An oscillation effect on MHD radiative Casson fluid flows in an asymmetric channel through Group theoretical analysis

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An oscillation effect on MHD radiative Casson fluid flows in an asymmetric channel through Group theoretical analysis

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Abstract

The flow has been made by considering oscillation and radiation effects for the Magneto-hydrodynamics (MHD) Casson fluid model within an asymmetric wavy channel. Oscillation occurs during the flow by taking into account the pressure gradient across the ends of the channel. The governed mathematical statement is handled analytically by choosing Group theoretical method. The partial differential equations (PDEs) of the governed system are transformed into ordinary differential equations (ODEs) by calculating the symmetries. Further, the mathematical problem is concluded and showed the graphical results for the following emerging parameters: Casson fluid parameter $\beta$, wave length $\lambda$, Oscillation parameter $\omega$, Reynolds number $Re$, Hartmann number $M$, radiation parameter $R$, heat source/sink parameter $Q$ and Peclet number $Pe$. The magnitude of velocity profile $f(\eta)$ increased with an increase in $\beta$, $\lambda$, $Re$ and $K$. Due to the variations of $H$ and $\omega$, $f(\eta)$ is decreased. The temperature profile $\theta(\eta)$ is increased when the values of $Pe$, $Q$ and $R$ are increased.

Keywords: Asymmetric channel flow; Oscillation parameter; Hartmann number; Pressure gradient; Heat source/sink parameter.
1 Introduction

Lie Group approach or technique has played a vital role in all the branches of mechanics especially in fluid mechanics. Now a days, we deal with a lot of mathematical problems in fluid mechanics which are non linear. Symmetry method provides an expertise for the dealing of the non linear partial differential equations which are not handled traditionally. The author Cantwell [1] presented the different types of the symmetries as: Lie Baacklund symmetries, variational symmetries, potential and other non-local symmetries. Therefore, Sivasankaran et. al. [2] utilized the Lie Group approach for the heat and mass transfer in natural convection. Atalik et. al. [3] promoted the boundary layer theory for the wedge through similarity analysis. Bhuvaneswari et. al. [4] analyzed the effect of the chemical reaction in an inclined surface with heat and mass transfer by using Group theoretic technique. Afify [5] governed the results about the analysis of similarity for the effect of thermal diffusion, suction/injection, free convection, heat and mass transfer over a stretching sheet. Further, Afify [6] presented the symmetry analysis for the creeping flow of the non-Newtonian fluid with MHD and heat transfer effect. Tapanidis et. al. [7] analyzed the results of the second grade fluid flow by scaling Group of transformations. The study of different types of the channels for example: asymmetric and symmetric walls, asymmetric and symmetric wavy channel walls, grooved, corrugated channels, flexible wavy channels, converging or diverging channels and tapered channels etc. is incorporated by the researchers. A lot of work have been done in all these different types of channels for the field of fluid mechanics [8-12]. All these studies give us an obvious spoken outcome about the channels as these are impervious. But there are many studies for which this condition is not appropriate as: studying blood flows in capillaries, diffusion of the digestive food particles and suction/injection of the particles through the walls of the blood capillaries. Umavathi et. al. [13] analyzed fluid flow by taking a hori-
zontal composite channel in which half surface of the channel is filled with unsteady viscous porous medium. Pramanik [14] focused on the porous stretching sheet exponentially for the non-Newtonian Casson fluid with thermal radiation effects. Jha et. al. [15] chose the parallel plates with the effect of suction/injection and oscillatory fluid flow. Moreover, we could see Jha et. al. [16] extended his work [15] by added heat generating/absorbing effects to the fluid flow. On the other hand, Jha et. al. [17] presented another fluid flow model with these properties as: viscous dissipation, free convection and time dependent boundary conditions. Heat transfer analysis has a vital role in our daily life. The study of Magnetohydrodynamic (MHD) is taken by considering the magnetic field. In engineering field, examples are: the process of heating, drying, radiators, cooling circuits, nuclear reactors, polymer industry and metallurgy. These applications have congruent relation with the fluid mechanics. Furthermore, Rashidi et. al. [18] considered MHD and the heat generating effects in a rectangular porous medium along with the free convection. Although, Sheikholeslami et. al. [19] analyzed forced convection with the effects of non-uniform MHD. Again, Rashidi et. al. [20] presented MHD effects by using artificial neural network and particle swarm optimization algorithm along with these considerations as: entropy generation, and stretching rotating disk. Rashidi et. al. [21] investigate the MHD and slip effects for the rotating porous disk. Moreover, Rashidi et. al. [22] considered nanofluid with steady MHD flow. Researchers also have done a lot of work to utilize the properties of Magnetohydrodynamic (MHD) as in [23-30]. Casson fluid model is the most common non-Newtonian fluid model due to its yield stress rheological properties (shear thinning). Furthermore, when the shear stress is too much increased then Casson fluid become a Newtonian fluid. Practically, the flow behavior of pigment oil suspensions of printing ink is the best example of the Casson fluid other than blood. Jelly, honey, tomato sauce and concentrated fruit juices are also the common useful
examples of Casson fluid. Mukhopadhyay et. al. [31] analyzed the rheological properties of Casson fluid flow. Animasaun et. al. [32] utilized the method of homotopy for the problem in which Casson fluid flows with thermophysical property along exponentially stretching sheet. Mukhopadhyay et. al. [33] showed the Casson fluid flow passing the stretching sheet with the radiative permeable surface. Nadeem et. al. [34] presented MHD effects exponentially by considering Casson fluid model. Further, Ur-Rehman et. al. [35] gave the study of double stratification with the effects of mixed convection and chemical reaction for the Casson fluid model. Muthuraj et. al. [36] notified the heat transfer and MHD oscillatory effects in an asymmetric wavy channel. Lastly, pondering on the literature, we can reach on the result that the oscillation effect for the MHD, radiative and incompressible Casson fluid flow in an asymmetric wavy channel is still inculcate. Pressure gradient is an oscillatory type across the end of the channel for this type of flow. The governed non linear system of the partial differential equations are solved by using Lie theoretic method. After getting the system of ordinary differential equation from the Lie method, these equations are solved analytically. The system has the unique solution that correspond to the boundary conditions as we have \( t = 0 \) for these boundary equations. Prominent parameters in the momentum and energy equations are graphically presented.
2 Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$a, b(m)$</td>
<td>wave amplitudes</td>
<td>$Re$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$B_o(A/m)$</td>
<td>magnetic field intensity</td>
<td>$T_1(K)$</td>
<td>fluid temperature at upper wall</td>
</tr>
<tr>
<td>$C_p(J/kg.K)$</td>
<td>specific heat a constant pressure</td>
<td>$T_2(K)$</td>
<td>fluid temperature at lower wall</td>
</tr>
<tr>
<td>$d_1(m)$</td>
<td>width of the channel</td>
<td>$u(m/s)$</td>
<td>velocity component in the x-direction</td>
</tr>
<tr>
<td>$g(m/s^2)$</td>
<td>gravitational acceleration</td>
<td>$U(m/s)$</td>
<td>free stream velocity of the fluid</td>
</tr>
<tr>
<td>$H_1$</td>
<td>upper wall</td>
<td>$v(m/s)$</td>
<td>velocity component in the y-direction</td>
</tr>
<tr>
<td>$H_2$</td>
<td>lower wall</td>
<td></td>
<td></td>
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<tr>
<td>$k(W/m.K)$</td>
<td>thermal conductivity</td>
<td>$\rho(kg/m^3)$</td>
<td>fluid density</td>
</tr>
<tr>
<td>$k'(H/m)$</td>
<td>porous permeability</td>
<td>$\nu(m^2/s)$</td>
<td>kinematic viscosity</td>
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<tr>
<td>$K$</td>
<td>the ratio of Darcy number</td>
<td>$\sigma_e(S/m)$</td>
<td>electrical conductivity</td>
</tr>
<tr>
<td>$M$</td>
<td>Hartmann number</td>
<td>$\varphi$</td>
<td>phase difference</td>
</tr>
<tr>
<td>$Pe$</td>
<td>Peclet number</td>
<td>$\beta$</td>
<td>Casson fluid parameter</td>
</tr>
<tr>
<td>$P(k.Pa)$</td>
<td>fluid pressure</td>
<td>$\alpha$</td>
<td>term due to thermal radiation</td>
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<tr>
<td>$Q$</td>
<td>heat source/sink parameter</td>
<td>$\eta$</td>
<td>similarity parameter</td>
</tr>
<tr>
<td>$Q &lt; 0$</td>
<td>heat sink</td>
<td>$\theta$</td>
<td>dimensionless temperature</td>
</tr>
<tr>
<td>$Q &gt; 0$</td>
<td>heat source</td>
<td>$\lambda_1(m)$</td>
<td>wave length</td>
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<tr>
<td>$R$</td>
<td>Radiation number</td>
<td>$\omega(H)$</td>
<td>oscillation number</td>
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3 Mathematical Statement

Asymmetric upper and lower wavy channel walls are as follows in (Husaain et. al. [12] and Umavathi et. al. [13]):

\[ H_1 = d_1 + a_1 \cos \left( \frac{2\pi x}{\lambda_1} \right), \]  \hspace{1cm} (1)

\[ H_2 = -d_2 + b_1 \cos \left( \frac{2\pi x}{\lambda_1} + \frac{\pi}{2} \right). \]  \hspace{1cm} (2)
where the constants $a_1$, $b_1$, $d_1$, $d_2$ and $\varphi$ are satisfied by the $a_1^2 + b_1^2 + 2a_1b_1 \cos \varphi \leq (d_1 + d_2)^2$.

The phase difference $\varphi$ with the range $0 \leq \varphi \leq \pi$ and $\varphi = 0$ represents the symmetric channel with waves out of phase and $\varphi = \pi$ the waves are in phase. These asymmetric walls having constant temperatures $T_1$ and $T_2$ and with the following properties as: unsteady, incompressible, transversely magnetic field, radiative filled fluid, laminar flow and porous medium. The fluid flow is set due to oscillatory pressure gradient which is chosen across the ends of the walls. Makinde (2005) presented the physical aspects of the channel flow by taking both plates at rest. Now, let the Cartesian coordinate system $(x, y)$ where $x$ lies along the center and $y$ is the distance measured such that $y = H_1$ and $y = H_2$ (see Fig. 1).

![Flow diagram for Casson fluid model](image)

Fig. 1: Flow diagram for Casson fluid model

The walls of the channel are fixed with temperatures $T_1$ and $T_2$ so that the channel is placed so high enough to induce radiative heat transfer. Applied magnetic field and Darcy’s resistance are also chosen to enhance the properties of the Casson fluid flows. The Casson rheological model and the constitutive equations can be used as follows in (Animasaun et...
al. [32], Mukhopadhyay et. al. [33] and Nadeem et. al. [34]):

\[
\tau_{ij} = \begin{cases} 
2 \left( \mu_B + \frac{p_y}{\sqrt{2} \pi} \right) e_{ij}, & \pi > \pi_c \\
2 \left( \mu_B + \frac{p_y}{\sqrt{2} \pi} \right) e_{ij}, & \pi < \pi_c
\end{cases}
\]

(3)

where, \( \pi = e_{ij} e_{ij} \) the product of the component of the rate of deformation with itself, \( \pi_c \) a critical value of this product based upon Casson fluid model, \( e_{ij} \) the \((i, j)\)th component of the deformation rate, \( \tau_{ij} \) Cauchy stress tensor, \( \mu_B \) plastic dynamic viscosity and \( p_y \) yield stress.

After using the constitutive laws, we have to break through a light on the governed boundary layer equations as (Mukhopadhyay et. al. [31]):

\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} = -\frac{1}{\rho \tilde{d}} \frac{dp}{dx} + \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \frac{\nu}{k} \tilde{u} - \frac{\sigma_e B_0^2 \tilde{u}}{\rho},
\]

(4)

\[
\frac{\partial T}{\partial \tilde{t}} = \frac{k}{\rho C_P \tilde{d} \tilde{y}^2} \frac{\partial^2 T}{\partial \tilde{y}^2} + 4 \frac{\rho \tilde{C}_P}{\rho C_P} \left( T - T_1 \right) + \frac{Q_u}{\rho C_P} \left( T - T_2 \right),
\]

(5)

with the resulted boundary conditions

\[ \tilde{u} = 0, \ T = T_1 \quad \text{on} \quad \tilde{y} = H_1, \]

(6)

\[ \tilde{u} = 0, \ T = T_2 \quad \text{on} \quad \tilde{y} = H_2. \]

(7)

Now, we use the following dimensionless quantities for the system of Eqs. (4) – (5) along with these boundary conditions in Eqs. (6) – (7).

\[
x = \frac{x}{\lambda_1}, \quad y = \frac{\tilde{y}}{d_1}, \quad u = \frac{\tilde{u}}{U}, \quad t = \frac{\tilde{t}}{d_1}, \quad h_1 = \frac{H_1}{d_1}, \quad h_2 = \frac{H_2}{d_1},
\]

(8)

\[
d = \frac{d_2}{d_1}, \quad a = \frac{a_1}{d_1}, \quad b = \frac{b_1}{d_1}, \quad p = \frac{d_1^2 \tilde{p}}{\rho \nu \lambda_1 U}.
\]

Let us consider the pressure gradient as:

\[
-\frac{\partial \tilde{p}}{\partial x} = \lambda e^{\omega t},
\]

(9)

and the system of equations after using Eqs. (8) – (9)

\[
\text{Re} \frac{\partial u}{\partial t} = \lambda e^{\omega t} \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} \left( 1 + \frac{2}{\alpha} \right) u,
\]

(10)
\[ P_{e} \frac{\partial \theta}{\partial t} = \frac{\partial^{2} \theta}{\partial y^{2}} + (R + Q) \theta, \quad (11) \]

with these dimensionless boundary conditions

\[ u = 0, \quad \theta = 0 \quad \text{on} \quad y = h_{1}, \quad (12) \]

\[ u = 0, \quad \theta = 1 \quad \text{on} \quad y = h_{2}, \quad (13) \]

where

\[ h_{1} = 1 + a \cos (2\pi x). \quad (14) \]

and

\[ h_{2} = -d - b \cos(2\pi x + \varphi), \quad (15) \]

are the dimensionless form of the Eqs. (1) - (2). Here, \( a, b, d \) and \( \varphi \) satisfied the condition

\[ a^{2} + b^{2} + 2ab \cos \varphi \leq (1 + d)^{2}. \]

The Reynolds number, Hartmann number, Darcy number, the ratio of Darcy number, the temperature, Peclet number, radiation parameter and heat source/sink parameter are as follow:

\[
\begin{align*}
\text{Re} & = \frac{Ud_{1}}{v}, \quad M^{2} = \frac{\sigma_{e}B_{0}^{2}d_{1}^{2}}{\rho v}, \quad Da = \frac{k'}{d_{1}^{2}}, \quad K = \frac{1}{Da}, \quad \theta = \frac{T - T_{2}}{T_{1} - T_{2}}, \\
P_{e} & = \frac{\rho C_{p}d_{1}U}{k}, \quad R = \frac{4\alpha^{2}d_{1}^{2}}{k}, \quad Q = \frac{Q_{w}d_{1}^{2}}{k}. \quad (16)
\end{align*}
\]

4 Symmetry Group of transformations and its Lie Algebra

Lie Group theory is an inevitable method as it is recommendable to every type of the differential equation. As one application is used to transform the partial differential equations to the ordinary differential equations in a very smooth way. Lie algebra theory utilized to find out the symmetries of the system of differential equation (DE). These equations have different
types of the symmetries. After calculation, we can obtain different types of the solutions by using these symmetries. Here, we have done a work to calculate Group symmetries admitted by Eqs. (10) – (11). After calculating these Lie Group symmetries, we will solve these Eqs. (10) – (11) to get an analytic solution according to the boundary conditions (12) – (13).

Taking a one parameter Lie group of infinitesimal transformations for the governed system of equations:

\[
t^* = t + \varepsilon \xi_1 (t, y, u, \theta) + o(\varepsilon^2),
\]
\[
y^* = y + \varepsilon \xi_2 (t, y, u, \theta) + o(\varepsilon^2),
\]
\[
u^* = u + \varepsilon \phi_1 (t, y, u, \theta) + o(\varepsilon^2),
\]
\[
\theta^* = \theta + \varepsilon \phi_2 (t, y, u, \theta) + o(\varepsilon^2).
\]

In Eq. (17), \( \varepsilon \) is a group parameter of the infinitesimal transformations. Eqs. (10) – (13) remain unchanged after applying Eq. (17). The symmetry group of infinitesimal generator can be written as

\[
\widetilde{V} = \xi_1 \partial_t + \xi_2 \partial_y + \phi_1 \partial_u + \phi_2 \partial_\theta.
\]

The corresponding one-parameter group \( \exp(\varepsilon \widetilde{V}) \) is a symmetry group of the system. The prolongation for the governing system can be written as:

\[
P^{(2)}\widetilde{V} = \widetilde{V} + \phi_1^t \frac{\partial}{\partial t} + \phi_1^y \frac{\partial}{\partial y} + \phi_1^u \frac{\partial}{\partial u} + \phi_1^\theta \frac{\partial}{\partial \theta} + \phi_2^t \frac{\partial}{\partial t} + \phi_2^y \frac{\partial}{\partial y} + \phi_2^u \frac{\partial}{\partial u} + \phi_2^\theta \frac{\partial}{\partial \theta}.
\]

Now utilizing the \( P^{(2)}\widetilde{V} \) on the system of Eqs. (10) – (11) by substituted the values of \( \phi_1^t, \phi_1^y, \phi_2^t \) etc. into the Eq. (19). After equating, we get the coefficients of various monomials. The equations invariance under the transformations of infinitesimal leads to the following expressions of the functions \( \xi_1, \xi_2, \phi_1 \) and \( \phi_2 \).

\[
\xi_1 = k_1; \quad \xi_2 = k_2; \quad \phi_1 = k_3; \quad \phi_2 = k_4 \theta + k_5.
\]
where, $k_1$, $k_2$, $k_3$ and $k_4$ are arbitrary parameters and these parameters represent the scaling transformations. We get $V_1$ by choosing $k_1 = 1$, $k_2 = k_3 = k_4 = 0$ in Eq. (18). Similarly, we will get the values of $V_2$, $V_3$, $V_4$ and $V_5$. We will consider the basis of the corresponding Lie algebra as:

$$V_1 = \partial_t, \ V_2 = \partial_y, \ V_3 = \omega u \partial_u, \ V_4 = \theta \partial_\theta, \ V_5 = \partial_\theta. \quad (21)$$

## 5 Group Invariants solutions

We will have the following classifications of the symmetry solutions for the mathematical governed problem. The cases of the optimal system are:

$I_1$) $V_1 + V_3,$ $I_7$) $V_1 + V_2 + V_3,$ $I_{13}$) $V_2 + V_3 + V_5$

$I_2$) $V_1 + V_4,$ $I_8$) $V_1 + V_2 + V_4,$ $I_{14}$) $V_1 + V_2 + V_3 + V_4$

$I_3$) $V_1 + V_5,$ $I_9$) $V_1 + V_2 + V_5,$ $I_{15}$) $V_1 + V_2 + V_3 + V_5$

$I_4$) $V_2 + V_3,$ $I_{10}$) $V_1 + V_3 + V_4,$ $I_{16}$) $V_i \ i = 1, 2, 3, 4, 5.$

$I_5$) $V_2 + V_4,$ $I_{11}$) $V_1 + V_3 + V_5,$

$I_6$) $V_2 + V_5,$ $I_{12}$) $V_2 + V_3 + V_4,$

We will find out the analytically unique solution of the following cases:

$I_1$) In this case, symmetries and functions are

$$\eta = y, \ u = e^{\omega t} f(\eta), \ \theta = \theta(\eta). \quad (22)$$

Using Eq. (22), the system of Eqs. (10) – (13) with the boundary conditions are:

$$\text{Re} \ i \omega f(\eta) - \left( 1 + \frac{1}{\beta} \right) f''(\eta) + \left( M^2 + \frac{1}{K} \right) f(\eta) + \lambda = 0, \quad (23)$$

$$\phi''(\eta) + (R + Q) \phi(\eta) = 0. \quad (24)$$
\[ f(h_1) = 0, \quad \theta(h_1) = 0, \quad \text{at} \quad \eta = h_1, \quad (25) \]
\[ f(h_2) = 0, \quad \theta(h_2) = 1, \quad \text{at} \quad \eta = h_2. \quad (26) \]

After calculation, we can get an analytically unique solution.

\( I_2 \) In this case, symmetries and functions are

\[ \eta = y, \quad u = f(\eta), \quad \theta = e^t\theta(\eta). \quad (27) \]

Using Eq. (27), the system of Eqs. (10 \(- 13) are as:

\[ \left( 1 + \frac{1}{\beta} \right) f''(\eta) - \left( M^2 + \frac{1}{K} \right) f(\eta) + \lambda e^{\omega t} = 0, \quad (28) \]
\[ Pe e^t\theta(\eta) - e^t\theta''(\eta) - (R + Q) \theta(\eta) = 0. \quad (29) \]

The system of equations is not in the appropriate form. Therefore the resulting system according to the boundary conditions is not having a unique solution.

\( I_3 \) In this case, symmetries and functions are

\[ \eta = y, \quad u = f(\eta), \quad \theta = t + \eta. \quad (30) \]

After pondering for the solution, the resulting system according to the boundary conditions is not having unique solution.

\( I_4 \) In this case, symmetries and functions are

\[ \eta = t, \quad u = e^{\omega t} f(\eta), \quad \theta = \theta(\eta). \quad (31) \]

After pondering for the solution, the resulting system according to the boundary conditions is not having unique solution.

\( I_5 \) In this case, symmetries and functions are

\[ \eta = t, \quad u = f(\eta), \quad \theta = e^t \theta(\eta). \quad (32) \]
In this case, these transformations are not suitable for the unique solution of the system.

$I_6$) In this case, symmetries and functions are

$$\eta = t, \quad u = f(\eta), \quad \theta = y + \eta. \quad (33)$$

In this case, these transformations are not suitable for the unique solution of the system.

$I_7$) In this case, symmetries and functions are

$$\eta = y - t, \quad u = e^{\omega t} f(\eta), \quad \theta = \theta(\eta). \quad (34)$$

In this case, these transformations are not suitable for the unique solution of the system.

$I_8$) and $I_9$) The similarity variables and functions are not possible.

$I_{10}$) In this case, symmetries and functions are

$$\eta = y, \quad u = e^{\omega t} f(\eta), \quad \theta = e^t \theta(\eta). \quad (35)$$

Using Eq. (35), the system of Eqs. (10) – (13) are as:

$$\text{Re} \ i \omega f(\eta) - \left(1 + \frac{1}{\beta}\right) f''(\eta) + \left(M^2 + \frac{1}{K}\right) f(\eta) + \lambda = 0, \quad (36)$$

$$\theta''(\eta) + (R + Q - Pe) \theta(\eta) = 0, \quad (37)$$

and the boundary conditions are

$$f(h_1) = 0, \quad \theta(h_1) = 0, \quad \text{at} \quad \eta = h_1, \quad (38)$$

$$f(h_2) = 0, \quad \theta(h_2) = 1, \quad \text{at} \quad \eta = h_2. \quad (39)$$

After calculation, we can get the required unique analytic solution of the system (36) – (39) with all the emerging parameters.

$I_{11}$) In this case, symmetries and functions are

$$\eta = y, \quad u = e^{\omega t} f(\eta), \quad \theta = \theta(\eta). \quad (40)$$
In this case, these transformations are not suitable for the unique solution of the system.

\(I_{12})\) In this case, symmetries and functions are

\[ \eta = t, \quad u = e^{i\omega t} f(\eta), \quad \theta = e^{\theta(t)}(\eta). \] (41)

In this case, these transformations are not suitable for the unique solution of the system.

\(I_{13})\) In this case, symmetries and functions are

\[ \eta = t, \quad u = e^{i\omega y} f(\eta), \quad \theta = e^{\theta(t)}(\eta). \] (42)

In this case, these transformations are not suitable for the unique solution of the system.

\(I_{14})\) In this case, symmetries and functions are

\[ \eta = y - t, \quad u = e^{i\omega t} f(\eta), \quad \theta = e^{\theta(t)}(\eta). \] (43)

In this case, these transformations are not suitable for the unique solution of the system.

\(I_{15})\) In this case, symmetries and functions are

\[ \eta = y - t, \quad u = e^{i\omega t} f(\eta), \quad \theta = t + \eta. \] (44)

In this case, these transformations are not suitable for the unique solution of the system.

\(I_{16})\) \(V_1\)

In this case, symmetries and functions are

\[ \eta = t, \quad u = f(\eta), \quad \theta = \theta(t). \] (45)

In this case, these transformations are not suitable for the unique solution of the system. For \(V_2\), symmetries and functions are

\[ \eta = y, \quad u = f(\eta), \quad \theta = \theta(t). \] (46)

This case is also not suitable for the system. For \(V_3, V_4, \) and \(V_5\), we cannot construct the symmetries and functions. We can see that the \(I_1\) and \(I_{10}\) have unique solutions but the most
appropriate case is the $I_{10}$. In the $I_{10}$ case, Eqs. (36) – (37) are the ordinary differential equations which leads to the solution of the system of Eqs. (10) – (13). Eqs. (36) – (39) are solved analytically by the DSolve command in the MATHEMATICA.

6 Solutions and physical quantities

The differential form of the Eq. (36) can be written as:

$$Z^2 f''(\eta) - m^2 f(\eta) = -\lambda, \quad (47)$$

where $Z^2 = \left(1 + \frac{1}{\beta}\right)$ and $m^2 = (M^2 + \frac{1}{K} + \omega \text{Re})$ with these boundary conditions

$$f(\eta) = 0 \quad \text{on} \quad \eta = h_1 \text{ & } \eta = h_2. \quad (48)$$

The solution is

$$f(\eta) = -\frac{4\lambda}{m^2 e^{h_1/h_2^m + h_2/h_1^m}} \sinh m \left(\frac{h_1 - \eta}{2Z}\right) \sinh m \left(\frac{h_2 - \eta}{2Z}\right). \quad (49)$$

One can obtain easily

$$f(\eta) = \left(\frac{\lambda}{m^2} + \frac{\lambda}{m^2} \frac{\sinh m(h_2 - y) + \sinh m(y - h_1)}{\sinh m(h_1 - h_2)}\right), \quad (50)$$

and

$$u(y, t) = f(\eta) \ast e^{i\omega t}.$$  

after taking $Gr \rightarrow 0$ in (Muthuraj et. al. (2010) and $Z^2 = 1$ means $\beta \rightarrow \infty$) for viscous case in Eq. (47).

The differential form of the Eq. (37) is

$$\theta''(\eta) + n^2 \theta(\eta) = 0, \quad (51)$$

where $n^2 = (R + Q - Pe)$ with these boundary conditions

$$\theta(\eta) = 1 \quad \text{on} \quad \eta = h_1 \text{ & } \theta(\eta) = 0 \quad \text{on} \quad \eta = h_2. \quad (52)$$
The solution is
\[ \theta(\eta) = \frac{\sin n(\eta - h_2)}{\sin n(h_1 - h_2)}. \]  
(53)

The skin friction coefficient across the asymmetric wavy channel walls is defined as:
\[ C_f = \frac{\tau_w}{\rho U^2}, \]  
(54)

where
\[ \tau_w = \left( \mu_B + \frac{p_y}{\sqrt{2\pi}} \right) \frac{\partial \bar{u}}{\partial \bar{y}}, \]  
(55)

The rate of heat transfer across the asymmetric wavy channel walls is defined as:
\[ Nu = \frac{\bar{x}}{T_1 - T_2} \left( - \frac{\partial T}{\partial \bar{y}} \right)_{\bar{y}=H_1,H_2}. \]  
(56)

After using Eq. (16) in Eqs. (54) – (56), the physical quantities skin friction coefficient and the Nusselt number are:
\[ \text{Re} C_f = \left( 1 + \frac{1}{\beta} \right) (f'(\eta))_{\bar{y}=h_1,h_2}, \]  
(57)
\[ \text{Re}(\text{Re}_x)^{-1} Nu = -(\theta'(\eta))_{\bar{y}=h_1,h_2}. \]  
(58)

The local Reynolds number is
\[ \text{Re}_x = \frac{U \bar{x}}{\nu}. \]  
(59)

7 Graphical results and discussion

Here, the graphical effects for the incompressible, MHD, radiative and oscillatory pressure gradient across the asymmetric wavy walls are incorporated. The graphs of the velocity and temperature profiles are presented to show the effects of various parameters. In Figs. 2-10, we set \( a = 0.3, \ b = 0.4, \ d = 0.5, \ \omega = 0.1, \ \varphi = 0.1 \) and \( x = -\pi \) and \( 0.1 \leq \text{Re}, \ K, \ M, \ Pe, \ Q, \ R, \ \beta, \ \lambda \leq 1 \). One can analyze that the range of the Reynolds number...
is very low. It means that viscous forces are dominant which leads to the non-Newtonian Casson fluid flow effects. Fig. 2 depicts the effect of Casson fluid parameter on the velocity profile parabolically. It seems to be very nice form of the parabola which predicts that it has the maximum fluid transfer at the ends of the walls rather than at the center of the channel. The velocity profile increased as Casson fluid parameter is increased. In the Fig. 3, the permeability has also the increasing effects on the velocity profile. Permeability variations at the ends of the walls do not exist but at the center of the channel it allows the fluid for the maximum transfer. Fig. 4 has increasing behaviour not only at the center but also at the ends of the walls when the values of the wave length is increased. Fig. 5 analyzed the result of velocity profile for the variation of magnetic field parameter. Velocity profile has decreasing behaviour with an increase in Hartmann number. It is because of the effect of the Lorentz force. This force turned to diminish the behaviour of the motion of the fluid. Figs. 6 presented the decreasing behaviour for the velocity profile with an increase in oscillation parameter. The Reynolds number showed the increasing behaviour for the velocity profile in the Fig. 7. Fig. 8 sketched for the temperature profile in which temperature profile having decreasing behaviour with an increase in Peclet number. Fig. 9 showed the temperature profile for the various values of heat source/sink parameter. It is obvious from the Fig. 9, temperature profile is increased. It allow the heat to transfer at the high rate. Fig. 10 depict the behaviour of the radiation parameter. It also has increasing effects on the temperature profile and rise the heat transfer rate. The Tables. 1-2 presented the values of the skin friction coefficient and the Nusselt number respectively for the upper wall. These tables have the same values with difference of the negative sign for the lower wall. It showed the channel asymmetric property for the various values of the physical parameters, namely \( \beta \), \( \text{Re} \), \( \omega \), \( \lambda \), \( \text{Pe} \), \( R \), and \( Q \). The values of the \( C_f \) for the physical parameter \( \beta \), are decreasing. On the other hand,
for the Re, \( \omega \) and \( \lambda \) the values of the \( C_f \) are increasing. The values of the Nusselt number are decreasing for the \( Pe \). The values of the \( R \) and \( Q \) having increasing effect on the Nusselt number.

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**Fig. 2:** Profile of \( f(\eta) \) for \( \beta \)

**Fig. 3:** Profile of \( f(\eta) \) for \( K \)

**Fig. 4:** Profile of \( f(\eta) \) for \( \lambda \)

**Fig. 5:** Profile of \( f(\eta) \) for \( M \)
Fig. 6: Profile of $f(\eta)$ for $\omega$

Fig. 7: Profile of $f(\eta)$ for $Re$

Fig. 8: Profile of $\theta(\eta)$ for $Pe$

Fig. 9: Profile of $\theta(\eta)$ for $Q$
Fig. 10: Profile of $\theta(\eta)$ for $R$
Table. 1 Effect of various parameters on the skin friction coefficient

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<th>$\lambda$</th>
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Table. 2 Effect of various parameters on $\theta'(\eta)$

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8 Concluding remarks

This article analytically presented the oscillation effects with magnetic field, permeability and radiation in an asymmetric wavy channel. The partial differential equations are converted into the ordinary differential equation by the method of Lie group algebra. The system of ordinary differential equations corresponding to the boundary conditions has a unique solution analytically. Solution is analyzed through the graphs by discussing various ongoing
parameters. The main points for these graphical effects are:

- Velocity profile $f(\eta)$ is a parabola. In all the velocity profile $f(\eta)$ graphical results, it has high magnitude along the channel centerline. On the other hand it has very low magnitude at the ends of the channel walls. We are noticed that when the Casson fluid parameter $\beta$, Permeability parameter $K$, wavelength $\lambda$ and Reynolds number $Re$ are increased then the magnitude of the velocity profile $f(\eta)$ is increased. On the other hand the Hartmann number $M$ and Oscillation parameter $\omega$ has decreasing effects on the magnitude of the velocity profile $f(\eta)$.

- The oscillatory flow has increased the magnitude of the heat transport of a system. As we see that the temperature profile is increased when the values of the Peclet number $Pe$, Heat source/sink parameter $Q$ and the radiation parameter $R$ are increased. It was seen experimentally and analytically that a large amount of heat is transported due to these effects.

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## References


