmetric intensity profile about the center axis to an asymmetric off-axis profile before it retains its initial symmetric profile again.

Again, the evolution of the central spot from a straight to curved path, and vice versa, is achieved without violating the conservation of energy or momentum by the virtue of the outer rings. In other words, the contributions of the FW modes ($\psi_1$ and $\psi_{-1}$) — though not pronounced over the two space intervals $5 \text{ cm} \leq z \leq 25 \text{ cm}$ and $65 \text{ cm} \leq z \leq 85 \text{ cm}$ — are still contained in the outer rings of the beam and are moved to the central spot of the beam via the morphological function $F_\ell(z)$. Therefore, the global energy and momentum of the beam is always conserved. This can also be interpreted as an experimental realization of the famous Hilbert Hotel paradox — as discussed in details in Chapter 4 and Refs. [115, 116, 117].

7.5 Summary

In this chapter, we have demonstrated a class of curved non-diffracting and self healing beams which are based on a generalization of the FW method. These beams are composed of superposition of zero order and higher order co-propagating Bessel modes suitably weighted to exhibit a desired longitudinal profile. Notably, such CFWs can be designed so that their central spot follows a spiral trajectory with the possibility to reverse its handedness and longitudinal intensity pattern with propagation. The beam can also be made to traverse a snake-like path over a finite distance, on demand. We expect this advanced level of control to be beneficial for many emerging applications in optical trapping, particle manipulation, and material processing.
Chapter 8

Experimental Demonstration of Wide-Range Tunable Refractometer Based on OAM of Longitudinally Structured Light

In this Chapter, we show one of the many possible applications that can benefit from Longitudinally Structured Light; namely Refractive Index (RI) sensing. In most RI sensing schemes, there is a trade-off between providing high-resolution measurements and covering a wide range of RIs. Here, we propose and experimentally demonstrate a novel mechanism for RI sensing by utilizing Orbital Angular Momentum (OAM) of structured light. Using a superposition of co-propagating higher order Bessel beams with equally-spaced longitudinal wavenumbers, we generate rotating light structures in which the rotation speed is sensitive to the RI of the medium. In principle, the sensitivity of this scheme can exceed $\sim 2700^\circ/\text{RIU}$ with a resolution of $\sim 10^{-5}$ RI unit (RIU). Furthermore, we show how the unbounded degrees of freedom of OAM can be deployed to offer wide dynamic range for sensing by generating light structures that evolve to different modes based on the RI change — thus extending the dynamic range to cover RI values from 1 to over 2.9 RIU. The rotating light structures are generated by a programmable Spatial Light Modulator (SLM), thus providing dynamic sensitivity which can be tuned to perform coarse and fine measurements, in real-time. This allows high sensitivity and resolution to be achieved simultaneously over a very wide dynamic range, which is a typical tradeoff in all RI sensing schemes\textsuperscript{1}.

\textsuperscript{1}The results presented in this chapter were first published in Refs. [114, 163] and adapted in this chapter under the Creative Commons Attribution 4.0 International License (https://creativecommons.org/licenses/by/4.0/).
8.1 Overview

The interactions of light with a medium can be exploited to sense, measure, and study important properties of the medium [164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174]. One important property is the index of refraction, which plays a crucial role in the design and classifications of most optical materials and devices. As such, modalities that can provide accurate, efficient, economical, and dynamic measurements of the index of refraction are always needed. In the past, various approaches to measure the index of refraction have been proposed and utilized. For instance, in laser-based refractive index (RI) sensing, the change in RI in a medium can be inferred from the angle of refraction of the beam in the medium. This has been accomplished in various ways; it was originally done by measuring the critical angle in the medium using a prism [165] or measuring the displacement of a beam that is obliquely incident on the sample [166]. More recently, other properties of light (e.g., diffraction) have been utilized to measure the index of refraction [167]. Although these previous techniques are relatively simple to implement, they lack reconfigurability, and their resolution is typically limited to \( \sim 10^{-3} \) RI unit (RIU).

In some other cases, the frequency response can be more informative where a change in RI can be linked to a shift in the transmission spectrum of a broadband source. This has been manifested by relating the RI change to a shift in the transmission spectrum of micro-ring resonators [175, 176], micro-fiber resonators [177, 178, 179], Mach-Zehnder interferometers [180, 181], a shift in surface plasmon resonance [182, 183], or detecting the shift in the reflection spectrum of Fabry-Perot resonators [184, 185]. Polarization is another property of light that has been used to diversify the RI measurements, thereby improving the sensing precision [186, 187]. Although these later techniques can detect the index of refraction with high resolution (reaching \( \sim 10^{-6} \) RIU), they often suffer from a lack of tunability and reconfigurability and have a narrow dynamic range, high cost, sensitive interfacing and packaging requirements, and involve complex device fabrication processes. To overcome these limitations, in this chapter, we propose and experimentally demonstrate a novel RI sensing mechanism that relies on two important but unexplored degrees of freedom of light: light’s OAM and the ability to pattern light’s intensity along its direction of propagation.

8.2 Background

Light beams with OAM possess helical phase-fronts due to their azimuthal phase dependency that follows \( e^{i\ell \phi} \), where \( \ell \) is the topological charge or winding index of the helical phase-front. Notably, OAM differs fundamentally from the spin angular momentum (SAM) associated with polarization [44, 45, 54, 60]: unlike SAM, which is limited to a value of \( \pm \ell \hbar \) per photon, OAM can acquire unbounded values of \( \ell \hbar \) per photon (\( \ell \) is an integer), thus offering additional degrees of freedom. These new degrees of freedom of light, namely, the OAM modes, have been utilized in imaging [60], optical trapping [50], material processing [61], data communications [55, 188], and motion sensing [171, 173]. Here, we show, for the
Chapter 8. Wide-Range Tunable Refractometer Based on OAM of Light

First time (to the best of our knowledge), the unexplored advantages of using light’s OAM to measure the index of refraction of a given medium. Such a development can open new directions in refractometry and remote sensing using structured light.

8.3 Concept

OAM beams are characterized by twisted phase-fronts with a singularity in the beam’s center. As such, they carry zero intensity in the beam’s center, whereas their intensity is distributed over a cylindrical surface along the beam’s axis. An OAM beam with topological charge \( \ell \) possesses \( \ell \) inter-twined helices in its phase-front [44, 45, 54, 60]. For the same value of \( \ell \), the amount of phase twist in each helix, over a finite distance, depends on the wavenumber and the RI of the medium. In essence, the helical phase can encounter different amounts of stretching (or compression) if the same beam propagates in different media (with different RI), as shown below. Although characterized by a helical phase-front, when looking at the transverse intensity profiles of an OAM mode, the intensity is distributed over a continuous ring. Hence, the amount of phase helicity is not readily detected by simply looking at the beam’s intensity profile, and its detection typically requires a wavefront sensing apparatus. However, with a judicious superposition of two OAM beams of opposite helicities such that the two OAM beams carry topological charges with opposite signs, it becomes possible to directly map the helicity in the beam’s phase-front to a modulation in the beam’s intensity profile, which can be easily detected by a CCD camera. This occurs as a result of introducing singularities into the phase-front along the azimuthal direction \( \phi \), which in turn creates discontinuities in the beam’s transverse intensity profile — often producing intensity patterns in the form of flower petals as discussed in Chapter 4 and Refs. [128, 129].

When the longitudinal wavenumbers of the superimposed OAM modes are slightly different, the beating between the spatial frequency harmonics will result in a light structure whose intensity profile can rotate along its optical path. The rotation of light’s intensity pattern along its propagation direction has been previously reported in Refs. [130, 131, 132, 133, 99]. The angular orientation of the rotating beam petals is a function of both its propagation distance \( z \) and its optical length. In other words, at a fixed detection plane along \( z \), the beam orientation will also vary if the RI of the medium is changed. This variation in the beam’s angular orientation can be interpreted based on the fact that the angular velocity of the rotating intensity pattern (petals) is directly linked to the amount of its phase helicity, which, in turn, depends on the RI. As such, it is possible to develop a laser-based sensing scheme using OAM modes such that the change in the RI can be linked to the change in the angular orientation of the beam’s transverse intensity profile that, in turn, is easily detected by a CCD camera. In short, at a given transverse plane, by measuring the angular orientation of the rotating intensity pattern (petals) in an unknown medium with respect to its orientation in air (as a reference), the RI of the unknown medium can then be accurately measured.
8.4 Theoretical Formulation

The rotating intensity pattern $\Psi(\rho, \phi, z, t)$ is composed of multiple OAM modes, where each OAM mode $\psi_\ell$ is a superposition of equal frequency co-propagating Bessel beams of different transverse and longitudinal wavenumbers. The resulting waveform, $\Psi(\rho, \phi, z, t)$, is thus given by

$$\Psi(\rho, \phi, z, t) = \sum_{\ell=-\infty}^{\infty} \psi_\ell = e^{-i\omega t} \sum_{\ell=-\infty}^{\infty} \sum_{m=-N}^{N} A_{\ell,m} J_{\ell}(k_{\ell,m}^\rho \rho) e^{i\ell\phi} e^{ik_{\ell,m}^z z}. \quad (8.1)$$

In Eq. (8.1), each OAM mode $\psi_\ell$ carries a specific charge $\ell$ and is composed of $2N + 1$ Bessel beams of equal order ($\ell$). For the $m^{th}$ Bessel beam in $\psi_\ell$, the transverse wavenumber $k_{\rho}^{\ell,m}$ is related to the longitudinal wavenumber $k_{\ell,m}^z$ via the consistency relation:

$$k_{\rho}^{\ell,m} = \sqrt{k^2 - (k_{\ell,m}^z)^2}. \quad (8.2)$$

An important property of these OAM modes is that via the superposition of $2N + 1$ Bessel beams with different spatial frequencies, the longitudinal intensity profile of the resultant beam can be controlled along the $z$-direction (i.e., along the beam’s axis) in a controlled manner — as discussed in Chapter 3 and Ref. [92]. This is achieved, in part, by the coefficients $A_{\ell,m}$ in Eq. (8.1), which represent different complex weighting factors for each Bessel beam in the superposition, calculated according to

$$A_{\ell,m} = \frac{1}{L} \int_{0}^{L} F(z) e^{-(i2\pi m)z} dz. \quad (8.2)$$

Function $F_\ell(z)$ in Eq. (8.2) is the desired longitudinal (axial) intensity profile. With the proper definition of the topological charges $\ell$ and the associated function $F_\ell(z)$, Eq. (8.1) can be deployed to generate a rotating OAM light structure with a predefined longitudinal extent that is independent of the transverse beam’s dimensions. The longitudinal control over the beam’s intensity profile is also an important property that will be utilized to extend the dynamic range of the proposed sensor, as discussed later in this chapter. Finally, a summation of two (or more) OAM modes with opposite signs for the topological charge $\ell$ transforms the regular rings associated with the Bessel beam’s transverse intensity profile into petal-like shapes whose rotation per RI change can be detected by a CCD camera.

The longitudinal wavenumbers of each OAM mode are equally spaced in the $k$–space around a constant parameter $Q$, in a comb-like setting. More specifically, $k_{\ell,m}^z = Q + 2\pi m/L$, where $m \in [-N, N]$, and $L$ is the distance over which the desired beam is generated. Figure 8.1(a) depicts the spatial frequencies of two OAM modes with opposite helicities, that is $\psi_{-1}$ and $\psi_{1}$, where the longitudinal wavenumbers are centered at slightly shifted constants $Q_{-1}$ and $Q_1$, respectively. Figure 8.1(a) also depicts the weighting factors $A_{\ell,m}$ for each Bessel beam of the superposition of Eq. (8.1), as obtained from Eq. (8.2). The phase and amplitude of the coefficients $A_{\ell,m}$ are evaluated such that the resulting beam extends for 50 cm (as defined by $F_\ell(z)$). This approach allows flexible control over the beam’s range without altering its transverse localization.

For this scenario, $F_\ell$ was set equal to unity for a distance of $0 \text{ cm} \leq z \leq 50 \text{ cm}$ ($\ell = \pm 1$) with
Figure 8.1: Schematic diagram illustrating the longitudinal wavenumbers of OAM modes $\psi_{-1}$ and $\psi_{1}$. a) Longitudinal wavenumbers in air ($n = 1$). b) Longitudinal wavenumbers in a medium with unknown refractive index $n$. Here, each OAM mode consists of 9 Bessel beams whose longitudinal wavenumbers are equally-spaced, in a comb-like setting, around a constant parameter $Q_{-1}$. Additionally, $F_{t}(z) = 1$, for $0 \text{ cm} \leq z \leq 50 \text{ cm}$ and $Q \simeq n \times Q$. Figure reproduced from Dorrah et al. [114].

propagation in air. As the figure shows, the complex coefficients $A_{\ell,m}$ form a comb-like structure in the k-space. Analogous with optical frequency combs, in which the spectral range of frequencies is related to the laser temporal pulse width [189, 190], here, the span of the spatial frequency comb is related to the radial extent of the beam (beam localization). As such, a more radially localized rotating light structure yields a k-space comb that spans a wider range of spatial frequencies.

Furthermore, in optical frequency combs, the spectral components are equally spaced in accordance to the laser repetition rate [189, 190], whereas in the case of our k-space comb, the teeth are equally spaced by a factor of $2\pi$. A powerful property of frequency combs is their ability to link the precision of optical frequencies with microwave frequencies [190], thus providing an accurate and precise spectral ruler that can be interfaced (accessed) with electronic circuitry. Here, by interfering two k-space combs, we provide a tool that can map the shift in the spatial frequencies of an OAM mode in a given medium, which is also linked to the helicity of the phase-front in that medium, into a rotation in the transverse
intensity profile that can be easily detected by a CCD camera. As shown in Chapter 4, the orientation of the rotating light patterns (petals) resulting from the superposition of two OAM modes, $\psi_1^\ell$ and $\psi_2^\ell$, is given by [99]

$$\Phi_{\ell_1,\ell_2}(\Delta Q) \propto \frac{(\Delta Q)(z)}{|\ell_1| + |\ell_2|},$$  \hspace{1cm} (8.3)

where $\Delta Q = Q_{\ell_1} - Q_{\ell_2}$ in the medium, and $z$ is the detection plane. Figure 8.1 (a) shows the case when the beam is propagating in air ($n = 1$). The longitudinal wavenumbers $k_{z_{1,m}}^\ell$ and $k_{z_{-1,m}}^\ell$, associated with $\psi_1$ and $\psi_{-1}$, are centered around $Q_1$ and $Q_{-1}$ and are equally spaced by a factor of $2\pi$. Interestingly, when the same beam is allowed to propagate in a different medium with an unknown refractive index $n$ ($n > 1$), the longitudinal wavenumbers $k_{z_{\rho,m}}^\ell$ are shifted and are centered around larger values, $\tilde{Q}_1$ and $\tilde{Q}_{-1}$, as depicted in Fig. 8.1 (b). This is a consequence of the consistency relation:

$$k_{\rho,m}^\ell = \sqrt{k^2 - (k_{z_{\rho,m}}^\ell)^2},$$

in which $k = \omega n/c$, whereas the values of $k_{\rho,m}^\ell$ are preserved at the boundaries. In a medium with refractive index $n$, the wavenumbers are still equally spaced but with smaller spacing, equal to $2\pi/n$. Figure 8.1(b) shows, for this case, the spacing between the centers of the two spatial frequency combs becomes: $\Delta \tilde{Q} = \Delta Q/n$ (more detailed derivation can be found in Appendix B).

According to Eq. (8.3), the change in $\Delta Q$, which is a function of RI, modifies the angular orientation of the beam petals ($\Phi$). Therefore, in any unknown medium, the rotating intensity pattern exhibits a specific angular orientation $\Phi$ that depends on the RI of the medium. By comparing the angular orientation of the intensity pattern in a medium with an unknown refractive index $n$ — denoted as $\Phi_{\ell_1,\ell_2}(\Delta \tilde{Q})$, with respect to its orientation in air as a reference ($n = 1$) — denoted as $\Phi_{\ell_1,\ell_2}(\Delta Q)$, at the same propagation distance, one can then accurately determine the refractive index of the unknown medium.

### 8.5 Experimental Setup

In order to test the proposed sensing mechanism, we carried out the following experimental procedure: first, the Bessel Beam superposition in Eq. (8.1) was computed and transformed into a 2D Computer Generated Hologram (CGH). The holograms were designed for the case of air, assuming $n = 1$. We adopted an amplitude mask to express the complex transmission function at the origin of propagation (i.e. $\Psi(\rho, \phi, z = 0, t)$). The transmission function at the SLM plane is given by

$$H(x, y) = \frac{1}{2}\{\beta(x, y) + \alpha(x, y) \cos[\Theta(x, y) - 2\pi(u_0 x + v_0 y)]\},$$  \hspace{1cm} (8.4)

where, $\alpha(x, y)$ and $\Theta(x, y)$ represent the amplitude and phase of $\Psi(\rho, \phi, z = 0, t)$, respectively, and $\beta(x, y)$ is a bias function chosen as a soft envelope for the amplitude $\alpha(x, y)$ according to $\beta(x, y) = [1+\alpha(x, y)^2]/2$ [120]. The pattern was interfered with a plane wave $\exp[2\pi i(u_0 x + v_0 y)]$. This shifts the encoded pattern
off-axis (in the Fourier plane) to the spatial frequencies \((u_0, v_0)\); thus making it easier to filter out the shifted pattern from the undesired on-axis noise by simply using an iris. In our experiment, \(u_0\) and \(v_0\) were set to \(1/(4\Delta x)\); where \(\Delta x\) is the SLM pixel pitch \((\Delta x = 36 \mu m)\).

Given the twisted nematic nature of our SLM (Holoeye LC2012 Amplitude SLM), which makes it operate with maximum efficiency on linearly polarized light, we used a polarizer-analyzer combination oriented at \((0^\circ)\) and \((90^\circ)\) with respect to the SLM axis, as depicted in Fig. 8.2. The CGH was encoded onto a 532 nm expanded and collimated laser beam with a Guassian profile, and the generated waveform was then imaged using a 4-f optical system and an iris which filters the first diffraction order while blocking the undesired diffraction orders and on-axis noise. The generated waveform was then transmitted into a glass tank at normal incidence. The tank was placed in the focal plane of the imaging system \((z = 0\) plane\), as shown in Fig. 8.2. Finally, the generated waveform was recorded inside the unknown fluid using a CCD camera at a fixed detection plane along \(z\). To establish the wide dynamic range capability of our sensing mechanism, we considered the following fluids: a) Deionized water \((n = 1.335)\), b) Vegetable oil \((n = 1.475)\), and c) Cinnamon Oil \((n = 1.57)\).

\[
F_\ell(z) = \begin{cases} 
F_1 = F_{-1} = 1 & 0 \text{ cm} \leq z \leq 50 \text{ cm}, \\
F_1 = F_{-1} = 0 & \text{elsewhere.}
\end{cases}
\]  

(8.5)

8.6 Experimental Results

To examine the performance of the proposed sensing scheme, we generated and tested multiple scenarios in which the rotating beam structures are composed of the superposition of OAM modes \(\psi_{-1}\) and \(\psi_{1}\), that is \(\Psi(\rho, \phi, z = 0, t) = \psi_1 + \psi_{-1}\). The function \(F_\ell(z)\) was chosen such that

Figure 8.2: Experimental setup used to generate and detect the rotating structured beam patterns for sensing. A computer generated hologram was sent onto a transmissive SLM that encodes the desired pattern on a 532 nm collimated laser beam. The SLM was sandwiched in a polarizer-analyzer configuration as it operates with maximum efficiency on vertically polarized incident light. The generated pattern was then filtered and imaged using a 4-f imaging system. Finally, the beam evolution was recorded inside the fluid using a CCD camera on a translation stage, where the \(z = 0\) plane lays to the right of Lens2 (at its focal plane). Figure reproduced from Dorrah et al. [114].
Using Eq. (8.5) and the fact that $\Delta \tilde{Q} = \Delta Q/n$, it follows that the differential angular orientation of the beam in a given medium with an unknown index of refraction ($n$) with respect to its orientation in air ($n = 1$), denoted as $\theta$ (for $\ell = 1, -1$) is given by

$$\theta = \Phi_{1,-1}(\Delta Q) - \Phi_{1,-1}(\Delta \tilde{Q}) = \frac{\Delta Q(1 - 1/n)z}{2}. \quad (8.6)$$

Figure 8.3: Measured and simulated transverse beam profiles of the rotating beams in air, water, vegetable oil, and cinnamon oil at propagation distance $z = 22$ cm for different cases of $\Delta Q$. a) $\Delta Q = 23.62 \text{ m}^{-1}$, b) $\Delta Q = 47.24 \text{ m}^{-1}$, c) $\Delta Q = 53.14 \text{ m}^{-1}$, d) $\Delta Q = 59.05 \text{ m}^{-1}$. The blue arrows represent the orientation of the rotating light structures. The corresponding theoretical and measured quantities ($\theta$ and $n$), are listed for each case. Figure reproduced from Dorrah et al. [114].

The waveforms were transmitted in air as a reference ($n = 1$), in addition to water ($n = 1.335$), vegetable oil ($n = 1.475$), and cinnamon oil ($n = 1.57$), and then detected at the propagation distance $z = 22$ cm within the medium. The choice of these media was made to offer a wide range of RIs. Figure 8.3 illustrates the measured and calculated variations of $\theta$ as a function of the RI of the medium. In each medium, the differential angle $\theta$ represents the orientation of the rotating light petal in that medium with respect to its orientation in air. The measured values of $\theta$ are obtained after identifying the centroids of the detected petal-like structures based on locating the local maxima. From Eq. (8.6), it can be observed that the sensitivity of this scheme ($\partial \theta/\partial n$) is directly proportional to $\Delta Q$. This is confirmed in Fig. 8.3(a-d), which correspond to cases in which $\Delta Q$ is equal to 23.62, 47.24, 53.14, and...
59.05 m\(^{-1}\), respectively. In each case, the unknown index of refraction \(n\) is evaluated from

\[
n = \frac{1}{1 - 2\theta/(z\Delta Q)},
\]

(8.7)

Figure 8.4: Performance of the proposed sensing scheme at \(z = 22\) cm. a) Differential orientation \(\theta\) as a function of the RI for different values of \(\Delta Q\) at \(z = 22\) cm, \(\Delta Q\) has the units of m\(^{-1}\). The vertical dash lines correspond to the refractive indices of Water (1.335), Vegetable Oil (1.475), and Cinnamon Oil (1.57). b) Measured refractive indices as a function of \(\Delta Q\). The markers represent the measured RI values, and the dashed lines represent the nominal values from the vendor. The average values were 1.331, 1.476, and 1.570 for water, veg. oil, and cinn. oil, respectively. c) Resolution of the proposed scheme as a function of \(\Delta Q\) under different scenarios of the standard deviation \(\sigma_\theta\) in estimating \(\theta\). d) Sensitivity and Dynamic range of the proposed scheme as a function of \(\Delta Q\) at three different detection planes; \(z = 18, 22,\) and 27 cm. Figure reproduced from Dorrah et al. [114].

Figure 8.4(a) displays the differential orientation angle \(\theta\) as a function of the RI for various values of \(\Delta Q\). The markers on the figure correspond to the measured \(\theta\) in water (blue), vegetable oil (green), and cinnamon oil (red). Here, the detection plane is still at \(z = 22\) cm. The slope of each curve corresponds to the sensitivity \((\partial \theta/\partial n)\). From the figure, it is evident that cases with larger \(\Delta Q\) possess larger slopes and thus exhibit higher sensitivity, in agreement with Eq. (8.6). Further, the measured refractive indices for each \(\Delta Q\) scenario, evaluated from Eq. (8.7), are shown in Fig. 8.4(b). Each marker represent an average of at least 5 different measurements. The measured RI values averaged over all \(\Delta Q\) scenarios in each medium are also listed in the figure, and the nominal values (from the vendor) are shown in
the dashed lines. The average measured RIs are: 1.331, 1.476, and 1.570. The standard deviation in estimating the angular orientation $\Delta \theta$ in water, vegetable oil, and cinnamon oil for water, vegetable oil, and cinnamon oil, respectively. The corresponding standard deviations in estimating $\theta$ are: $\sigma_\theta = 1.9^\circ$, 1.474$^\circ$, and 0.644$^\circ$, respectively.

Figure 8.4(c) shows the resolution of the proposed sensing mechanism as a function of $\Delta Q$. Resolution of a sensor is the smallest change in the measurand (here, RI) that can be detected. Resolution ($\sigma_r$) is inversely proportional to sensor’s sensitivity and directly proportional to the standard deviation of the output variable ($\sigma_\theta$), according to: $\sigma_r = \sigma_\theta / (\partial \theta / \partial n)$. For the current setup, sensor’s resolution is on the order of $10^{-3}$ RIU, corresponding to sensitivity of $270^\circ$/RIU. As Fig. 8.4(c) indicates, by further improving the optical setup (e.g., a better camera and less noisy laser) and hence reducing $\sigma_\theta$, or by increasing the value of $\Delta Q$, the sensor’s resolution can be further improved (reaching $\simeq 10^{-5}$ RIU). We also note that a larger separation in $\Delta Q$ typically implies that longitudinal wavenumbers $k^\ell,m_z$ are also more widely separated in the spatial frequency domain. Through the consistency relation, $k^\ell,m_\rho = \sqrt{k^2 - (k^\ell,m_z)^2}$, a larger separation in $k^\ell,m_z$ implies a wider span in the spatial frequency $k^\ell,m_\rho$, and hence a more localized beam (if $k^\ell,m_\rho$ acquire large values).

The ability to generate highly localized structured light depends on the SLM’s pixel pitch $\Delta x$; where $k^\ell,m_\rho$ and $\Delta Q$ are inversely proportional to $\Delta x$. Using commercially available SLMs with pixel pitch of approximately 4 $\mu$m (the SLM used in our experiment had a pixel pitch of 36 $\mu$m), it is possible to achieve $\Delta Q \approx 600$ m$^{-1}$. Therefore, by using currently available SLM technology, it is possible to achieve resolutions in the order of $10^{-5}$ RIU. This can be further improved by an order of magnitude ($\sigma_r \approx 10^{-6}$) when using metasurfaces or phase masks to generate beams with higher values of $\Delta Q$.

From Eq. (8.6), it is clear that, in addition to increasing $\Delta Q$, the sensitivity of our scheme depends on the distance $z$ (i.e., increasing $z$ yields higher sensitivity). This is depicted in the green curves of Fig. 8.4(d). In principle, the sensitivity of our scheme can exceed $2600^\circ$/RIU at the plane of detection $z = 27$ cm for $\Delta Q = 600$ m$^{-1}$. It should be noted that, in this case, higher sensitivity is achieved at the expense of reducing the sensor’s dynamic range. For the data depicted in Fig. 8.3, the upper bound on the dynamic range is dictated by value of $n$ associated with $\theta = 180^\circ$, after which the rotating pattern reproduces itself (becomes degenerate). The dynamic range is evaluated by solving $1/[1 - 2\pi/(z\Delta Q)]$, and is plotted as blue curves in Fig. 8.4(d) as a function of $\Delta Q$ at three different detection planes along $z$. It is observed that, in contrast to the sensitivity, the dynamic range becomes smaller at larger values of $\Delta Q$ (and distance $z$). Therefore, there is a trade-off between simultaneously achieving high sensitivity (and high resolution), while maintaining a large dynamic range for sensing. We will show later on how higher OAM modes can be exploited to address this problem.

In all the previous cases, we have presented scenarios in which the rotating beam structures were detected at a fixed plane, $z = 22$ cm. In the next section, we will show the effect of the detection plane $z$ on sensitivity and resolution of the proposed scheme.
8.7 Improving Sensitivity by Increasing the Interaction Length

In addition to the dependence on the spatial frequency separation $\Delta Q$, the sensitivity of the proposed sensing scheme also depends on the location of the detection plane along $z$. In the previous cases, we considered a detection plane that was fixed at $z = 22$ cm. Here, we show that higher sensitivity can be achieved by setting the detection plane at further distances.

![Graph](image)

Figure 8.5: a) Differential orientation ($\theta$) as a function of the RI for different values of $\Delta Q$ at $z = 27$ cm, $\Delta Q$ has the units of m$^{-1}$. The vertical dash lines correspond to the refractive indices of Water ($1.335$), Vegetable Oil ($1.475$), and Cinnamon Oil ($1.57$). A degeneracy in detection appears at $n = 1.489$ for $\Delta Q = 70.86$ m$^{-1}$ (red dashed curve). b) Measured refractive indices as a function of $\Delta Q$ for different values of $\nu$ in water (blue), vegetable oil (green), and cinnamon oil (red). The markers correspond to the measured RIs and the dashed lines represent the nominal values from the vendor. The averaged measured values were $1.342$, $1.480$, and $1.576$ for water, vegetable oil, and cinnamon oil, respectively. c) Resolution of the proposed scheme at $z = 27$ cm as a function of $\Delta Q$, under different scenarios of $\sigma_\theta$. d) Accuracy of the proposed sensor at $z = 27$ cm as a function of $\Delta Q$ under different scenarios of the mean absolute error in $\theta$ ($\nu_\theta$). Figure reproduced from Dorrah et al. [114].

Figure 8.5(a) depicts the sensor response ($\theta$) as a function of RI when the detection plane is set at $z = 27$ cm. The markers correspond to the measured $\theta$ in water (blue), vegetable oil (green), and cinnamon oil (red). It is observed that, at $z = 27$ cm, the differential angle $\theta$ acquires larger values for the same RI change. Hence, the sensitivity of the scheme can be improved by setting the detection plane at longer distance along $z$, in agreement with Eq. (8.6). The measured RIs at $z = 27$ cm, for each $\Delta Q$ scenario, evaluated from Eq. (8.7), are plotted in Fig. 8.5(b). Each marker represents an average of at
least 5 different measurements. The measured RI values averaged over all $\Delta Q$ scenarios in each medium are listed in the figure, denoted by $\tilde{n}$, whereas the nominal values (from the vendor) are shown in the dashed lines. The measured RI at $z = 27$ cm — averaged over all scenarios of $\Delta Q$ — were: 1.342, 1.480, and 1.576 for water, vegetable oil, and cinnamon oil. The standard deviations in estimating the angular orientation $\theta$ in water, vegetable oil, and cinnamon oil were: $\sigma_\theta = 1.24^\circ$, $1.91^\circ$, and $1.17^\circ$, respectively. Again, this uncertainty represents the main limiting factor for the resolution of the proposed scheme.

Figure 8.5(c) depicts the resolution as a function of $\Delta Q$ for different scenarios of $\sigma_\theta$. Resolution is calculated according to: $\sigma_r = \sigma_\theta/(\partial \theta/\partial n)$, thus obtaining the minimum detectable change in the index of refraction ($n$). With $\Delta Q = 60 \text{ m}^{-1}$ and steps of $\sigma_\theta = 2^\circ$, the resolution is $7.5 \times 10^{-3}$ RIU, which is better than the case of $z = 22$ cm; whereas with $\sigma_\theta = 0.1^\circ$, the resolution can reach $3.7 \times 10^{-4}$. Furthermore, the resolution can be improved by an order of magnitude by generating more localized beams, in which $\Delta Q$ is ten times larger, as shown in Fig. 8.5(c). For example, the proposed scheme can be utilized to measure RI with a resolution in the order of $10^{-5}$, when $\Delta Q$ is set to $600 \text{ m}^{-1}$ — which is feasible using SLMs with 4-$\mu$m pixel pitch.

Accuracy is another important metric in RI sensing. In our proposed scheme, accuracy is determined by the reliable detection of the centroids’ maxima in the rotating petal-like structures. Hence, the error in determining $\theta$ represents the main limiting factor for the accuracy of the proposed scheme. By taking the derivative of Eq. (8.7) with respect to $\theta$, the accuracy can be expressed as

$$\epsilon_n = \frac{2z\Delta Q}{[z\Delta Q - 2\theta]^2} \nu_\theta$$

(8.8)

where $\nu_\theta$ denotes the mean absolute error in $\theta$. In our experiments, $\nu_\theta$ was $\sim 2^\circ$ at $z = 27$ cm. Fig. 4.1(d) depicts the sensor’s accuracy as a function of $\Delta Q$ under different scenarios of $\nu_\theta$. Similar to the resolution and sensitivity, the sensor’s accuracy can be dramatically improved by using beams with larger values of $\Delta Q$. We also note that because beams with larger values of $\Delta Q$ are more localized, they lead to lower values of $\nu_\theta$. Equation (8.8) also characterizes the sensor’s precision (i.e., the repeatability of RI measurements over time). This is readily performed by replacing $\nu_\theta$ with the standard deviation ($\sigma_\theta$). A more detailed analysis on the sensor’s tolerance to the deviations in $\theta$, $z$, and $\Delta Q$ can be found in Appendix B.

### 8.8 Expanding the Dynamic Range of RI Sensing

In the previous section, we showed that sensitivity and resolution can be improved by extending the length over which the beam interacts with the medium. The improvement is achieved at the expense of limiting the dynamic range of the scheme. For instance, consider Fig. 8.6(d) which depicts the case when the rotating light pattern, with $\Delta Q = 70.86 \text{ m}^{-1}$, is detected at $z = 27$ cm. It is observed that, in the case of vegetable oil, $\theta = 175^\circ$. This implies that the rotating light pattern is very close to acquiring
a degenerate orientation when compared to the beam propagating in the air. Furthermore, in the case of cinnamon oil, $\theta$ exceeds $180^\circ$; hence, mapping the orientation to the index of refraction is no longer unique. In other words, a measured value of $\theta = 203^\circ$ is degenerate with $\theta = (203^\circ - 180^\circ) = 23^\circ$. Such degeneracy can also be verified from Fig. 8.5(a) in the dashed red curve. It is thus not clear, in this case, if the RI value should be mapped to 1.59 or 1.044 (corresponding to the measured orientations $\theta = 203^\circ$ and $\theta = 23^\circ$, respectively). As previously mentioned, the dynamic range is constrained when $\theta$ reaches $180^\circ$, where it can readily be calculated from $1/|1 - 2\pi/(z\Delta Q)|$. This suggests that, at $\Delta Q = 70.86 \text{ m}^{-1}$ and $z = 27 \text{ cm}$, the dynamic range of this sensing scheme only spans the range from $n = 1$ to $n \sim 1.489$.

Consequently, there is a clear trade-off between achieving high sensitivity and maintaining a wide dynamic range for RI measurement. However, by incorporating larger topological charges ($\ell > 1$) in the superposition of Eq. (8.1), it becomes possible to mitigate this trade-off. OAM modes with larger $\ell$ have $\ell$-helices in their phase-front. When these OAM modes — with opposite signs of $\ell$ — are superimposed, the rotating pattern will possess more than two rotating petals as a result of introducing additional phase singularities in the azimuthal direction ($\phi$) [99]. In this case, it is possible to generate rotating beams that can change their number of petals (evolving from two to three petals, for instance) as the optical length (index of refraction) is increased. This, in turn, can be utilized to break the degeneracy of the two-petal patterns and extend the dynamic range of the sensor while maintaining high sensitivity and resolution.

For example, consider the waveform $\Psi(\rho, \phi, z = 0, t) = \psi_{-1} + \psi_1 + \psi_2$, in which the function $F_\ell(z)$ is defined as

$$
F_\ell(z) = \begin{cases} 
F_{-1} = 1 & 0 \text{ cm} \leq z \leq 50 \text{ cm}, \\
F_2 = 1.5 & 0 \text{ cm} \leq z \leq 18 \text{ cm}, \\
F_1 = 1 & 18 \text{ cm} \leq z \leq 50 \text{ cm}, \\
F_{-1} = F_1 = F_2 = 0 & \text{elsewhere.}
\end{cases}
$$

(8.9)

Here, the beam is composed of OAM modes with $\ell = 1, -1,$ and 2. As such, the propagating beam (in air) is designed to possess three petals over the range $0 \text{ cm} \leq z \leq 18 \text{ cm}$ before it reduces to two petals over the range $18 \text{ cm} \leq z \leq 50 \text{ cm}$. Note that this behavior assumes the beam propagation in the air.

The ability to control the topological charge of structured light along the beam’s axis is depicted in Figure 8.6(b) which illustrates the evolution of the beam from 3 petals to 2 petals with propagation. This is made possible via the careful constructive and destructive interference among the 9 co-propagating Bessel beams in the superposition. By the virtue of Eq. (8.9), only OAM modes with non-zero intensity
Figure 8.6: a) Schematic diagram showing the evolution of the generated beam from 3 to 2 petals with propagation in the different media. b) Measured and simulated angular orientation of the rotating pattern at propagation distance \( z = 27 \text{ cm} \). The rotating beam is designed to evolve into 3 petals (instead of 2, as in case a) when the intensity profile is degenerate with the case of air, thus extending the dynamic range of the sensing scheme. Figure reproduced from Dorrah et al. [114].

will contribute to the beam center, while the contributions of the other OAM modes are distributed over the outer rings of the beam. Those contributions stored in the outer rings can then be restored to the beam’s center at the prescribed locations defined by Eq. (8.9). In principle, one can discern this effect as a practical manifestation of the Hilbert’s hotel paradox concept [117, 116, 101]. As the beam is allowed to propagate in different media, the distance over which the beam carries 2 or 3 petals changes depending on the refractive index (from the relation \( \Delta \tilde{Q} = \Delta Q/n \)). Figure 8.6(c) represents the measured and simulated transverse intensity profiles of the rotating light structure (with \( \Delta Q = 70.86 \text{ m}^{-1} \)) in air, water, vegetable oil, and cinnamon oil — at a propagation distance \( z = 27 \text{ cm} \). Here, once the sensing scheme approaches the limit of its dynamic range (i.e., \( \theta \) approaching 180°), the beam evolves into a new intensity profile with three petals instead of two. In this case, the dynamic range is no longer constrained by the condition \( \theta = 180^\circ \). In other words, there is one-to-one mapping between the beam orientation and the index of refraction of the medium. Accordingly, the dynamic range is now extended from RI = 1 to 2.91 as opposed to the previous case (Fig. 8.6(a)), where the span of dynamic range was from RI = 1 to 1.489. Interestingly, the dynamic range of the beam can be even further extended by incorporating higher OAM modes in the superposition of Eq. (8.7).
8.9 Discussion

In this chapter, we showed how structured light with OAM can be utilized to measure the real part of the RI. Future considerations include the following: First, the proposed scheme can be deployed to estimate the imaginary part of the RI associated with propagation losses. This can be performed by detecting the intensity level of the measured images. By quantifying the amount of attenuation in the intensity pattern with respect to the reference medium (air) at a given detection distance \( z \), the imaginary part of the RI can then be estimated by applying Beer’s law \([191]\). Since the developed waveform allows control over the longitudinal intensity profile, it is possible to generate rotating light structures that are immune to the propagation losses effects, as discussed in Chapter 3 and Refs. \([90, 92]\). Second, the sensitivity and resolution can be dramatically enhanced by deploying OAM modes with faster rotation rates. This is readily achieved in various ways, such as using accelerated OAM modes \([137]\) or generating rotating beams with larger values of \( \Delta Q \), as explained earlier. Another approach is to replace the second lens in the 4-f imaging system of Fig. 8.2 by a lens with much shorter focal length. This scheme compresses the beam’s longitudinal extent and thus acts as a photonic gear that amplifies the beam’s rotation rate along the \( z \)-direction and, hence, boosts the sensitivity and resolution of the entire scheme. Third, similar to OAM, polarization can be exploited to extend the dynamic range of sensing. This can be realized by utilizing beams with a polarization state that is dependent on the optical path following the concept introduced in Chapter 5 and Refs. \([138, 139, 140]\). Finally, it is also interesting to extend the current sensor to characterize the RI of non-homogeneous media. This can be the subject of future work.

8.10 Summary

We proposed and experimentally demonstrated a novel tunable RI sensing scheme based on the OAM of light. By adding OAM modes with opposite helicities, we created petal-like light structures that rotate along their optical length. The angular orientation of the rotating petal-like structure depends on the RI of the medium. The rotation in the transverse intensity profile can be measured easily with respect to a reference (air) using a CCD camera, and the differential measurement can then be utilized to accurately identify the RI change. Hence, the proposed sensing scheme is based on a simple setup that only requires an SLM for beam generation and a CCD camera for detection. The sensitivity is only limited by the available SLM bandwidth and can in principle exceed \( 2700^{\circ}/\text{RIU} \) with a resolution of \( 10^{-5} \) RIU using SLMs with a 4 \( \mu \text{m} \) pixel pitch, which are widely available commercially. We also proposed a novel mechanism to expand the dynamic range of sensing by incorporating higher OAM modes in the transmitted beam, thus scanning the range from \( \text{RI} = 1 \) to over 2.91. The programmability of the SLM allows the sensitivity, resolution, and dynamic range of the sensor to be reconfigured on demand, thus providing a tunable sensing mechanism that can provide coarse and fine RI measurements in real time. We thus envision this method to open new directions in refractometry and remote sensing.
Chapter 9

Discussion

Recent advances in laser beam shaping tools have redefined the domain of Structured Light thus opening new venues of innovation in light science and applications. The goal of this dissertation was to develop a systematic approach to design and experimentally generate a broad class of customized Structured Light beam patterns in which almost all degrees-of-freedom of light are readily controlled along the axis of propagation. To do so, we expanded on the Frozen Wave method. Such method provides a systematic approach to shape the longitudinal intensity profile of non-diffracting waveforms via a judicious superposition of co-propagating Bessel beams with different spatial frequencies. With suitable weighting factors for each Bessel beam in the superposition, the intensity profile of the envelope can be modulated via controlled interference. While the concept of the Frozen Wave has been introduced in 2004 [70, 71] to control the longitudinal intensity profile, controlling other degrees-of-freedom of light along the beam’s axis has been scantly investigated before this work — despite the potential this may merit. In this regard, we expanded on the Frozen Wave formulation in order to control other degrees-of-freedom of light along the propagation axis including: light’s intensity, orbital angular momentum (OAM), polarization, wavelength, and even its propagation trajectory. This chapter is organized as follows: in section 9.1, we highlight the main contributions of this thesis in the context of relevant efforts in the literature. Afterwards, we present the unified approach that governs all classes of longitudinally structured light developed in this thesis. Notably, throughout this thesis, longitudinal shaping of each degree-of-freedom has been performed via some modification to the original Frozen Wave theory. Each longitudinally structured beam pattern presented in this thesis can thus be regarded as a special case of such generalized method. In Section 9.2, we provide such unified method — the Generalized Frozen Wave method.
9.1 Thesis Contributions and Significance

1. First Experimental Demonstration of Attenuation-Resistant Frozen Waves Inside an Absorbing Fluid (Reported in Chapter 3 and Refs. [92, 93, 94, 95]).

Chapter 3 presented the first experimental demonstration of a Frozen Wave that can maintain the intensity profile of its modulated envelope even in the presence of medium absorption. Before this work, few efforts have tried to mitigate the attenuation suffered by non-diffracting beams in absorbing media. This has been attempted by deploying an exponential intensity axicon (exicon) to generate constant intensity Bessel beams in absorbing dye solution [73], or to generate loss-proof self-accelerating beams in the presence of two photon absorption [96]. More recently, shape preserving surface-plasmon polariton beams have been demonstrated in the presence of plasmon losses [97]. Although these efforts are promising steps to alleviate beam attenuation in lossy media, the ability to freely shape the longitudinal intensity profile of the beam (for example, turning it on and off with propagation distance) in an absorbing medium has not been demonstrated before the work presented in this thesis. Generating attenuation-compensated non-diffracting beams can solve many challenges in light-sheet microscopy inside thick biological specimens [74].

2. Controlling the Topological Charge and OAM of Non-Diffracting Beams with Propagation (Reported in Chapter 4 and Refs. [99, 100, 101]).

Chapter 4 introduced unusual scenarios in which optical vortices can be engineered to undergo non-trivial, yet fully controlled, topological transitions in free space. In particular, we investigated a paradoxical scenario in which a beam carrying (OAM) can be designed to locally change the sign and/or the magnitude of its topological charge under unperturbed propagation in air; without violating any OAM conservation laws. We started by providing the theoretical approach for designing, creating, and analyzing such beams and presented two case studies in which the beam was engineered to change the sign and magnitude of its topological charge with propagation. We measured the OAM along the beam’s axis and showed that it is possible to control the OAM within the center of the beam (locally) with propagation. In this process, we studied the topological transitions at the boundaries and showed that while OAM can vary locally, the global OAM is always conserved, when measured across the entire cross section of the beam. We also described the physical mechanism via which topological transition occurs; reporting cases that manifest creation, movement, and annihilation of singularities across the beam; transforming its topology without disturbing its net topological charge. Our study thus revealed new dynamics that advanced our prior knowledge of the field of singular optics and topological transitions. Before this work, non-trivial topological deformations have been deliberately realized under very special conditions: originally by interfering vortex modes with Gaussian beams [106], by realizing charge flipping induced in a non-linear medium [107, 108], and in noncanonical vortices generated by an
astigmatic optical setup [109, 110]. Following our work [192, 99, 100, 101], other research groups reported some cases in which the topological charge of non-diffracting beams can be designed to vary along the propagation direction [111, 112]. The ability to control both the sign and magnitude of the topological charge of non-diffracting beams under unguided propagation in air has rarely been investigated before our work. Such control offers many new possibilities in refractive index sensing as we demonstrated in Chapter 8 and in dense data communications as reported in Ref. [76].

3. Controlling the Polarization State and OAM of Non-Diffracting Beams Simultaneously Inside a Lossy Fluid (Reported in Chapter 5 and Refs. [138, 139, 140]).

Chapter 5 presented the first experimental demonstration of a non-diffracting waveform that can be engineered to change its polarization and topological charge independently while overcoming the propagation losses inside an absorbing medium. We started by generating linearly polarized beams in which the SoP is altered from vertical to horizontal and back to vertical, while maintaining the same intensity level. Then we present scenarios in which the intensity level can be shaped while the polarization was simultaneously altered from vertical to horizontal polarization. Afterwards, more complicated polarization states have been generated; for instance, a case in which a linearly polarized beam in the horizontal direction becomes radially polarized and eventually evolves to linear polarization in the vertical direction. Finally, we demonstrate complex beam structures in which multiple degrees of freedom can be controlled simultaneously along the beam’s axis; in essence, creating an attenuation-resistant and non-diffracting beam that can change both its polarization state and topological charge with propagation, independently and on-demand. Few other studies have reported control over the polarization state of non-diffracting beams along the propagation direction. However, those earlier studies either did not provide a systematic approach to design the desired beams [111, 142, 143, 144, 145] or did not allow special polarization states — such as azimuthal and radial — to be generated [144]. Controlling the polarization state of optical beams along the propagation direction may offer new degrees-of-freedom in our existing light manipulation tool set. In material processing, for instance, this can be utilized to control the shape and size of laser-machined structures by inducing a polarization-dependent ablation effect along the structure [78], which can be deployed in waveguide writing or to create periodic structures. In addition, spatially varying SoP can be used to modulate the absorption profile of polarization-dependent optically pumped medium [64], or to tailor the spectrum profile of quantum emitters over a given volume[77].

4. Shaping the Wavelength and OAM Profile of Non-Diffracting Beams along their Axis of Propagation (Reported in Chapter 6).

Chapter 6 presented, for the first time to the best of our knowledge, a class of non-diffracting
Bessel beams in which two degrees-of-freedom of light, namely its wavelength and topological charge, can be changed independently ("at-will") along the beam’s axis of propagation, when traveling in air and without incorporating any non-linear effects. Such control is qualitatively different from earlier efforts that demonstrated achromatic (broadband) Bessel modes [148]. The ability to generate multi-chromatic Bessel beams, in which the wavelength (linear momentum) can be controlled along the beam’s axis, can unlock many possibilities. For instance, beams with spatially varying wavelengths can be deployed to control the excitation properties of quantum emitters, which are wavelength-dependent [80]. It can also be utilized to shape the spatial profile of optically pumped media [79], or to control the shape and size of laser-machined structures by inducing wavelength-dependent ablation effects [81]. On the other hand, longitudinal control of the topological charge (orbital angular momentum) can offer new degrees-of-freedom in micro-manipulation [53, 75] and remote sensing [114], and can dramatically enhance communication channel capacities [76]. We thus envision that simultaneous longitudinal control of the beam’s wavelength and topological charge can lead to new advances in the field of optical sciences and applications of light.

5. **Experimental Generation of Curved Frozen Waves (Reported in Chapter 7 and Refs. [149, 150])**

Chapter 7 presented a class of non-diffracting and self-healing curved beams, denoted as Curved Frozen Waves (CFW), whose central intensity spot can be designed to exhibit a non-periodic off-axis spiraling or snake-like behavior over a predefined space region. The generated beams exhibit a spiraling behavior in which the beam’s sense of rotation can be reversed and the longitudinal intensity profile can be controlled, on demand. Furthermore, the beam can be structured to evolve from straight to curved snake-like, and then retain its original straight path, at-will. Such control has been realized while conserving the energy and global momentum of the beam. Few earlier studies have attempted to generate Bessel-like beams whose intensity follows curved (off-axis) trajectories [157] such as snake-like [158] and/or spiral paths [159, 160, 161, 162]. However, when such a behavior was sought, the generated beams typically exhibited periodic off-axis profile over the entire beam range; with no ability to simultaneously control the longitudinal behavior of Bessel beams while propagating along a curved trajectory (for example, changing the sense of spiraling, turning the beam intensity on and off, evolving from a straight to curved path at-will). The additional capabilities contributed in this thesis can be beneficial for many applications in optical trapping and material processing.

6. **First Demonstration of Wide-Range Tunable Refractometer Based on OAM of Light (Reported in Chapter 8 and Refs. [114, 163])**

In Chapter 8 we proposed and experimentally demonstrated a new technique for Refractive Index
(RI) sensing based on Orbital Angular Momentum (OAM) of longitudinally structured light. We exploited the OAM of light to generate non-diffracting and rotating light structures in which the angular orientation is sensitive to the refractive index of the medium. By interfering two spatial-frequency combs of OAM modes, we provided a tool that can map the shift in the spatial-frequencies of an optical beam in a given medium, as a result of the refractive index change, into a rotation in the transverse intensity profile, which can then be easily detected by a CCD camera. In principle, the sensitivity of this scheme can exceed \(3000^\circ/\text{RIU}\), with a resolution of \(\sim 10^{-5}\ \text{RIU}\). Furthermore, in order to expand the dynamic range of sensing, we generated OAM modes that not only change their angular orientation but that can also modify their mode profile, as a result of the refractive index change. In essence, we presented a scenario in which the OAM mode changes both its angular orientation and mode profile, evolving from two rotating petals to three rotating petals, as a result of changing the refractive index of the medium. This dramatically extends the dynamic range of RI measurement, covering from 1 to over 2.9 thus outperforming many RI sensing schemes in the literature. A powerful advantage of our method lies in using a programmable Spatial Light Modulator (SLM) for the beam generation. This provides dynamic tunability over the sensitivity and resolution of the proposed scheme. With such tunability, a coarse RI measurement can first be performed over a wide dynamic range. Then the SLM can be refreshed to perform a finer measurement with much higher resolution. As such, the proposed scheme addresses the challenge of simultaneously achieving high sensitivity, high resolution, while covering a wide dynamic range of RI measurement, which are typical trade-offs in all RI sensing schemes.

9.2 Generalized Frozen Wave Method

The original Frozen Wave method introduced in Refs. [70, 71] relied on the superposition of co-propagating Bessel beams of the same wavelength (and order) but with different wavenumbers to modulate the longitudinal intensity profile. As such, each Bessel beam was weighted by a different complex coefficient. Chapter 3 introduced the concept of attenuation-resistant Frozen Waves inside absorbing fluids. Loss compensation has been carried out in Eq. (3.1) by accounting for the medium losses when calculating the coefficients of each Bessel beam in the superposition. This led to creating Frozen Waves with growing intensity thus counteracting the exponential loss profile of the medium. Afterwards, in Chapter 4, higher order Bessel beams were incorporated in the Frozen Wave superposition to control the topological charge of the beam. This has been carried out in Eq. (4.1) by adding another summation over the order of the Bessel beams in the superposition. Similarly, Curved Frozen Waves in Chapter 7 have been engineered by adding the summation over the topological charges in Eq. (7.1). Chapter 5, on the other hand, expanded the scalar Frozen Wave representation to the \(x\) and \(y\) components of the waveform in Eq. (5.8) and introduced the function \(G_{uv}^{n}(\phi)\) to control the State of Polarization along the beam’s axis. Finally,
Chapter 6 extended the Frozen Wave superposition to include Bessel beams of different wavelengths to achieve longitudinal wavelength division multiplexing of chromatic non-diffracting beams. This has been carried out by adding another summation over the wavelengths of the Bessel beams in the superposition of Eq. (6.1). As such, every chapter has considered a special case of a more generalized method through which all the aforementioned degrees-of-freedom of light can be longitudinally controlled (simultaneously and independently). We refer to this as the Generalized Frozen Wave Method in which the longitudinally structured waveform is given by the following unified closed form expression

$$E_{u}(\rho, \phi, z, t)|_{u=x,y} = \sum_{\lambda=\lambda_1}^{\lambda_{\infty}} e^{-i \frac{2\pi}{\lambda_0} t} \sum_{\ell=-\infty}^{\infty} \sum_{v=1}^{M} \sum_{m=-N}^{N} A_{\lambda,\ell,v,m} J_{\ell}(k_{\lambda,\ell,v,m} \rho) e^{i k_{\lambda,\ell,v,m} z}$$  \hspace{1cm} (9.1)$$

Equation (9.1) represents the Generalized Frozen Wave Method in which almost any degree-of-freedom of light can be controlled along the propagation axis of the beam. Here, $G_{\lambda,\ell,v}(\phi)|_{u=x,y}$ defines the azimuthal dependence over each subregion of the Frozen Wave and the superscript $u$ denotes the $x$ and $y$ components of the waveform. Additionally, the coefficients $A_{\lambda,\ell,v,m}$ are obtained by solving the integral

$$A_{\lambda,\ell,v,m} = \frac{1}{L} \int_{0}^{L} F_{\lambda,\ell,v}(z) e^{-(i \frac{2\pi}{\lambda_0} m - Im\{k_{\lambda,\ell,v,m=0}\}) z} dz,$$  \hspace{1cm} (9.2)$$

where $F_{\lambda,\ell,v}(z)$ defines the predetermined longitudinal intensity profile over each subregion of the Frozen Wave. Equation (9.1) together with Eq. (9.2) can be used to control various properties of longitudinally structured light beams. This includes control over the longitudinal intensity profile (value and pattern), OAM, polarization, and wavelength. Hence, Eqs. (3.1,4.1,5.8,6.1,7.1) can all be interpreted as special cases of the unified expression given by Eq. (9.1) — the Generalized Frozen Wave Method.
Chapter 10

Summary and Outlook

10.1 Thesis Summary

This thesis presented a systematic method through which different degrees-of-freedom of Structured Light — namely its intensity profile, orbital angular momentum, polarization state, and wavelength — can all be independently controlled along the beam’s axis of propagation. In this process, new and intriguing classes of non-diffracting beams have been designed, modeled, experimentally generated, and analyzed. For instance, with regard to longitudinal intensity control, several patterns of shape-preserving beams have been reported where the envelope was engineered to carry an arbitrary intensity profile while overcoming propagation losses inside absorbing fluids (as opposed to ordinary beams that suffer from an exponential decay), thus tackling a fundamental limitation of propagation in lossy media, as discussed in Chapter 3. Chapter 4, on the other hand, established a systematic methodology to engineer vortex beams so that they undergo non-trivial topological transitions without violating momentum conservation or any physical laws. In this process, several interesting light structures were reported; demonstrating how a rotating beam pattern can reverse its sense of rotation (sign of topological charge) or modify the amount of its phase-front helicity (magnitude of topological charge) as it propagates. Polarization is another fundamental degree-of-freedom of light that has been controlled along the axial direction of non-diffracting beams, as explained in Chapter 5. In addition to controlling polarization, longitudinal control over the wavelength has been demonstrated in Chapter 6; unlocking new possibilities in light-matter interactions. Later on, in Chapter 7, an intriguing class of non-diffracting beams that can follow a curved snake-like or spiral trajectories under free space propagation has been reported; offering new degrees-of-freedom in materials processing. In Chapter 8, a class of non-diffracting light structures carrying orbital angular momentum has been exploited as a possible solution in refractive index sensing; providing a compromise between yielding high resolution measurements while covering a wide dynamic sensing range. Lastly, Chapter 9 outlined the main contributions of this work in the context of existing
literature and presented the Generalized Frozen Wave Method that governs all structured light patterns reported throughout this thesis. This provides a unified approach within which advanced classes of non-diffracting beams can be designed and experimentally generated, thus enriching our existing optics toolkit, and addressing the emerging need for advanced Structured Light.

While handful of questions have been addressed in the current work, new questions were also raised; hence, opening new directions of research. In the following, we highlight some of the potential areas that can be the subject of future investigation.

10.2 Future Work

10.2.1 Longitudinally Structured Light in Arbitrary Media

This thesis has primarily dealt with Structured Light inside a linear isotropic homogeneous medium of propagation. In some cases, we generated Structured Light patterns in absorbing media in which the loss profile of the absorbing fluids followed the Beer-Lambert’s law. In many practical scenarios, however, it is important to generate customized patterns of Structured Light inside more complicated media in which the absorption profile is non-linear, or where the waveform suffers from scattering, or non-homogeneity. This is a very promising area with many potential implications for imaging [193] and can be the subject of future research.

10.2.2 Engineering Structured Light Over Micro-Metric Regions (Non-Paraxial Regime)

The theoretical formulations presented in this thesis were presented under the scalar and paraxial regime. Following this assumption, the $z$-component of the developed waveform has been neglected for simplicity. In this regard, various scenarios of Structured Light have been designed, modeled, and generated over relatively long space regions (of few centimeters). However, in many real-life applications, such as optical trapping and micro-manipulation, it is desirable to shape the optical beam over micrometer space regions. This can be done more efficiently through a continuous (rather than discrete) superposition of Bessel beams which can be expressed as a discrete superposition of Mackinnon type beams [194]. However, longitudinal intensity control over small regions, few times larger than the wavelength, will result in highly non-paraxial beams. Accordingly, the previous scalar paraxial approximation — provided by the Frozen Wave method — breaks down and it becomes necessary to formulate the solution in an exact vectorial form that satisfies the Maxwell’s equations. As such, a possible extension of the current work is to design, model, and experimentally generate customized patterns of Structured Light over a micro-metric scale. This can be approached by considering the continuous superposition of Bessel modes as explained in Ref. [195] or by incorporating extraordinary transmission effects using plasmonic
meta-surfaces [196]. Experimental demonstration of such patterns has not been reported yet and could therefore be the subject of future work using high resolution SLMs or fabricated masks.

10.2.3 Enhancing the Performance of the Proposed OAM Sensor

A wide range tunable refractometer that exploits orbital angular momentum of light has been demonstrated in Chapter 7. The sensing scheme was based on transmitting a class of rotating light structures into a fluid with an unknown refractive index. Because the angular velocity of such rotating structures depends on the medium of propagation, it was possible to map the angular orientation of the beam (recorded by a CCD camera) into the unknown index of the medium. As shown in Chapter 7, the sensitivity and resolution of the proposed sensing scheme can be further improved by incorporating high resolution SLM devices to generate more localized light structures. This compresses the beam’s longitudinal extent and amplifies the beam’s rotation rate along the $z$-direction; thus improving the sensitivity and resolution of the entire scheme. Another approach for improving sensitivity and resolution is to replace the 4-$f$ imaging system with a telescope that acts like a photonic gear; scaling down the rotating beam into micro-metric scale; hence improving its sensitivity at the same refractive index change. Further, in the same chapter, we showed how Structured Light with OAM can be utilized to measure the real part of the refractive index. An area of future considerations could be estimating the imaginary part of the refractive index (associated with propagation losses). This can be performed by detecting the intensity level of the measured images. By quantifying the amount of attenuation in the intensity pattern with respect to the reference medium (air) at a given detection distance $z$, the imaginary part of the RI can then be estimated by applying Beer’s law [191]. Additionally, similar to OAM, polarization can be exploited to extend the dynamic range of sensing. This can be realized by utilizing beams with a polarization state that is dependent on the optical path following the concept introduced in Chapter 4 and Refs. [138, 139, 140]. Finally, it can also be useful to extend the current sensing scheme to characterize the RI of non-homogeneous media.

10.2.4 Developing a Generalized Methodology to Create Arbitrary Curved Beams

Chapter 6 of this dissertation demonstrated the possibility of generating non-diffracting beams following curved snake-like and spiral paths in air. This has set the scene for the development of a more generalized methodology through which curved waveforms can be designed to follow arbitrary predefined trajectories. A recently reported approach achieve this via a judicious superposition of weighted plane-waves [197]. Experimental generation of those beams can be interesting in many applications such as optical trapping and atom guiding, to name a few.
10.2.5 Exploiting the Developed Beam Manipulation Tool-Set in Micro-manipulation

Optical trapping and micro-manipulation are two rich areas of research that can benefit from the new beam structures and degrees-of-freedom developed in this dissertation. A possible direction of future multidisciplinary research is to integrate our proposed beam generation modules into a micro-manipulation tool kit and investigate the new possibilities than can be unlocked in physics, medicine, and chemistry using Structured Light.

10.2.6 Longitudinally Structured Light in the Single Photon Limit

Throughout this dissertation, various degrees-of-freedom of Structured Light have been controlled along the propagation direction of non-diffracting beams. Our approach relied on a scalar solution to the wave equation — formulated in the classical regime. In this sense, several intriguing classes of light have been reported. For example cases in which rotating light structure was engineered to reverse the sense of its rotation or change its speed with propagation. Similarly, we reported other scenarios in which the polarization state of the beam can be managed along its axis. It would be interesting to extend these phenomena to the quantum regime and the single photon limit. For instance, a curious research problem would be to generate entangled pairs of Longitudinally Structured Light patterns (with changing states of polarization or orbital angular momentum) and study the longitudinal behavior of the entangled states of such pair along the axis of propagation[113]. Furthermore, another promising area is to investigate the dynamics associated with spin-orbit coupling [103, 104] using three dimensional Structured Light.
Appendices
Appendix A

Modal Decomposition and Reconstruction

This Appendix contains supplementary information for Chapter 4. It describes the detailed steps of modal decomposition and reconstruction of 3D structured light fields described by $U(\rho, \phi, z, t)$.

A.1 Procedure

To examine the physical dynamics associated with the spatial evolution of OAM, full modal decomposition of $U(\rho, \phi, z, t)$ was performed along its propagation direction. Recall that any optical field can be expressed as a coherent superposition of linearly independent basis functions $V_j$ such that

$$\tilde{U}(\rho, \phi) = \sum_{j=1}^{P} c_j V_j(\rho, \phi),$$  \hspace{1cm} (A.1)

where $c_j = |c_j|e^{i\Delta \theta_j}$, whereas $\Delta \theta_j$ is the intermodal phase between $V_j$ and a reference mode $V_0$ such that $\Delta \theta_j = \theta_j - \theta_0$. Further, here, the field is decomposed at a fixed $z$-plane (i.e. $z_k$) such that $U(\rho, \phi, z = z_k, t) = \tilde{U}(\rho, \phi)e^{-i\omega t}$. The modal power coefficients, $|c_j|^2$, are mathematically obtained by evaluating the inner product between $\tilde{U}$ and each basis $V_j$, i.e. $|c_j|^2 = |\langle \tilde{U} | V_j \rangle|^2$, given by

$$|c_j|^2 = \frac{\left| \int_{\mathbb{R}^2} V_j^*(\rho, \phi)\tilde{U}(\rho, \phi)\rho d\rho d\phi \right|^2}{\sum_{j=1}^{P} |c_j|^2},$$  \hspace{1cm} (A.2)

and normalised such that $\sum_j |c_j|^2 = 1$. Importantly, the inner product given by Eq. (A.2) can be performed all-optically by projecting $\tilde{U}(\rho, \phi)$ onto correlation filters digitally encoded on a programmable SLM — a technique that is also compatible with our setup [121, 198, 199, 200, 201, 202]. Figure A.1 depicts the experimental setup used to generate and detect the 3D structured beams.
In this setup, SLM-2 acts as a digital filter that performs an inner product between $\hat{U}(\rho, \phi)$ and the basis function $V_j$. This requires the complex conjugate of $V_j$ to be encoded on the display of SLM-2, as follows

$$H_j(\rho, \phi) = V_j^*(\rho, \phi). \quad \text{(A.3)}$$

In practice, the outcome of this inner product is obtained by performing an on-axis intensity measurement in the far field of the reflected beam — realized here by placing Lens-5 in a 2f configuration thus transforming the plane of SLM-2 into the far field (k-space) to be detected by CCD-2. As such, the modal power coefficients $|c_j|^2$ can be obtained. Note that the absolute weights $|c_j|$, solely, are not sufficient to fully reconstruct the pattern $\hat{U}(\rho, \phi)$. The intermodal phases $\Delta \theta_j$ are also needed, and are readily obtained from two interferometric measurements that are feasibly enabled using SLM-2 as well [121, 198, 199, 200, 201, 202]. This requires two additional correlation functions to be encoded on SLM-2; namely, $H_j^{\cos} = [V_j^* + V_0^*]/\sqrt{2}$ and $H_j^{\sin} = [V_j^* + iV_0^*]/\sqrt{2}$. Accordingly, the intermodal phases $\Delta \theta_j$ are evaluated from

$$\Delta \theta_j = -\arctan \left[ \frac{\sqrt{2} I_j^{\sin} - |c_j|^2 - |c_0|^2}{\sqrt{2} I_j^{\cos} - |c_j|^2 - |c_0|^2} \right]. \quad \text{(A.4)}$$

Here, $I_j^{\sin}$ and $I_j^{\cos}$ depict the measured on-axis intensity signals resulting from the inner products with $H_j^{\sin}$ and $H_j^{\cos}$, respectively [121, 198, 199, 200, 201, 202].

A key benefit in using SLMs as digital filters is the ability to combine multiple correlation filters, encoded with different grating periods, and to multiplex them into a single hologram. This allows to spatially separate the respective inner products, in the Fourier plane, thus reducing the total number of required measurements. By adding all the elements of the bases functions $V_j$ weighted by their respective complex coefficients $|c_j| e^{i\Delta \theta_j}$, as per Eq. (A.1), the transverse pattern at $z = z_k$, (i.e. $\hat{U}(\rho, \phi)$), can be reconstructed. This process can then be repeated at different $z$-planes to fully reconstruct the 3D structured field, $U(\rho, \phi, z, t)$.

We note that precise alignment between SLM-1 and SLM-2 is particularly critical when performing
the optical correlations (inner products). For this purpose, both SLMs were mounted on 3D translational stages with micron-scale resolution and were fixed throughout the measurements. As such, beam propagation has been realized digitally by updating the distance \( z \), in the propagation term of Eq. (4.1), with increments of 1 cm. This approach provides an accurate realization of beam propagation within the interval \( z \leq L \), under the paraxial regime. It also ensures consistency in the alignment when recording the successive planes, as opposed to mechanically displacing the detection system.

### A.2 Choice of Bessel Functions

In principle, one may represent the optical field \( U(\rho, \phi, z, t) \) in terms of a given set of basis functions. In our case, at first glance, the Bessel functions may seem a natural choice for basis functions, given that \( U(\rho, \phi, z, t) \) itself is constructed from a discrete superposition of Bessel modes. However, this choice is accompanied with an inherent challenge: a necessary condition for the all-optical inner product discussed previously is to adopt orthonormal basis functions \([202]\) — a requirement that is not generally satisfied for Bessel functions with different spatial frequencies \( k_\rho \). Orthogonality implies minimal overlap between the modes such that

\[
\int \int_{\mathbb{R}^2} V_j^* V_{j'} \rho d\rho d\phi = \delta_{j,j'},
\]  

(A.5)

whereas normalization ensures that the total energy of the input field is conserved when transformed from one function space to another. In this case \( \sum |c_j|^2 = 1 \). Violating the orthonormality condition can lead to inaccurate reconstruction of the field and may violate the conservation of energy in some cases \([202]\). To satisfy the orthogonality requirement, the overlap between the Bessel functions was first studied at different values of \( k_\rho \). In this regard, multiple cross-talk measurements were carried out among the Bessel modes at different separations between their wavenumbers \((k_\rho)\) \([203]\). More details on the cross-talk analysis can be found in Appendix B. Based on this analysis, we were able to set a limit on the minimum separation between the values of \( k_\rho \) so that the generated beam \( U(\rho, \phi, z, t) \) contained a discrete set of *pseudo-orthogonal* Bessel functions — which were then encoded in the correlation filters at the detection stage. In essence, prior knowledge of \( U(\rho, \phi, z, t) \) allowed us to narrow down the number of digital filters encoded on SLM-2, thus making the modal decomposition process more efficient. Additionally, the Bessel bases were normalized with respect to their total energies (which are mode dependent) to satisfy the normality condition.

A powerful advantage in modal decomposition is that it provides full access to key physical quantities of the reconstructed beam, such as its Poynting vector, OAM density, and effective charge \( \ell \). Furthermore, the intermodal phases provide detailed insights into the mechanisms governing the 3D structured beam evolution and topological deformation.
Appendix B

Cross Talk Analysis for Bessel Modes

This appendix contains supplemental data that is useful for the proper implementation of modal decomposition and reconstruction of optical fields using the Bessel modes. In particular, it presents the cross-talk analysis carried out to assess the orthogonality and overlap between the Bessel functions pertaining to the beams reported in Chapter 4.

B.1 Cross-Talk Measurements

A necessary requirement for performing the all-optical modal decomposition of structured beams is to adopt orthonormal basis functions [198, 199, 200, 201, 202, 121]. In essence, orthogonality defines the degree of distinguish-ability of each mode (basis function) at the detection system, whereas normalization ensures that the total energy of the optical field is conserved when transformed from one basis system to another. The Bessel functions deployed in our 3D structured beams, however, are in general not orthogonal in nature. Nevertheless, one can still obtain a normalized pseudo-orthogonal set of Bessel functions provided that their transverse wavenumbers \(k_\rho\) are sufficiently separated, as will be discussed shortly.

To assess the degree of orthogonality within the Bessel functions, we performed multiple cross-talk analyses. Here, cross-talk represents the amount of overlap between one mode and another, and is mathematically represented by

\[
\int \int_{\mathbb{R}^2} V_j^* V_{j'} \rho d\rho d\phi. \quad (B.1)
\]

Ideally, when the modes in the function space are orthogonal, the cross-talk should be zero as long as \(j \neq j'\). In our case, each Bessel function \(J_\ell(k_\rho)e^{i\ell\phi}\) is defined by its radial dependence \(k_\rho\) and azimuthal dependence \(e^{i\ell\phi}\). To create a discrete set of pseudo-orthogonal Bessel functions, we start by
analyzing the cross-talk among the Bessel modes with different values of $\ell$, while fixing the parameter $k_\rho$.

Figure B.1 depicts the measured cross-talk between the Bessel modes with different quantized charges $\ell$. The maximum cross-talk is less than -10 dB in this case, which implies that the adopted Bessel modes are sufficiently distinguishable in the azimuthal direction.

In addition to the radial dependence, the vortex Bessel modes are also characterized by a radial dependency described by the continuous variable $k_\rho$. We introduce the metric $\delta k_\rho$ to describe the separation between Bessel functions with different $k_\rho$, in percentage form, such that

$$\delta k_\rho = \frac{k_\rho^j - k_\rho^j'}{k_\rho^j} \times 100. \quad (B.2)$$

This defines the distance between one Bessel function and another in the k-space. Figure B.2 depicts the cross-talk measurements under different separations of $k_\rho$. It is observed that, while the cross-talk is remarkable in case (a), it is significantly reduced as the separation $\delta k_\rho$ is increased. For instance, at $\delta k_\rho = 9.2\%$, the cross-talk is less than -10 dB which implies that the Bessel functions are distinguishable in the radial direction in that case. Accordingly, in all the 3D structured beam profiles presented in Chapter 4, we ensured that the separation among the Bessel functions in the k-space always satisfies the condition $\delta k_\rho \geq 9.2\%$.

Consequently, it is possible to obtain a discrete set of Bessel functions that are pseudo-orthogonal in both the radial and azimuthal directions. Once this set is defined, each Bessel mode can then be normalized with respect to its total energy to satisfy the orthonormality condition and perform the modal decomposition efficiently.
Figure B.2: Cross-talk measurements for Bessel beams as function of $\delta k_\rho$: a) $\delta k_\rho = 3.8\%$, b) $\delta k_\rho = 4.6\%$, c) $\delta k_\rho = 5.35\%$, d) $\delta k_\rho = 6.12\%$, e) $\delta k_\rho = 6.8\%$, f) $\delta k_\rho = 7.6\%$, g) $\delta k_\rho = 8.4\%$, h) $\delta k_\rho = 9.2\%$. Figure reproduced from Dorrah et al. [101] (© 2018 American Physical Society).
Appendix C

Supplementary information for Chapter 8

The first section of this appendix includes the derivation needed to characterize the shift which occurs to the longitudinal wavenumbers of the rotating light structure as it propagates inside a medium with refractive index \((n > 1)\). Additionally, in the second section, an analysis for the proposed sensor's tolerance to the deviations in \(\theta\), \(\Delta Q\), and \(z\) is presented.

C.1 Shift of Spatial Frequencies (k-comb) in the Sensed Medium

In this section, we derive the relations \(\tilde{Q} \simeq n \times Q\) and \(\Delta \tilde{Q} = (Q_1 - Q_{-1})/n\), as discussed in Chapter 7. We start from the consistency relation \((k_{z}^{\ell,m})^2 + (k_{p}^{\ell,m})^2 = k_0^2\) in air and \((\tilde{k}_{z}^{\ell,m})^2 + (k_{p}^{\ell,m})^2 = k^2\) in the medium, where \(k = \omega n/c\). These relations can be re-written as \(k_{z}^{\ell,m} = \sqrt{k_0^2 - (k_{p}^{\ell,m})^2}\) and \(\tilde{k}_{z}^{\ell,m} = \sqrt{k^2 - (k_{p}^{\ell,m})^2}\). Note also that \(k_{z}^{\ell,m} = Q_{\ell} + 2\pi m/L\) in air. Without loss of generality, let us consider the central term \(k_{z}^{\ell,m=0}\) such that

\[Q_{\ell} = k_{z}^{\ell,m=0} = k_0 \sqrt{1 - \left(\frac{k_{p}^{\ell,m=0}}{k_0}\right)^2}\]  \hspace{1cm} (C.1)

In the paraxial regime, \(k_{p}^{\ell,m=0} \ll k_0\), the above expression can be expressed as

\[Q_{\ell} = k_{z}^{\ell,m=0} = k_0 \left[1 - \frac{1}{2} \left(\frac{k_{p}^{\ell,m=0}}{k_0}\right)^2 - \frac{1}{8} \left(\frac{k_{p}^{\ell,m=0}}{k_0}\right)^4 + ...\right]\]  \hspace{1cm} (C.2)

Similarly,
\[
\tilde{Q}_\ell = k_z^{\ell,m=0} = k_0 n \left[ 1 - \frac{1}{2} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^2 - \frac{1}{8} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^4 + \ldots \right] \tag{C.3}
\]

The spacing between the central longitudinal wavenumbers in air and in the medium, denoted as \(\Delta Q\) and \(\Delta \tilde{Q}\), are thus given by

\[
\Delta Q_\ell = Q_\ell - Q_{-\ell} = k_0 \left[ \frac{1}{2} \left( \frac{k_\rho^{\ell,m=0}}{k_0} \right)^2 - \frac{1}{2} \left( \frac{k_\rho^{\ell,m=0}}{k_0} \right)^2 + \frac{1}{8} \left( \frac{k_\rho^{\ell,m=0}}{k_0} \right)^4 - \frac{1}{8} \left( \frac{k_\rho^{\ell,m=0}}{k_0} \right)^4 + \ldots \right] \tag{C.4}
\]

and

\[
\Delta \tilde{Q}_\ell = \tilde{Q}_\ell - \tilde{Q}_{-\ell} = k_0 n \left[ \frac{1}{2} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^2 - \frac{1}{2} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^2 + \frac{1}{8} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^4 - \frac{1}{8} \left( \frac{k_\rho^{\ell,m=0}}{k_0 n} \right)^4 + \ldots \right] \tag{C.5}
\]

By only neglecting the higher order terms (raised to the fourth power) in Eq. (C.4) and Eq. (C.5), the relation \(\Delta \tilde{Q} \simeq \left( Q_1 - Q_{-1} \right)/n\) is established. Furthermore, by neglecting all the higher order terms (raised to the power of two and four) in Eq. (C.2) and Eq. (C.3), it follows that \(Q \simeq k_0\) and \(\tilde{Q} \simeq k_0 n\); thus the relation \(\tilde{Q} \simeq n \times Q\) holds true.

### C.2 Sensor’s tolerance to deviations in \(\theta\), \(\Delta Q\), and \(z\)

In this section, we characterize the sensor’s tolerance to the deviations in the angular orientation \((\theta)\), in the separation \(\Delta Q\), and in the detection plane \((z)\), using a closed form expression. We start from Eq. (7.6), which states that

\[
n = \frac{1}{1 - 2\theta/(z\Delta Q)} . \tag{C.6}\]

By taking the partial derivatives with respect to \(\theta\), \(z\), and \(\Delta Q\), one can then establish the following relation

\[
\delta n = \frac{2z\Delta Q}{(z\Delta Q - 2\theta)^2} \delta \theta + \frac{2\Delta Q \theta}{(z\Delta Q - 2\theta)^2} \delta z + \frac{2z\theta}{(z\Delta Q - 2\theta)^2} \delta \Delta Q. \tag{C.7}
\]

Equation (C.7) shows that the proposed sensing scheme is more tolerant to the deviations in the angular orientation, denoted by \(\delta \theta\), when either the parameter \(\Delta Q\) or the detection plane \((z)\) are set to larger values. Stated otherwise, by generating OAM modes with larger separation \(\Delta Q\) and increasing the interaction length \(z\), our sensing scheme records smaller errors when identifying the refractive index \((n)\), for the same deviation in \(\theta\). This is readily discerned from the quadratic dependency on \(\Delta Q\) and \(z\) in the denominator of Eq. (C.7). Hence, at larger \(\Delta Q\) and/or \(z\), the accuracy and precision of the proposed sensor is improved and it becomes more tolerant to errors in estimating \(\theta\) as well as the uncertainty in \(z\).
Appendix D

Thesis Publications

D.1 Articles Published in Refereed Journals


D.2 Papers Published in International Conference Proceedings


Bibliography


