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Bi–global Stability Analysis in Curvilinear Coordinates

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A method is developed to solve bi–global stability functions in curvilinear systems which avoids reshaping of the airfoil or remapping the disturbance flow fields. As well, the bi–global stability functions for calculation in a curvilinear system are derived. The instability features of the flow over a NACA (National Advisory Committee for Aeronautics) 0025 airfoil at two different angles of attack, corresponding to flow with a separation bubble and fully separated flow, are investigated at a chord-based Reynolds number of 100,000. The most unstable mode was found to be related to the wake instability, with a dimensionless frequency close to one. For flow with a separation bubble, there is an instability plateau in the dimensionless frequency ranging from 2 to 5.5. After the plateau and for increasing dimensionless frequency, the growth rate of the most unstable mode decreases. For fully separated flow, the plateau is narrower than that for flow with a separation bubble. After the plateau, with increased dimensionless frequency, the growth rate of the most unstable mode decreases and then increases once again. The growth rate of the upstream shear layer instability was found to be larger than the downstream shear layer instability.

I. INTRODUCTION

Airfoil performance is often limited or degraded by flow separation, which is usually associated with loss of lift, increased drag, and kinetic energy losses. Thus, many methods of flow control methods have been developed to suppress or avoid separation entirely.1 Exploiting the instability of separated flow, periodic flow control methods improve on steady control while maintaining the same energy input.2 As a precursor to developing control strategies, it is important to understand and quantify the flow stability characteristics of separated flow.

There are three kinds of linear stability analysis approaches: (a) classical stability, (b) bi–global stability and (c) tri–global stability. Classical stability approaches, often based on the Orr–Sommerfeld equation, assume the basic flow is nonhomogeneous in only one spatial direction. In the past, many studies have investigated flow stability by utilizing the Orr–Sommerfeld equation.3,4 This is limited by an assumption of locally parallel flow, which is not altogether applicable to separated flow. Tri–global stability considers the three-dimensionality of the base flow and perturbations.5,6 However, there is a non–trivial memory requirement to solve the eigenvalue problem limiting its practical use: e.g., for a case with 64 mesh points in each direction, 17.6 terabytes of memory is required.7

The bi–global stability approach considers the non–uniformity of the flow variables in two spatial directions and is a good option for airfoil analysis, as the variation of the flow parameters along the spanwise direction is significantly weaker than the other two directions. Taking advantage of this, the bi–global stability method assumes the perturbation as a wavelike mode in the spanwise direction. Compared with tri–global stability analysis, the solution process of the bi–global stability method is simplified and the required memory is significantly reduced. Despite this simplification, bi–global stability analysis is still computationally expensive, as very large partial-derivative eigenvalue problems (EVP) must be solved. Exploiting the sparsity or developing high-order finite-difference schemes are helpful to speed up the solution process. The high-order finite-difference scheme of order-q (FD-q) method was found to significantly outperform all other finite difference schemes in solving classic linear local, bi–global, and tri–global eigenvalue problems based on both memory and CPU time requirements.

Bi–global stability analysis has been used in many applications, e.g., channel flow,9–11 flat plate12 and bluff body flows13,14. Bi–global stability analysis has also been applied to flow past a NACA 0012 airfoil at high attack angle at chord-based Reynolds numbers (Re = u∞c/v) ranging from 400–1000, where u∞ is the freestream velocity, c is the chord length, and v the kinematic viscosity.15 It was found that the near wake and far wake instabilities are the two dominant unstable modes and with increasing wavenumber, the unstable modes are suppressed. Kitsios et al.16 performed bi–global stability analysis for flow past a NACA 0015 airfoil at an angle of attack of 18 degrees and Re = 200. Most of the airfoil flows analyzed are at very low Reynolds number, around

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200–2000\textsuperscript{16,17}. Non-classical stability analyses of Reynolds number flows around $1 \times 10^{5}$ are still very limited, and are important for unmanned aerial vehicles (UAVs), small-scale wind turbines, and low-speed aircraft, where flow separation is often encountered.

Typically, the input for bi–global stability analysis is the steady or time-periodic base flow that is non-homogeneous in two spatial directions at a given Reynolds number\textsuperscript{18}. The base flow can be stable or unstable. For stable flow, the base flow can be obtained through solving the governing function of the base flow. However, it is difficult to compute the base flow for inherently unstable flows. Normally, selective frequency damping method is sometimes unable to identify an unstable steady state (\textsuperscript{18}) or the mean flow can be used to obtain the base flow for the analysis of global stability. However, the selective frequency damping method is sometimes unable to identify an unstable steady state\textsuperscript{20}. Although the use of the mean flow is not strictly correct with respect to the formulation of the linearized Navier-Stokes equations, solving the eigenmodes of perturbations about the mean flow helps to identify how perturbations can grow or delay with respect to the time-average flow. These growing modes are helpful to aid in the design of flow control strategies\textsuperscript{20}.

For the flow past an airfoil, the bi–global stability equations cannot be directly discretized. To solve the stability equations in the curvilinear coordinate system, conformal mapping techniques are often used\textsuperscript{16}. The mapping between the curvilinear and physical coordinates for an airfoil geometry is very involved, requiring 4 mappings. The first maps the rectangular curvilinear grid to cylindrical coordinates. A Joukowsky transformation is then used to convert the cylindrical grid to an airfoil shaped mesh. Then appropriate parameters are selected to best represent the airfoil. This airfoil shaped mesh is then translated, scaled and finally rotated by the angle of attack to align with the finite volume mesh to return the coordinates in physical space. Sometimes, it is difficult to find suitable parameters to best represent the shape of a certain airfoil. There is a mapping relationship between the velocity in physical coordinates and curvilinear coordinates. Thus, after solving the bi–global stability equations in curvilinear coordinates, to show the spatial structure in physical space, a remapping process is needed. To avoid the reshaping of the geometry of the airfoil and remapping of the flow field, a new method for solving the stability equation in the curvilinear coordinate system is proposed.

Post processing techniques, such as proper orthogonal decomposition (POD)\textsuperscript{21} and dynamic mode decomposition (DMD)\textsuperscript{22}, have been widely applied to analyze the unsteady characteristics of the coherent structure in different applications\textsuperscript{18,23,24}. POD can be applied to both instantaneous numerical and experimental snapshots and the mode of POD is hierarchically ordered with respect to their energy content. The frequency information is contained in the POD temporal coefficient. The spatial distribution of the first two POD modes was found to be typically coupled in pairs\textsuperscript{25}. And the first two modes are also found to be representative of a vortex shedding phenomenon which is identified to be induced by Kelvin–Helmholtz instability\textsuperscript{26,27}.

The present work introduces a new method for solving the bi–global stability equations in curvilinear coordinates. This method is applied to the flow over a NACA 0025 airfoil to demonstrate its validity. Two states are considered: laminar separation bubble and fully separated flow, corresponding to angles of attack (AOA) of 5 and 12 degrees, respectively, at a chord-based Reynolds number of 100,000. Stability characteristics and flow features are presented in a frequency range of $F^{+} = 0 - 11 (= \omega c/2\pi u_{\infty})$, where $\omega$ is the angular frequency of the perturbation. The new method is compared with POD analysis of the fully separated flow.

II. BASE FLOW COMPUTATION METHOD

The numerical computations of the base flow were performed using three dimensional large–eddy simulation (LES). The subgrid scale stress tensor, $\tau_{ij}$, was modeled with an eddy viscosity approach, $\tau_{ij} = -2v_{t}\tilde{S}_{ij}$, where $v_{t}$ is the eddy viscosity and $\tilde{S}_{ij}$ is the filtered strain rate tensor. A subgrid scale turbulence kinetic energy model was employed. The temporal and convective terms were discretized using a second order backward implicit time stepping scheme and second order total variation diminishing (TVD) scheme, respectively. An adaptive time stepping scheme was employed to maintain a Courant—Friedrichs—Lewy (CFL) number of $C_{l} < 0.7$ throughout the domain. The Pressure-Implicit with Splitting of Operators (PISO) algorithm was used for the pressure–momentum coupling. The airfoil surface was defined as a no-slip boundary condition and a periodic boundary condition was applied to the lateral boundaries, spaced $c/3$ apart, where $c$ is the chord length. The inlet and outlet were assigned laminar inflow and zero-gradient outletflow conditions, respectively.

The computational domain with respect to the mid-section plane of the airfoil and bi–global stability analytic region are shown in Figure 1. The downstream section of the mesh extends 11 chord lengths (11c) to the outlet boundary. The half-height of the mesh is 6c and the lateral boundaries (not shown) are at a distance of $c/3$. Extending the downstream section to 15c resulted in no appreciable change ($<1 \%$). The computations were performed on 64-128 processors using the Blue Gene/Q (BGQ) and General Purpose Cluster (GPC) at Scinet\textsuperscript{28}. A block-structured mesh with $32 \times 10^{6}$ cells was employed with mesh refinement concentrated in the wake and around the NACA 0025 airfoil which had a chord length $c = 0.3$m. For wall-resolved LES, it is well accepted that the required mesh resolution, which has been achieved in all cases presented, is $\Delta x^{+} \approx 100$, $\Delta y^{+} \approx 2$, and $\Delta z^{+} \approx 20$\textsuperscript{29,30}. A detailed discussion of the validation of the computed flow can be found in Ziada and Sullivan\textsuperscript{31}.

III. A NEW METHOD TO SOLVE THE BIGLOBAL STABILITY FUNCTION

The state variables ($\phi$) can be decomposed into the base state ($\overline{\phi}$) and the perturbation($\phi'$):\textsuperscript{6}

$$\phi(x,y,z,t) = \overline{\phi}(x,y) + \varepsilon \phi'(x,y,z,t)$$ \hspace{1cm} (1)
where $\bar{\phi}(x,y)$ indicates the two-dimensional steady base flow, in this study chosen as the time-averaged flow along the midspan obtained by three-dimensional large-eddy simulation, and $\phi'(x,y,z,t)$ is the perturbation. The perturbation was assumed to have a form of

$$\phi'(x,y,z,t) = \phi(x,y) e^{i(\beta z - \omega t)} + \phi^*(x,y) e^{i(-\beta z + \omega t)} \quad (2)$$

where the $^*$ superscript denotes the complex conjugate. The second term is required because $\phi$ and $\omega$ in general are complex, while $\phi'$ must be real. $\beta$ is the wavenumber of the structure of the perturbation in the spanwise direction $z$. The real part of complex value $\omega$ represents the angular frequency and the imaginary part of $\omega$ corresponds to the growth/damping rate of the associated amplitude function. A positive value of the imaginary component, $\text{Im}(\omega)$, indicates the exponential growth of the perturbation, whereas a negative value of $\text{Im}(\omega)$ corresponds to the damping of the unstable mode. In the context of bi-global stability, the base flow in the spanwise direction $w$, is assumed to be zero. When the above base flow simplifications and modes of perturbation are substituted into the Navier-Stokes equations and higher order terms ($O(\epsilon^2)$) are neglected, the linearized Navier-Stokes equations are obtained:

$$\hat{u}_x + \hat{v}_y + i\beta \hat{w} = 0 \quad (3)$$

$$-\hat{n}u_x - \hat{n}u_y - \hat{n}u_z - \hat{n}v_y - \hat{n}v_x - \hat{n}w_x + (\hat{u}_{xx} + \hat{u}_{yy} - \beta^2 \hat{u})/Re_c = -i\omega \hat{u} \quad (4)$$

$$-\hat{n}v_x - \hat{n}v_y - \hat{n}v_z - \hat{n}v_y - \hat{n}v_x + (\hat{v}_{xx} + \hat{v}_{yy} - \beta^2 \hat{v})/Re_c = -i\omega \hat{v} \quad (5)$$

$$-\hat{n}w_x - \hat{n}w_y - \hat{n}w_z - i\beta \hat{p} + (\hat{w}_{xx} + \hat{w}_{yy} - \beta^2 \hat{w})/Re_c = -i\omega \hat{w} \quad (6)$$

where the subscripts denote partial differentiation with respect to the indicated variable. The bi-global stability equations are cast as a partial derivative eigenvalue problem.

$$A\hat{\phi} = \omega M\hat{\phi} \quad (7)$$

A matrix-based approach is the most used method for bi-global stability analysis. $A$ is the spatial discretization operator, which is a function of the mesh, base flow, Reynolds number ($Re_c$), and wavenumber ($\beta$). Finite difference methods are often used to determine the expression for $A$. The finite difference method is performed on a set of discrete grid points. For the case of flow around the airfoil, the above equations cannot be directly discretized.

To solve the stability equation in the curvilinear coordinate system, Kitsios et al. used conformal mapping to transform the airfoil to a rectangular domain. In the present work, base flows are generated in physical space and need to be transformed to the curvilinear calculation domain before performing the stability calculation. An O-grid geometry is used to generate the grid for the stability analysis, as shown in Figure 2. The basic velocity field is interpolated to the bi-global grid through a cubic spline interpolation method, on which the bi–global stability analysis will be undertaken.

$$\hat{\phi} = (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \quad (8)$$

In order to avoid adding boundary conditions on each adjacent grid block, a code was written to merge the grids of four grid blocks into one block, as shown in Figure 3(a). The corresponding mesh used for stability analysis in curvilinear coordinates is shown in Figure 3(b). The airfoil surface (L2) corresponds to $j = 1$ in the curvilinear coordinate system and the far field (L1) corresponds to $j = N$, where N is the number of mesh points in the $j$ direction. First, the relationship between the physical coordinate system $(x,y)$ and the calculation curvilinear coordinate system $(i,j)$ must be established.

$$i = i(x,y); \quad j = j(x,y) \quad (9)$$

For corresponding points, they share the same value and simplify to

$$u(x_A,y_A) = u(i_A,j_A) \quad (10)$$

so that when the bi–global stability equations are solved, the value of the perturbation in each grid can be obtained without remapping.
Using the approach above, the bi–global stability equations in the parametric form in the computed coordinate system are presented as a function of the curvilinear coordinates. For example, considering the continuity equation, $\dot{u}_x$ can be expressed in parametric form in the computed coordinate system as

$$\dot{u}_x = \hat{u}_i * i_x + \hat{u}_j * j_x$$ \hspace{1cm} (11)

Using the approach above, the bi–global stability equations in the new curvilinear coordinate system are

$$\dot{u}_i * i_x + \dot{u}_j * j_x + \hat{v}_i * i_y + \hat{v}_j * j_y + i\hat{\beta} \hat{\omega} = 0$$ \hspace{1cm} (12)

$$-\left(\hat{\pi}_i * i_x + \hat{\pi}_j * j_x\right) * \hat{u}$$
$$-\left(\hat{\pi}_i * i_x + \hat{\pi}_j * j_x\right) * \hat{v}$$
$$-\left(\hat{\pi}_i * j_x + \hat{\pi}_j * j_x\right) - \hat{\pi}(\hat{u}_i * i_x + \hat{u}_j * j_x) - \hat{\nu}(\hat{v}_i * i_x + \hat{v}_j * j_x) + \left\{\hat{u}_i * i_{xx} + \hat{u}_j * j_{xx} + \hat{u}_{ii} * (i_x)^2 + \hat{u}_{jj} * (j_x)^2 + 2\hat{u}_{ij} * i_x * j_x + \beta^2 \hat{\omega} / Re_c\right\} = -i\hat{\omega} \hat{\nu}$$ \hspace{1cm} (13)

$$-\left(\hat{\pi}_i * i_x + \hat{\pi}_j * j_x\right) * \hat{u}$$
$$-\left(\hat{\pi}_i * i_x + \hat{\pi}_j * j_x\right) * \hat{v}$$
$$-\left(\hat{\pi}_i * j_x + \hat{\pi}_j * j_x\right) - \hat{\pi}(\hat{u}_i * i_x + \hat{u}_j * j_x) - \hat{\nu}(\hat{v}_i * i_x + \hat{v}_j * j_x) + \left\{\hat{v}_i * i_{xx} + \hat{v}_j * j_{xx} + \hat{v}_{ii} * (i_x)^2 + \hat{v}_{jj} * (j_x)^2 + 2\hat{v}_{ij} * i_x * j_x + \hat{\omega} / Re_c\right\} = -i\hat{\omega} \hat{\nu}$$ \hspace{1cm} (14)

$$-\hat{\pi}(\hat{\nu}_i * i_x + \hat{\nu}_j * j_x) - \hat{\nu}(\hat{\nu}_i * i_x + \hat{\nu}_j * j_x)$$
$$i\hat{\beta} \hat{\nu} + \left\{\hat{\nu}_i * i_{xx} + \hat{\nu}_j * j_{xx} + \hat{\nu}_{ii} * (i_x)^2 + \hat{\nu}_{jj} * (j_x)^2 + 2\hat{\nu}_{ij} * i_x * j_x + \hat{\omega} / Re_c\right\} = -i\hat{\omega} \hat{\nu}$$ \hspace{1cm} (15)

It should be noted that the value of $i_x$ should be solved through an inverse transformation.

$$x = x(i, j); \quad y = y(i, j)$$ \hspace{1cm} (16)

Differentiating, it is possible to obtain

$$dx = x_i * di + x_j * dj$$ \hspace{1cm} (17)

$$dy = y_i * di + y_j * dj$$ \hspace{1cm} (18)

$$di = i_x * dx + i_y * dy$$ \hspace{1cm} (19)

$$dj = j_x * dx + j_y * dy$$ \hspace{1cm} (20)
As a matrix, this becomes
\[
\begin{bmatrix}
  i_x & i_y \\
  j_x & j_y \\
\end{bmatrix} = \begin{bmatrix}
  y_j - x_j & -x_j \\
  -y_i & x_i \\
\end{bmatrix}
\begin{bmatrix}
  x_j - x_i \\
  y_j - y_i \\
\end{bmatrix}
\]
(21)
and the coefficients in the above equations can be solved.

On the airfoil surface (L2), the boundary conditions \( \hat{u} = \hat{v} = \hat{w} = 0 \) are imposed on the perturbation velocities and the compatibility condition for pressure of a zero-wall-normal gradient is employed\(^{32} \). Since the properties of the perturbations are not known on the boundary L1 before solving the EVP, the amplitude functions are linearly extrapolated from within the domain\(^ {16} \). L3 and L4 share the same line, and the interior boundary condition is imposed.

The bi-global eigenvalue equations are cast as a partial-derivative eigenvalue problem in the curvilinear system, eq.7 and solved with a purpose-written code. With such large-scale partial-derivative eigenvalue problems, it is computationally more efficient to solve for the eigenvalues nearest a certain point in the complex plane using a shift-and-invert method, such as the implicitly restarted Arnoldi method (IRAM). Thus, the above eigenvalue problem is modified as shown, given a complex shift \( \sigma \):
\[
(\hat{A} - \sigma M)^{-1} M \hat{\phi} = \hat{\phi}/(\omega - \sigma) 
\]
(22)

The Arnoldi Package (ARPACK) library was used on a cluster with 500 GB memory. A study of domain size was conducted to ensure the convergence of eigenspectrum, and grid resolution refinement was performed to ensure accurate stability results. The code was validated against the results of Theofilis et al.\(^ {32} \) for Poiseuille flow in a rectangular domain at \( Re = 100 \) and \( \beta = 1 \) and showed good agreement.

IV. INSTABILITY FEATURES IN FLOW WITH SEPARATION BUBBLE

The base flow for an airfoil angle of attack of 5 degrees and \( Re_c = 1 \times 10^{5} \) is shown in Figure 4. There is a small separation bubble on the airfoil surface near the middle, and the flow reattaches prior to the trailing edge. The time-averaged flow is dominated by a small recirculation zone in this area.

When the Reynolds number is very low (below 2000), the angular frequency of the most unstable mode is close to 0 and there is only 1 or even no unstable mode in some cases, which means that setting the “shift value” \( \sigma \) to zero is the proper choice\(^ {16} \). When the Reynolds number is higher, as in this case, there are many unstable modes. To deploy the full power of the Arnoldi algorithm\(^ {33} \), several shift values \( \sigma \) are chosen to capture the eigenvalues with the dimensionless frequency \( F^{+} \) from 1 to 11. The variation of the growth rate of the most-unstable/least-stable eigenvalue with respect to the dimensionless frequency is shown in Figure 5. The nondimensional growth rate \( \omega_{in} \) is
\[
\omega_{in} = \frac{\omega \delta^{*}}{2\pi U_r} 
\]
(23)
where \( \delta^{*} \) is the thickness of the separation bubble and \( U_r \) is the characteristic velocity chosen as the maximum reverse flow velocity. The spanwise perturbation wavenumber was set to zero, which examines the stability characteristics of two-dimensional flow. The dimensionless frequency of the most unstable modes in the whole solved range was found to be around 1 and the spatial structure of this most unstable mode is shown in Figure 6. Normalizing the complex eigenfunctions using the corresponding maximum absolute values, contour plots are generated for the real components of these nondimensional eigenfunctions. The alternating velocity perturbation originates approximately 2 chord lengths downstream of the airfoil. The wake instability mode is dominant and its frequency is important for flow control. Many studies have shown that the best control effect occurs when the excitation frequency of an active control system is \( F^{+} = 1 \). Additionally, effective control can be achieved when the dimensionless excitation frequency range varies from 0.25 < \( F^{+} < 2.0 \)\(^ {34} \). In the dimensionless frequency range from 2 to 5.5, there is a plateau where the frequency has little effect on the growth rate of most unstable modes. After the plateau, with increased dimensionless frequency, the growth rate of the most unstable modes significantly decreases, as shown in Figure 5.

It can be seen that 39% of all unstable modes are located in the dimensionless frequency range of 0.5 to 1.5. With increased dimensionless frequency, both the largest growth rate and the number of unstable modes decrease. When the dimensionless frequency is larger than 6, there are only a few unsteady modes and the unsteady growth rate is significantly decreased. This confirms past experimental work that control is better achieved over this low frequency range.

Monotonically growing modes corresponding to \( F^{+} = 0 \), known as stationary modes, are also found at this Reynolds number. The spatial structure of this mode is shown in Figure 7. While this mode does not fluctuate in time (\( F^{+} = 0 \),
it does grow ($\omega_* \approx 0.032$). This kind of stationary mode is self–excited and was also identified in other studies$^{16,35}$. Rodriguez et al. $^{36}$ demonstrated the requirement of a minimum reverse flow magnitude within the separation bubble to induce the occurrence of such a global mode was between 7% and 8% of the local free-stream velocity. The maximum amplitude of the reverse flow encountered within the separation bubble is $\sim 18\%$ of the local free-stream velocity in this case.

V. INSTABILITY FEATURES IN A FULLY SEPARATED FLOW

The mean velocity magnitude at the midspan of the airfoil at an angle of attack of 12 degrees and $Re_c = 1 \times 10^5$ is shown in Figure 8. Flow separation occurs shortly downstream of the leading edge. A region of low velocity persists from separation well beyond the trailing edge of the airfoil. In this case, there is no boundary layer reattachment and flow is stalled. Again, the base flow validation is presented in Ziadé et al. $^{31}$.

The variation of the growth rate of the most–unstable eigenvalue with respect to the dimensionless frequency is shown in Figure 9. The spanwise perturbation wavenumber was also set to zero. The dimensionless frequency of the most unstable modes over the range was found in $F^+ = 1$ and $F^+ = 1.45$. The spatial structure of $F^+ = 1$ is shown in Figure 10(a) using the dimensionless real part of the streamwise velocity. The spatial structure of $F^+ = 1.45$ is similar with the spatial structure of $F^+ = 1$. This structure also exhibits features of the wake type mode. The alternating velocity perturbation originates approximately 1.2 chord lengths downstream of the airfoil, which is closer to the trailing edge than that in 5 degree flow fields.

The instability plateau ranges about from 2 to 3.5, which is shorter than that in the 5 degree flow field. However, the largest growth mode did not monotonically decrease with the dimensionless frequency after the plateau. When $F^+$ is near 5,
FIG. 8. The base flow of the NACA 0025 airfoil at AOA = 12°

the largest growth rate is found to reach a minimum. The spatial extent of this mode is shown in Figure 10(b). This mode is located along the downstream shear layer and is named the downstream shear layer mode, which has not received much attention. The largest magnitude of this mode exists in the streamwise location 0.73c and becomes weaker as it develops downstream. This modes mainly exists in the streamwise location 0.2–2.6c. Further increasing the frequency, the most unstable growth rate increases once again. This feature is different from that seen at 5 degrees. The number of unstable modes in this region is significantly greater for the 5 degree flow fields. The spatial structure of $F^+ \approx 10$ is shown in Figure 10(c). This mode is associated with the upstream shear layer instability. The upstream shear layer mode mainly exists in the streamwise location 0.08–1.32c, and the largest magnitude, located at the 0.38c, becomes weaker as it develops downstream. Compared with the downstream shear layer mode, the position of the largest magnitude of upstream shear mode is farther upstream. Most of the unstable modes are located in the dimensionless frequency range from 0.5 to 1.5, similar to the previous case. The number of unstable modes in the high frequency region ($F^+ > 6.5$) is significantly greater than that in the 5 degree flow field.

VI. POD ANALYSIS OF FULLY SEPARATED FLOW

The time-dependent flow field $V(x,y,t)$ can be approximately treated as the sum of average and fluctuating component:

$$V(x,y,t) = \overline{V}(x,y) + V'(x,y,t) \quad (24)$$

Using POD, the fluctuating component can be written as the following summation:

$$V'(x,y,t) = \sum_{i=1}^{n} \alpha_i(t) \Phi_i(x,y) \quad (25)$$

FIG. 9. The variation of the growth rate of the most-unstable/least-stable eigenvalue with respect to the dimensionless frequency for $AOA = 12°$

The temporal characteristics of mode $i$ are reflected by the POD temporal coefficients, $\alpha_i(t)$.

POD attempts to decouple the spatial and temporal structure of the unsteady flow field as the composition of the various modes with different amplitudes.

The distribution of energy among the POD modes with regards to the transverse velocity is presented in Figure 11. The first and second modes are energetically similar, containing roughly 10% of the energy each. Modes 1 to 10 are paired; i.e., odd–even modes with energy of similar magnitude. The energy ratio of the higher modes are at least one order of magnitude less. For example, the energy ratio of the 30th mode is less than 1%. POD provides an orthogonal set of minimal number of basis vectors, which is helpful in constructing a reduced-order model of the unsteady flow field. Compared with the bi–global stability analysis, POD arranges modes according to energy content. However, here the low energy associated with these modes minimizes the physical association of the flow structures to the mode. In this case, the flow modes do not directly correspond to the instabilities previously identified experimentally or computationally.

The spatial structure of the first and second POD mode are shown in Figure 12. It can be seen that these two modes are similar but with a phase difference, both positioned downstream for the trailing edge. With increasing mode number, the spatial structure moves upstream along the airfoil, as shown in Figure 13(a). The temporal coefficient of the first two modes is shown in Figure 14, where $t_{wake}$ is the period of the wake obtained in the stability analysis. It can be seen that the coefficients of these two modes also have shift value about $\pi/4$. Furthermore, the dominant frequency is equal to the wake frequency previously obtained. Further increasing the modes, the spatial structure breaks down and becomes
FIG. 10. Eigenfunction of $Re(\hat{u})/Re(\hat{u})_{max}$ (a) $F^+ = 1$ (b) $F^+ \approx 5$ (c) $F^+ \approx 10$

FIG. 11. Energy ratio of POD eigenvalues for wall normal velocity very noisy. Compared with POD, bi–global stability provides a clearer separation of the wake instability and shear layer instability modes.

FIG. 12. Spatial structure of POD mode 1 and mode 2
VII. CONCLUSIONS

A method is presented to solve the bi–global stability equations in curvilinear coordinates which avoids the reshaping of the airfoil or remapping the disturbance flow fields. The bi–global stability equations for a curvilinear system were derived. With the validated code, the instability features of airfoil flow at two different angles of attack, corresponding to flow with a separation bubble and fully separated flow, were investigated at a chord-based Reynolds number of \(10^5\). The most unstable mode was found to be related to the wake instability with a dimensionless frequency close to one. For flow with a separation bubble, there is an instability plateau in the dimensionless frequency range from 2 to 5.5. After the plateau, with increased dimensionless frequency, the growth rate of the most unstable mode decreases. For fully separated flow, the plateau is narrower than for flow with separation bubble. After the plateau, with increased dimensionless frequency, the growth rate of the most unstable mode reaches a minimum and then increases once again. The growth rate of the upstream shear layer instability was larger than the downstream shear layer instability. A stationary global mode was also found for these two cases. For stalled flow, the spatial structure of the wake instability mainly exists downstream, while the upstream shear layer instability zone was upstream 1.32 chord lengths. Proper orthogonal decomposition was performed on the fully separated flow. The dominant frequency obtained using proper orthogonal decomposition is in agreement with the result from the bi-global stability method. Bi–global stability analysis, in comparison to proper orthogonal decomposition, better separates the wake and shear layer instability modes allowing improved insight for control strategies.

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