# Dynamics of Viscous Dissipative Gravitational Collapse in f(R,T) Gravity With Full Causal Approach

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Dynamics of Viscous Dissipative Gravitational Collapse in $f(R, T)$ Gravity With Full Causal Approach

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Abstract

Here, we present a study of dissipative gravitational collapse in spherically symmetric gravitating spacetime by applying Misner-Sharp approach in metric $f(R, T)$ gravity. This work is extension of Herrera et al. [1] already done in general theory of relativity. To this end, we have formulated a set of equations of motion by using $f(R, T) = R + \lambda T$, (where $R$ is Ricci scalar, $T$ is trace of stress-energy tensor and $\lambda$ is coupling constant) and stress-energy tensor of viscous and heat conducting fluid. The conservation equations have been also derived for such set up. Both the conservation equations and equation of motion provide the dynamical equations in considered theory of gravity. Furthermore, the causal thermodynamics approach introduced by M"uller-Israel-Stewart has been used to formulate the transportation equations of heat flux and bulk viscosity. Then we perform the coupling of dynamical equations and transportation equations. The effects of bulk and shear viscosity have been investigated in detail. Finally, we have studied the impact of matter-curvature coupling parameter $\lambda$ on the dynamics of the gravitating source.

Keywords: Gravitational collapse, Dissipative quantities, Heat transport equations and $f(R, T)$ gravity theory.

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1 Introduction

The empirical data from the anisotropy of the Supernova type Ia (SNeIa), large scale structure, measurement of Cosmic Microwave Background (CMB) and baryon acoustic oscillations [2]-[5] indicate that the current expansion of the Universe is accelerating rather than it is decelerating.

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The present expansion in Universe is galvanized by Dark Energy (DE). We have two ways to access this cosmic acceleration. The first way is, in context of General theory (GR) is to present the vacuum/exotic matter energy component called DE. Many aspirants have been contemplated [6]-[11] to analyze the nature of DE in current situation. The second way, is the generalization of \( f(R) \) theory where the Einstein Lagrangian \( \mathcal{L}_{(m)} \) explicitly coupling \( R \), which is arbitrary function read [12]-[15]. According to the SNeIa investigation there is a third component of Universe other than the matter and radiation. To address such problems like interaction between dark energy and modified gravity produce more effective results rather than GR. GR is the fundamental gravity theory which deals with weak field approximation but to explore the strong field limit modified theories of gravity are sufficient. Due to the modifications in GR the strong field system can be done. In other theories of gravity Einstein-Hilbert (EH) action can be modified to get alternative gravity theories. Like other modified theories of gravity \( f(R) \) theory is most fundamental modification in GR, which can be studied in reference to spherical symmetric perfect fluid gravitational collapse read [16]. \( f(R) \) gravity theory is applied to get the equation of motion for massive objects, in solution of these equations an extra force is generated, the effects of this extra force are studied in [17].

Keeping in mind the end goal to give the decent way for existence of dark matter, several researchers contributed modified theories of gravity like \( f(G) \) gravity. In [18], the solution of DE problem in the form of models is presented and also late-time accelerating cosmology is generated with the help of ideal fluid. One of the most fascinating view in cosmological and astrophysical phenomena is the reconstruction scheme for modified theories of gravity. The problems related to the recent Universe expansion can be investigated by such reconstruction programs from the help of \( f(G), f(R), f(R,G), f(T) \) (\( T \) is torsion scalar) and \( f(R,T) \) gravities. Abbas et al. [19] apply \( f(T) \) to explore the hypothetical displaying of anisotropic smaller star by using exact solution of Karori and Barua metric to the spherically symmetric metric. Also Abbas et al. investigated [20] the possibility of formation of anisotropic compact stars in the the context of Gauss-Bonnet theory, so called \( f(G) \) gravity, where \( G \) is Gauss-Bonnet invariant.

In 2011, another appealing modified theory of gravity was produced by Harko et al. in [21], this modified theory of gravity is named as \( f(R,T) \) gravity, which is the extension of \( f(R) \) which is fundamental modified theory of gravity. In current theory EH action is replaced by an arbitrary function \( R \) and \( T \), which deals with matter and curvature coupling. As this theory was being introduced, it acquired compelling concentration and researchers discussed its various aspects on reconstruction programmes, wormholes, energy conditions, neutron stars, cosmological and astrophysical phenomena, perturbation schemes, anisotropic Universe models and analysis of compact stars (for detail see [22]-[32]). In \( f(R,T) \) gravity modify theory the gravitational collapse has been extensively studied. Sharif and Yousaf [33] studied dynamical examination of gravitating stars in \( f(R,T) \) theory. Meanwhile, Chakraborty established an alternate \( f(R,T) \) gravity theory and DE problem in [34]. He also studied the energy conditions in \( f(R,T) \) gravity in scenario of general and particular form of perfect fluid with the help of constant equation of state. Recently, Noureen and Zubair [35] investigated the development of gravitating spherical symmetric collapsing star with anisotropic fluid in the reference of \( f(R,T) \) gravity. Shabani and Ziaie [36] proposed the dependability of Einstein static Universe in \( f(R,T) \) gravity and also discussed in detail the results
obtained in context of $f(R, T)$ gravity are more stable rather than the $f(R)$ gravity.

Finally, Hondjo [37] has presented model $f(R, T) = f(R) + \lambda f(T)$ more manageable form and explored the development in matter governing era to an expansion of Universe. The application of scalar field is studied with the help of two known cases for scale factor relating with expansion of Universe. For the sake of unification of matter dominant and accelerated phase the ordinary matter is neglected in the first example of $f(R, T)$ model. The second example in $f(R, T)$ model shown impact of unification in calculation without neglecting ordinary matter and accelerated phase is obtained with the help of matter dominating phase. The results obtained by Herrera et al. in [1] by using GR and results obtained by applying model of $f(R, T)$ gravity are worthy of comparison.

In this manuscript the arrangement of working is as follows: In section-2, Einstein field equations are formulated in context of $f(R, T)$ gravity; specially adopted $R + \lambda T$ model. In section-3, we discuss the dynamics of the gravitating source, also Misner and Sharp approach is investigated. In section-4, the causal thermodynamics approach introduced by Müller-Israel-Stewart has been used to formulate the heat transportation equation. The last segment outlines the discussion.

### 2 The Field Equations in $f(R, T)$ gravity

Here, we take a spherical symmetric gravitational collapsing fluid distribution within a spherical surface $\Sigma$. We suppose that the fluid is dissipative as heat flow, bulk and shear viscosity. Taking co-moving coordinated inside $\Sigma$, the accepted form of interior metric is written as follows

$$\frac{d^2s}{c^2} = -A^2dt^2 + B^2dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(1)

where $A$, $B$ and $C$ are functions of $t$ and $r$. The Einstein-Hilbert action in the present frame work of $f(R, T)$ gravity is written as

$$S = \int \left[ \frac{f(R, T)}{16\pi G} + \mathcal{L}_{(m)} \right] d^4x \sqrt{-g},$$

(2)

where $T_{\alpha\beta}$ is stress-energy tensor and $\mathcal{L}_{(m)}$ is matter action. Also,

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g}\mathcal{L}_{(m)}}{\delta g_{\alpha\beta}},$$

(3)

Here, $T = g^{\alpha\beta}T_{\alpha\beta}$ and

$$T_{(m)}^{\alpha\beta} = g_{\alpha\beta}\mathcal{L}_{(m)} - \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_{(m)}}{\partial g^{\alpha\beta}}.$$

(4)

By varying the given action with respect to metric tensor, we get following form of field equations in $f(R, T)$ gravity

$$R_{\alpha\beta}f_R(R, T) - \frac{1}{2}g_{\alpha\beta}f(R, T) + (g_{\alpha\beta}\Box - \nabla_\alpha \nabla_\beta) f_R(R, T)$$

$$= 8\pi G T_{(m)}^{\alpha\beta} - f_T(R, T)T_{(m)}^{\alpha\beta} - f_T(R, T)\Theta_{\alpha\beta},$$

(5)
where
\[ \frac{\partial f}{\partial R} = f_R, \quad \frac{\partial f}{\partial T} = f_T, \]
\[ \Box = \nabla^\alpha \nabla_\beta \] and
\[ \Theta_{\alpha\beta} = \frac{g^{\mu\nu} \delta T_{\mu\nu}}{\delta g_{\alpha\beta}} = -2T_{\alpha\beta} + g_{\alpha\beta} \mathcal{L}_{(m)} - 2g^{\mu\nu} \frac{\partial^2 \mathcal{L}_{(m)}}{\partial g^{\alpha\beta} \partial g^{\mu\nu}}. \] (6)

By analyzing interesting aspects that the extra force depends on Lagrangian density. There are two choices for \( \mathcal{L}_{(m)} \) [38], either \( \mathcal{L}_{(m)} = -p \) [39] or \( \mathcal{L}_{(m)} = -\mu \). In our present study, we assume that \( \mathcal{L}_{(m)} = -\mu \), then Eq.(6) can be written as:
\[ \Theta_{\alpha\beta} = -2T_{\alpha\beta} - \mu g_{\alpha\beta} \] (7)

The more general form of the cosmology model \( f(R, T) \) is used, so that this study shows that results obtained by this formulation are very nearly equals to that in general relativity.

We consider the functional form as follow \( f(R, T) = R + \lambda T \) in Eq.(4), we get
\[ G_{\alpha\beta} = \frac{\lambda}{2} (T + 2\mu) g_{\alpha\beta} + (8\pi + \lambda) T_{\alpha\beta} \] (8)

where \( G_{\alpha\beta} \) represents the usual Einstein tensor.

The assumed matter energy momentum is
\[ T_{\alpha\beta}^- = (p + \mu + \Pi)V_\alpha V_\beta + (\Pi + p)g_{\alpha\beta} + q_\alpha V_\beta + \pi_{\alpha\beta} \] (9)

where \( \mu, p, \Pi, q^\alpha, \pi_{\alpha\beta}, V^\alpha \) and \( l^\alpha \) represent the energy density, pressure, bulk viscosity, heat flux, shear viscosity, four-velocity and radial null four-vector respectively. Under the co-moving relative motion, these quantities satisfy the following relations:
\[ V^\alpha q_\alpha = 0, \quad V^\alpha V_\alpha = -1, \quad l^\alpha V_\alpha = -1, \quad l^\alpha l_\alpha = 0, \quad V_{\mu\nu} V^{\nu} = 0, \]
\[ \pi^\alpha_\alpha = 0, \quad \pi_{[\mu\nu]} = 0. \] (10)

The \( \sigma_{\alpha\beta} \) is the shear and can be written as
\[ \sigma_{\alpha\beta} = V_\alpha^{\beta} + V^\beta_\alpha + a_\alpha (V_\beta) - \frac{1}{3} \Theta h_{\alpha\beta} \] (11)

where \( a_\alpha \) and \( \Theta \) are given by
\[ a_\alpha = V_{(\alpha\beta)} V^\beta, \quad \Theta = V^\alpha_\alpha \] (12)

and \( h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta \) is the projection tensor. Let us assume co-moving metric then
\[ V^\alpha = A^{-1} \delta^\alpha_0, \quad q^\alpha = q B^{-1} \delta^\alpha_1, \quad l^\alpha = A^{-1} \delta^\alpha_0 + B^{-1} \delta^\alpha_1 \] (13)
where \( q \) depends on \( t \) and \( r \). Additionally it takes after from Eq.(10) that

\[
\pi_{0\alpha} = 0, \quad \pi_1^\alpha = -2\pi_2^\alpha = -2\pi_3^\alpha
\]  

(14)

In a more conservative frame, we can compose

\[
\pi_{\alpha\beta} = \Omega \left( \chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right)
\]  

(15)

\( \chi^\alpha \) is unit four vector with the radial direction and satisfying

\[
\chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0, \quad \chi^\alpha = B^{-1} \delta^\alpha_1
\]  

(16)

and \( \Omega = \frac{3}{2} \pi_1^3 \).

With (13), we obtain from (11), then the non null components are

\[
\sigma_1^1 = \frac{2}{3} B^2 \sigma, \quad \sigma_2^2 = \frac{\sigma_{33}}{\sin^2 \theta} = -\frac{1}{3} C^2 \sigma,
\]  

(17)

where

\[
\sigma = \frac{1}{A} \left( \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right), \quad \Theta = \frac{1}{A} \left( 2 \frac{\dot{C}}{C} + \frac{\dot{B}}{B} \right)
\]  

(18)

Hence non-null components of field equations are

\[
\mu = \frac{4\pi}{(4\pi + \lambda)(8\pi + \lambda)} \left[ \frac{1}{A^2} \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} + \frac{1}{B^2} \left( \frac{2\dot{B}}{B} - \frac{C'}{C} \right) \frac{C'}{C} \right]
\]

\[
+ \frac{\lambda}{4\pi A^2} \left\{ \left( \frac{\dot{B}}{2B} + \frac{\dot{C}}{C} \right) \frac{\dot{A}}{A} \right\} + \left( \frac{\dot{B}}{2B} + \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \frac{\dot{B}}{2B} - \frac{\dot{C}}{C} \right\} + \frac{\lambda}{4\pi B^2} \left\{ \left( \frac{\dot{C}}{C} - \frac{B}{2B} \right) \frac{A'}{A} + \left( \frac{2\dot{B}}{B} - \frac{C'}{C} \right) \right\} \frac{C'}{C} + \frac{\lambda}{8\pi B^2} \left\{ \frac{A''}{A} \right\},
\]  

(19)

\[
p + \Pi = \frac{8\pi}{3(4\pi + \lambda)(8\pi + \lambda)} \left[ \frac{1}{B^2} \left\{ \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) \frac{\dot{C}}{C} - \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{2C} \right) \frac{C'}{C} \right\} + \frac{\lambda}{8\pi A^2} \left\{ \left( \frac{\dot{B}}{2B} + \frac{\dot{C}}{C} \right) \frac{\dot{A}}{A} - \left( \frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \frac{\dot{B}}{2B} - \frac{\dot{C}}{C} \right\} + \frac{\lambda}{8\pi B^2} \left\{ \frac{A''}{A} \right\} \right]
\]

\[
+ \frac{2\dot{C}''}{C} - \left( \frac{\dot{B}}{2B} - \frac{C''}{C} \right) \frac{A'}{A} - \left( \frac{2\dot{B}}{B} - \frac{C'}{C} \right) \frac{C'}{C} \right\},
\]  

(20)
\[ \Omega = \frac{1}{(8\pi + \lambda)} \left[ \frac{1}{A^2} \left\{ \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \frac{\dot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} \right\} \right. \\
- \frac{1}{B^2} \left\{ \left( \frac{B'}{B} + \frac{C'}{C} \right) \frac{A'}{A} + \left( \frac{B'}{B} + \frac{C'}{C} \right) \frac{C'}{C} + \frac{A''}{A} - \frac{C''}{C} \right\}, \] 
\tag{21} 
\]

\[ q = \frac{1}{8\pi + \lambda AB} \left[ \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \frac{\dot{C}}{C} - \frac{\dot{C}}{C} \right], \tag{22} \]

where \( \frac{\partial}{\partial t} = \cdot \) and \( \frac{\partial}{\partial r} = \prime \).

Now, the Misner and Sharp mass is characterized by

\[ m = \frac{C}{2A^2B^2} \left( A^2B^2 - C'^2A^2 + \dot{C}^2B^2 \right). \tag{23} \]

### 3 Dynamical equations

Bianchi Identities are

\[ G_{\alpha\beta}^{\gamma \beta} V_{\gamma} = 0, \quad G_{\alpha\beta}^{\gamma \beta} \chi_{\beta} = 0 \tag{24} \]

The above equations yield

\[
(8\pi + \lambda) \left\{ -\frac{1}{A} \left( \frac{\dot{B}}{B} - \frac{2\dot{C}}{C} \right) \mu - \frac{1}{A} \ddot{\mu} - 2 \frac{(AC)'}{ABC} q - \frac{1}{B} q' \right\} \\
- \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) (p + \Pi) - 2 \frac{1}{3A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \Omega \right\} - \frac{1}{A^2} (\ddot{\mu} + 3\ddot{p} + 3\Pi) = 0 \tag{25} \]

and

\[
(8\pi + \lambda) \left\{ \frac{1}{A} \dddot{q} + \frac{1}{A} 2(BC)'q + \frac{1}{B} \left( p' + \Pi' + \frac{2}{3}\Omega' \right) + \frac{1}{B} A' \mu + \frac{1}{B} A' P_{eff} \\
+ 2 \frac{1}{3B} \left( \frac{A'}{A} + \frac{3C'}{C} \right) \Omega \right\} + \frac{1}{B^2} (\dddot{\mu} + 3\dddot{p} + 3\Pi) = 0 \tag{26} \]

In order to discuss the dynamics of the gravitating source, we follow Misner and Sharp [40] approach and define the proper time and proper radial derivatives as under

\[ D_T = \frac{1}{A} \frac{\partial}{\partial t}, \tag{27} \]

\[ \]
The proper radial derivative is denoted by $D_C$ and defined as,

$$D_C = \frac{1}{C'} \frac{\partial}{\partial r},$$

where the radius $C$ of a spherically symmetric surface inside $\Sigma$ is measured from the area of interior spacetime. By using Eq.(27), we can get the velocity $U$ defined by

$$U = D_T C < 0 \quad \text{(In case of collapse).}$$

Now from Eq.(23), we can be re-written

$$E \equiv \frac{C'}{B} = \left[ \frac{1}{C'} + U^2 - \frac{2m(t,r)}{C} \right]^{1/2}.$$  

With the help of Eqs. (19-22) and (27-28)

$$D_T m = -\frac{32\pi^2}{(4\pi + \lambda)(8\pi + \lambda)} \left[ \left( p + \Pi + \frac{2}{3} \Omega \right) U + qE + \Delta_1 \right]$$

and

$$D_C m = -\frac{32\pi^2}{(4\pi + \lambda)(8\pi + \lambda)} \left[ \mu + qE + \Delta_2 \right].$$

where $\Delta_1$ and $\Delta_2$ are presented in appendix, these appear due to matter curvature coupling parameter $\lambda$. Also, $D_T m$ represents the rate of change of the total energy inside spherical surface of radius $C$. The expression $\left( p + \Pi + \frac{2}{3} \Omega \right)$ seems on R.H.S of Eq.(31) $(U < 0)$ shows effective radial pressure. In same way the second term $qE$ represents the matter energy permitted from the spherical surface $\Sigma$, it is highly effected by the coupling parameter $\lambda$. $\mu$ and $qE$ the last two terms appeared in Eq.(32) on R.H.S measures the energy density of fluid element and $(qE < 0)$ measures the outflow of the radiation and heat respectively. The outflow of the radiation is increased due to coupling parameter $\lambda$ which appears in the definition of $\Delta_2$.

$$D_T U = -\frac{m}{C^2} - \frac{(8\pi - \lambda)}{(4\pi + \lambda)(8\pi + \lambda)} \left[ \left( p + \Pi + \frac{2}{3} \Omega \right) U + \frac{EA'}{AB} \right] + E \left[ \frac{4qU}{C} + D_T q + \frac{\lambda}{2} D_C (\mu + 3p + 3\Pi) \right] - E^2 \left[ D_C \left( p + \Pi + \frac{2}{3} \Omega \right) \right] + \frac{2\Omega}{C}.$$  

Substituting the value of $\frac{A'}{A}$ from Eq.(33) into Eq.(26), we get speaks to the successful inertial mass/inactive gravitational mass in accordance with proportionality rule.
To assess the term \( (\mu + p + \Pi + \frac{2}{3}\Omega) \) suggested on L. H. S along with R.H.S of Eq.(34) gives effective inertial mass in accordance with equivalence principle. On the right hand side of Eq.(34), the first term produces the gravitational force. Also in this term the expression written in curly brackets indicates the dissipative effects of active gravitational mass on the spherical collapsing object in dark source in present theorem of gravity. The gradient of effective inertial mass density is highly affected due to the matter curvature coupling parameter \( \lambda \). The last term being the gradient of effective mass density is significantly influenced by the shear viscosity. Herrera et al.[1] discussed the same significance early in frame work of General relativity. The last square bracket contains two different subscriptions. The first contribution indicates the gradient of shear, bulk viscosity and radiation pressure. The second contribution is due to effective pressure.

4 Transport equations

The constitutive transport equations for heat, bulk and shear viscosity takes the following form [1]

\[
\tau_0 \Pi_{\alpha} V^\alpha + \Pi = \alpha_0 q_\alpha^\alpha - \frac{\zeta T}{T} \left( \frac{\tau_0 V^\alpha}{\zeta T} \right)_\alpha \Pi - \zeta \Theta
\]  

(35)

\[
\tau_1 h_\alpha^\beta q_{\beta,\mu} V^\mu + q_\alpha = -\kappa \left[ h_\alpha^\beta \Pi_{\beta} (1 + \alpha_0 \Pi) + T (a_0 - \alpha_0 \Pi_{\alpha} - \alpha_1 \tau_1^\mu) \right]
\]

\[
+ \alpha_1 \tau_1^\mu h_\alpha^\beta \Pi_{\beta} - \frac{\kappa T^2}{2} \left( \frac{\tau_1 V^\beta}{\kappa T^2} \right)_\beta q_\alpha
\]  

(36)

\[
\tau_2 h_\alpha^\mu h_\mu^\nu \pi_{\mu,\nu} V^\rho + \pi_{\alpha,\beta} = 2\eta \alpha_1 q_{<\beta,\alpha>} - 2\eta_\sigma_\alpha_\beta - \eta T \left( \frac{\tau_2 V^\nu}{2\eta T^2} \right)_\nu \pi_{\alpha,\beta}
\]  

(37)

with

\[
q_{<\beta,\alpha>} = h_\beta^\mu h_\alpha^\nu \left( \frac{1}{2} (q_{\mu,\nu} + q_{\nu,\mu}) - \frac{1}{3} q_{\sigma,\gamma} h_\sigma^\gamma h_{\mu,\nu} \right)
\]  

(38)

also,

\[
\tau_0 = \zeta \beta_0, \quad \tau_1 = \kappa T \beta_1, \quad \tau_2 = 2\eta \beta_2.
\]  

(39)

For vanishing thermodynamic coupling coefficient then equations (35-37) reduce to (2.21-2.23) in [41]. In current scenario the each of equations (35-37) has one independent component.

\[
\tau_0 \Pi = -\left( \zeta + \frac{\tau_0}{2} \Pi \right) A \Theta + \frac{A}{B} \alpha_0 \zeta \left[ q' + q \left( \frac{A'}{A} + 2 \frac{C'}{C} \right) \right]
\]

\[
- \Pi \left[ \frac{\zeta T}{2} \left( \frac{\tau_0}{\zeta T} \right) + A \right]
\]  

(40)
\begin{equation}
\tau_1 \dot{\theta} = - \frac{A}{B} \kappa \left\{ T' \left( 1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega \right) + \frac{T'}{A} \left( 1 - \frac{2}{3} \alpha_1 \Omega \right) - \alpha_0 \Pi' \right\} - \frac{C'}{C} \right\} - q \left[ \frac{\tau_1}{2} A \Theta + \frac{\kappa T^2}{2} \left( \frac{\tau_1}{\kappa T^2} \right) \right] \tag{41}
\end{equation}

and

\begin{equation}
\tau_2 \dot{\Omega} = - 2 \eta A \sigma + 2 \eta A \left( q' - q \frac{C'}{C} \right) - \omega \left[ \eta T \left( \frac{\tau_2}{2 \eta T} \right) - A + A \Theta \frac{\tau_2}{2} \right] \tag{42}
\end{equation}

Now for the sake of investigation of effects of several variables on spherically symmetric geometry, substitute Eq.(41) into Eq.(34) and get the result as

\begin{equation}
\left( \mu + p + \Pi + \frac{2}{3} \Omega \right) (1 - \Lambda) D_T U = F_{hyd} + (1 - \Lambda) F_{grav}
\end{equation}

\begin{equation}
+ E \left\{ \frac{\lambda}{2} D_c (\mu + 3p + 3\Pi) + \frac{\kappa T^2 q}{2 \tau_1} D_T \left( \frac{\tau_1}{\kappa T^2} \right) - \frac{3}{2} q \Theta - \frac{q}{\tau_1} - 2q \frac{U}{C} \right\}
\end{equation}

\begin{equation}
+ E^2 \left[ \frac{\kappa}{\tau_1} D_c T \left( \frac{2}{3} \alpha_1 \Omega + \alpha_0 \Pi + 1 \right) - \frac{\kappa T}{\tau_1} \left\{ \alpha_0 D_c \Pi + \frac{2}{3} \alpha_1 (D_c \Omega \right.
\end{equation}

\begin{equation}
+ \frac{3 \Omega}{C} \right\} \right\} \right\} \right\}
\end{equation}

(43)

where \( F_{grav} \) and \( F_{hyd} \) are written as

\begin{equation}
F_{grav} = \left( \mu + p + \Pi + \frac{2}{3} \Omega \right)
\end{equation}

\begin{equation}
\times \left[ \frac{m(t, r)}{C^2} + \frac{8\pi - \lambda}{2(4\pi + \lambda)(8\pi + \lambda)} \left( \mu + p + \Pi + \frac{2}{3} \Omega \right) \right], \tag{44}
\end{equation}

\begin{equation}
F_{hyd} = E^2 \left[ D_c \left( p + \Pi + \frac{2}{3} \Omega \right) + \frac{2 \Omega}{C} \right], \tag{45}
\end{equation}

and also \( \Lambda \) is defined by

\begin{equation}
\Lambda = \frac{\kappa T}{\tau_1} \left( \mu + p + \Pi + \frac{2}{3} \Omega \right)^{-1} \left( 1 - \frac{2}{3} \alpha_1 \Omega \right) \tag{46}
\end{equation}
Using Eqs. (35) and (45), we get

\[
\left( \mu + p + \Pi + \frac{2}{3} \Omega \right) (1 - \Lambda + \Delta) D_T U = F_{\text{hyd}} + (1 - \Lambda + \Delta) F_{\text{grav}}
\]

\[
+ E^2 \left[ \frac{\kappa}{\tau_1} D_C T \left( \frac{2}{3} \alpha_1 \Omega + \alpha_0 \Pi + 1 \right) - \frac{\kappa T}{\tau_1} \left\{ \alpha_0 D_C \Pi + \frac{2}{3} \alpha_1 (D_C \Omega + 3 \Omega C) \right\} \right] (\mu + p + \Pi + \frac{2}{3} \Omega) \Delta
\]

\[
+ \frac{3 \Omega}{C} \right\} \left( \mu + p + \Pi + \frac{2}{3} \Omega \right) \Delta \left( \frac{D_C q}{q} + \frac{2q}{C} \right)
\]

\[
+ E \left[ \frac{\lambda}{2} D_c (\mu + 3p + 3 \Pi) + \frac{\kappa T q}{2 \tau_1} D_T \left( \frac{\tau_1}{\kappa T q} \right) + \frac{q}{\tau_1} + 2q U \right.
\]

\[
+ \frac{\Delta}{\alpha_0 \zeta q} \left( \mu + p + \Pi + \frac{2}{3} \Omega \right) \left\{ 1 + \frac{\zeta T}{2} D_T \left( \frac{\tau_0}{\zeta T} \right) \right\} \Pi + \tau_0 D_T \Pi \right]\]

(47)

where the value of \( \Delta \) is given by

\[
\Delta = \frac{3 \alpha_0 \zeta q^2}{2 \zeta + \tau_0 \Pi} \left( \mu + p + \Pi + \frac{2}{3} \Omega \right)^{-1}. \quad (48)
\]

This is the main equation resulting from the coupling of dynamical equation and transportation equations, at that point the inertial energy density and the gravitational mass density are collectively affected by the factor \( 1 - \Lambda + \Delta \). The coupled transport equation appearing in this case involve the effects of matter-curvature coupling constant \( \lambda \). Along these lines the present analysis is generalization of [41], by the assistance of an entire causal treatment of every single dissipative variable and the addition of thermodynamics viscous/heat coupling coefficients in \( f(R, T) \) theory of gravity. The coupling constant \( \lambda \) affects the inertial mass density and overall dynamics of the gravitating source.

## 5 Conclusion

In this paper we have studied a modified theory of gravity named as \( f(R, T) \) gravity, especially considered a generalized gravity model which results a coupling between matter and geometry. The generalized model \( f(R, T) = R + 2\lambda T \) is applied to derive the field equations. The presented model is most important to discuss the particular cases of accessible problems of astrophysics and cosmology. It has been observed that the impact of \( f(R, T) \) gravity model on obtained results have more made a significant difference comparatively to General Relativity on results obtained in the field of propagation of gravitational waves, gravitational collapse and cosmology as well. To discuss the dynamics of gravitational collapse different models are necessary to built. Now a days \( f(R, T) \) gravity has great importance and considered as an alternated to the standard General Relativity. The problems related to accelerated universe can widely be explained also provides an approach to discuss the current cosmic acceleration without extra spatial dimension or an exotic dark energy component. The Eq. (47) can be restored in standard GR by putting the coupling constant \( \lambda \) equal to zero the same results of standard GR are obtained as discussed in [1].
The set of equations have been well-established in this work to discuss the dynamics of spherically symmetric dissipative viscous fluid collapse in $f(R,T)$ gravity. It has been observed that Sharif and Kausar [42] discussed in detail the effects of spherically symmetric locally anisotropic and experiencing dissipation in the form of heat flow and viscosity in fluid distribution within the frame work of metric $f(R)$ gravity. By applying Causal technique it is investigated from Eq.(34) that the dissipative variables justify the transport equations calculated from causal thermodynamics and viscid/heat coupling coefficients. In this way we can discuss the effects of dissipative variables and viscid/heat coupling coefficients on the amount of passive gravitational mass. It is also observed that one of the dissipative variable particularly $\kappa$ when approaches to infinity so as to cause a serious decreasing change in gravitational force term, which may present the reversal of gravitational collapse [43].

In the diffusion approximation of a gravitating source, it is generally sensible to propose that thermal profile of the source is proportional to variation of temperature. The main objective of the present work is to investigate the dynamical behavior of the gravitating in $f(R,T)$ gravity using the full causal approach implemented for the first time to relativistic viscous dissipative gravitating source. In Ref.[1] the full causal approach has been investigated to discuss the impacts of dissipative factors on the dynamics of spherical gravitational collapse, this investigation shed light on many astronomical situations. On applying the present analysis to some astrophysical environment, one would expect the occurrence of pre-supernovae explosion. Also, a sufficient increase in thermal profile of source may produce a reasonable descend in force of gravity of the source which contributes to prevent collapse rather than favor of gravitational explosion.

6 Apendix

$$\Delta_1 = \frac{\lambda}{\pi} \left[ \frac{1}{A^2} \left\{ \frac{A''C}{16BC} + \frac{A'B'C}{16A^2B} - \frac{B'C}{16AB} - \frac{A'B'A^2C}{16B^3C^2} - \frac{A'C^2}{8A^2C} - \frac{AC}{8AC} \right\} + \frac{A'C'C'}{8B^2C} \right]$$

$$\Delta_2 = \frac{\lambda}{\pi} \left[ \frac{1}{A^2} \left\{ \frac{AC}{8A} + \frac{B'C'C'}{2B} + \frac{C}{8} - \frac{C'C}{8} + \frac{A'C'C'}{4AC'} - \frac{C'C'C'}{4C'} - \frac{B'C^2}{16B} \right\} + \frac{1}{B^2} \left\{ \frac{A''C^2}{16A} - \frac{A'B'C^2}{16AB} + \frac{A'C'C'}{8A} + \frac{B'C'C'}{4B} + \frac{C^2}{8} - \frac{CC''}{4} \right\} \right]$$

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