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Accurate analytic solution for ideal boson gases in a highly anisotropic two-dimensional harmonic trap

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Motivated by quantum statistical mechanics, we have proposed an accurate analytical solution to the problem of Bose-Einstein condensation (BEC) of ideal bosons in a two-dimensional anisotropic harmonic trap. The study reveals that the number of noncondensed bosons is characterized by an analytical function, which relates to a series expansion of \( q \)-digamma functions in mathematics. The \( q \)-digamma function is a function of temperature, boson number, and anisotropic parameter. The analytical solution describes fully the experimental results in the BEC of ideal bosons in a two-dimensional anisotropic harmonic trap. We derive the analytical expressions of the critical temperature and the condensate fraction in the thermodynamic limit. The first main conclusion is that for a fixed temperature and boson number, there is a critical anisotropic parameter, which is the precise onset of BEC in this harmonically trapped two-dimensional system. The second main conclusion is that the critical temperature in a two-dimensional anisotropic harmonic trap is larger than that in a two-dimensional isotropic harmonic trap.

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I. INTRODUCTION

The boson systems in which the boson number is conserved possess a common quantum property that a macroscopically large number of bosons tend to occupy the ground state of a quantum mechanical system. This phenomenon is called Bose-Einstein condensation (BEC). It is not until 1995 that the three groups observed the BEC of trapped ultracold alkali atoms [1–3]. Bose-Einstein condensation also applies to quasiparticles of Bose type in solids. Recently, the BEC of excitons [4, 5], exciton-polaritons [6, 7], and magnons [8, 9] has been observed also in several solid-state systems. On the other hand, there has been an intense interest of research into the BEC of photons in low-dimensional optical microcavities [10]. Thermalization of a two-dimensional photon gas has been observed experimentally [11] in a dye-filled curved-mirror microcavity. Weitz and colleagues have observed the BEC of two-dimensional photons in a dye-filled curved-mirror microcavity [12, 13]. In this paper, we investigate the BEC of ideal boson gases in a two-dimensional anisotropic harmonic trap. The accurate analytical solution of the BEC state of such an ideal boson gas is elaborated.

Originally, the BEC of ultracold alkali atoms was observed in three-dimensional harmonic traps. This observation has stimulated much interest in the BEC of ultracold alkali atoms in two-dimensional harmonic traps. The two-dimensional physics has been always a fascinating subject since the advent of quantum mechanics [14]. The two-dimensional character lead to incredibly rich physics. Some exact analytical solutions in two-dimensional systems can be obtained using specific methods. The first theoretical study pointed out that ideal ultracold Bose atoms in a two-dimensional isotropic harmonic trap can form a Bose-Einstein condensate [15]. The BEC of sodium atoms in two dimensions has been realized in a magneto-optical trap [16]. The BEC in such effectively two-dimensional systems demonstrates new features: well below the transition temperature \( T_c \), the equilibrium state is a true condensate, whereas at intermediate temperatures \( T < T_c \), one has a quasicondensate (condensate with fluctuating phase). On the other hand, for a two-dimensional boson system with interactions in a homogeneous space, there is no BEC but the Kosterlitz-Thouless transition, which exhibits quasi-long range order and superfluidity. The Kosterlitz-Thouless theory associates this phase transition with the emergence of a topological order. The Kosterlitz-Thouless transition has been observed in an ultracold Bose atomic gas in a homogeneous quasi-two-dimensional optical trap [17].

The study subject is an ideal boson gas in a two-dimensional anisotropic harmonic trap. In fact, a noninteracting Bose-Einstein condensate can be prepared by means of a Feshbach resonance [18]. In order to treat an ideal boson gas in a two-dimensional anisotropic harmonic trap, in all the references [19] one utilizes the following manners: (1) the computer simulation is employed, (2) the thermodynamic limit is used, and (3) the quasiclassical approximation that the discrete level structure is approximated by a continuous density of states is adopted. In order to avoid the above manners, we present an accurate analytic solution for an ideal boson gas in a two-dimensional anisotropic harmonic trap, in all the references [19] one utilizes the following manners: (1) the computer simulation is employed, (2) the thermodynamic limit is used, and (3) the quasiclassical approximation that the discrete level structure is approximated by a continuous density of states is adopted. In order to avoid the above manners, we present an accurate analytic solution for an ideal boson gas in a two-dimensional anisotropic harmonic trap. Our solution is effective for arbitrary temperature, boson number and anisotropic parameter. It is found that the analytic solution relates to a series expansion of \( q \)-digamma functions in mathematics. The \( q \)-digamma function is a function of temperature, boson number and anisotropic parameter. We derive the analytic expressions of the critical temperature and the condensate fraction in the thermodynamic limit. The first main conclusion is that for a fixed temperature and boson number, there is a critical anisotropic parameter, which is the precise onset of BEC in this harmonically trapped two-dimensional system. The second main conclusion is that the critical temperature in a two-dimensional anisotropic harmonic trap is larger than that in a two-dimensional isotropic harmonic trap.
The remainder of the present paper is organized as follows. Section II gives the quantum description of single Bose atoms in a two-dimensional anisotropic harmonic trap. In Sec. III, we present an accurate analytic solution for an ideal boson gas in a two-dimensional anisotropic harmonic trap. In Sec. IV, we present an accurate analytic solution for phase transitions of harmonically trapped ideal bosons in the thermodynamic limit. The omnibus discussion is presented in Sec. V.

II. QUANTUM DESCRIPTION OF SINGLE BOSE ATOMS IN A TWO-DIMENSIONAL ANISOTROPIC HARMONIC TRAP

Let the $z$ axis be a symmetric axis of a harmonic potential for trapping an ultracold atomic gas. We describe a cold-atom condensate of dilute density $n$ in this harmonic trap by four length scales: the transverse radius $R_x$, the axial radius $R_z$, the scattering length $a$, and the length $\xi = (4\pi n a)^{-1/2}$. In most experiments on Bose-Einstein condensates, the radii and lengths satisfy the inequality $R_x, R_z \gg \xi \gg a$. In this case, a Bose-Einstein condensate is three-dimensional and is investigated by the various methods [20]. In this paper, we consider a special case when the healing length is larger than $R_x$. Since then the condensate becomes restricted to two dimensions, a qualitatively different behavior of BEC is expected. New phenomena that have been observed in this case are, for example, quasi-condensates with a fluctuating phase and the Kosterlitz-Thouless phase transition [17]. As a result, the study subject is an oval-shaped two-dimensional condensate with $R_x > \xi > R_z \gg a$.

Let us take into account single Bose atoms with zero spin. $m$ is the mass of Bose atoms. Single Bose atoms move in a two-dimensional anisotropic harmonic potential. In quantum mechanics, the canonical momentum of a Bose atom at position $r$ is given by

$$\mathbf{p} = -i\hbar \left( e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} \right),$$

where $e_x$ and $e_y$ are the unit vectors along the $x$ and $y$ axes, respectively, and $\hbar$ is Planck’s constant reduced. In nonrelativistic quantum mechanics, the Hamiltonian of a Bose atom in a two-dimensional anisotropic harmonic potential is given by

$$\hat{H} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} m \left( \omega_x^2 x^2 + \omega_y^2 y^2 \right),$$

where $\omega_x$ and $\omega_y$ are the angular frequencies of the trap along the $x$ and $y$ axes, respectively. The stationary state of a Bose atom at position $r$ is described by the wave function $\Psi_n(r) = \Psi_n(x)\Psi_n(y)$, where $n_x$ and $n_y$ are two quantum numbers. The wave functions $\Psi_n(x)$ and $\Psi_n(y)$ satisfy the Schrödinger equations,

$$\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega_x^2 x^2 \right) \Psi_n(x) = E_n \Psi_n(x),$$

$$\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_y^2 y^2 \right) \Psi_n(y) = E_n \Psi_n(y),$$

where $E_n = E_{n_x} + E_{n_y}$ is the energy eigenvalue of a Bose atom. Equations (3) and (4) are the one-dimensional harmonic oscillator equations and have the oscillator levels

$$E_n = (n_x + \frac{1}{2}) \hbar \omega_x + (n_y + \frac{1}{2}) \hbar \omega_y,$$

$$n_x, n_y = 0, 1, 2, \ldots, n_y = 0, 1, 2, \ldots. \tag{5}$$

The eigenfunctions $\Psi_n(x)$ and $\Psi_n(y)$ corresponding to the oscillator levels are given by

$$\Psi_n(x) = \frac{1}{\sqrt{n_x!}} e^{-\frac{x^2}{2a_x^2}} \left( \frac{\hbar}{\sqrt{2m} \omega_x} \right)^{n_x/2} H_n \left( \frac{x}{a_x} \right), \tag{6}$$

$$\Psi_n(y) = \frac{1}{\sqrt{n_y!}} e^{-\frac{y^2}{2a_y^2}} \left( \frac{\hbar}{\sqrt{2m} \omega_y} \right)^{n_y/2} H_n \left( \frac{y}{a_y} \right), \tag{7}$$

where $a_x = \sqrt{\hbar/m \omega_x}$ is a length characterizing the spread of the wave function in the $x$ direction and $a_y = \sqrt{\hbar/m \omega_y}$ is a length characterizing the spread of the wave function in the $y$ direction. $H_n(\xi)$ is the Hermite polynomial of the $n$th degree in $\xi$. The states described by the wave function $\Psi_n$ are called oscillator states.

III. QUANTUM STATISTICAL DESCRIPTION OF A FINITE NUMBER OF IDEAL BOSONS

Since the boson number in a two-dimensional anisotropic harmonic trap is conserved, the two-dimensional boson gas has a nonzero chemical potential $\mu$. At arbitrary temperature $T$, $N_\mu$ signifies the mean occupation number of the $\mu$th oscillator level. In the standard way, we find that

$$N_\mu = \frac{1}{e^{E_\mu/k_B T} - 1},$$

where $k_B$ is Boltzmann’s constant. Equation (8) is the well-known Bose-Einstein distribution. The chemical potential $\mu$ must obey the law of conservation of particle number:

$$\sum_n N_n = N,$$
where \( \beta = 1/k_B T \). Here \( z \) is the fugacity defined by \( z = \exp(\beta \mu^*) \) and we have introduced an effective chemical potential \( \mu^* = \mu - \frac{1}{2} \hbar \omega_x + \omega_x \). The ground state of the system corresponds to \( n_x = n_y = 0 \). Because the effective chemical potential includes the zero-point energy, the energy of the ground state is zero. From Eq. (10), the number of Bose atoms in the ground state is \( N_0 = z/(1 - z) \). \( N_0 \) diverges as \( z \to 1 \), and therefore the BEC happens at \( z = 1 \). As a result, \( N_0 \) is called the number of condensed atoms. After putting Eq. (10) into Eq. (9) and completing the summation over \( n \), then one obtains

\[
\sum_{j=1}^{\infty} z^j \frac{1}{(1 - q^j_x)(1 - q^j_y)} = N, \tag{11}
\]

where \( q_x = \exp(-\beta \hbar \omega_x) \) and \( q_y = \exp(-\beta \hbar \omega_y) \).

Employing the identity

\[
\frac{z}{1 - z} = \sum_{j=1}^{\infty} z^j,
\]

we can rewrite Eq. (11) as

\[
\sum_{j=1}^{\infty} \frac{z^j}{(1 - q^j_x)(1 - q^j_y)} = N. \tag{12}
\]

Under the assumption of \( \omega_x \geq \omega_y \) and after a simple arrangement, Eq. (13) is reduced to

\[
\sum_{j=1}^{\infty} \frac{z^j q^j_x}{(1 - q^j_x)(1 - q^j_y)} = N. \tag{14}
\]

In the \( q \)-analog theory in mathematics, the two infinite series in Eq. (14) are \( q \)-series. The first \( q \)-series converges and its analytic expression can be derived as

\[
F_q(x_a) = \sum_{j=1}^{\infty} \frac{z^j q^j_x}{(1 - q^j_x)} = \frac{\ln(1 - q_x) + \psi_q(x_a)}{\ln q_x}, \tag{15}
\]

where the definition \( z = \exp(\beta \mu^*) \) is utilized and \( x_a = 1 - \mu^*/\hbar \omega_x \). \( \psi_q(x) \) is called the \( q \)-digamma function and is defined by \( \psi_q(x) = d[\ln \Gamma_q(x)]/dx \). Here \( \Gamma_q(x) \) is called the \( q \)-gamma function and is defined by

\[
\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^n x}, \tag{16}
\]

where \( |q| < 1 \) and \( x \neq 0, -1, -2, \ldots \). Jackson introduced the \( q \)-gamma function \([21]\) and Krattenthaler and Srivastava introduced the \( q \)-digamma function \([22]\). Recently, scientists have widely applied the \( q \)-gamma function and the \( q \)-polygamma function \([23]\).

The second \( q \)-series in Eq. (14) converges too but its analytic expression is hard to get. We want to derive its analytic expression, and then we can rewrite the second \( q \)-series in Eq. (14) as

\[
G_q(x_a) = \sum_{j=1}^{\infty} \frac{q_x^j (x_a-1+\omega_y/\omega_x)}{(1 - q^j_x)(1 - q^j_y)} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{q_x^j (x_a-1+k\omega_y/\omega_x)}{(1 - q^j_x)(1 - q^j_y)}. \tag{17}
\]

With the aid of Eq. (15), the analytic expression of Eq. (17) can be derived as

\[
G_q(x_a) = \frac{k_c \ln(1 - q_x) + 1}{\ln q_x} \sum_{k=1}^{\infty} \psi_q(x_a-1+k\omega_y/\omega_x), \tag{18}
\]

where the upper limit \( \infty \) of summation is replaced by an upper cutoff \( k_c \) and in practice we set \( k_c = 200 \). The numerical simulation displays that the upper cutoff \( k_c = 200 \) is sufficient for a high precision calculation.

Since the fugacity \( z = \exp(\beta \mu^*) \) contains the temperature \( T \), the fugacity \( z \) does not represent the chemical potential \( \mu^* \) by much and so \( z \) is not a good physical quantity. The quantity \( x_a \) represents the chemical potential \( \mu^* \) a lot and so \( x_a \) is a good physical quantity. As a result, the quantity \( x_a \) is called the reduced chemical potential. In the same way, the quantity \( q_x \) represents the temperature \( T \) a lot and so \( q_x \) is a good physical quantity. By using the good physical quantities \( x_a \) and \( q_x \), we can cast Eq. (14) into an equation of state:

\[
\frac{q_x^{x_a-1}}{1 - q_x^{x_a-1}} + H_q(x_a) = N, \tag{19}
\]

\[
H_q(x_a) = F_q(x_a) + G_q(x_a), \tag{20}
\]

where \( H_q(x_a) \) denotes the number of noncondensed bosons. We must seek the numerical solution of Eq. (19) to obtain the reduced chemical potential \( x_a \). When the axial angular frequency \( \omega_x \) is fixed, \( x_a \) is a function of temperature \( T \), particle number \( N \) and anisotropic parameter \( \omega_y/\omega_x \). In case one acquires \( x_a \), one can obtain the number of Bose atoms in the ground state from the relation \( N_0 = q_x^{x_a-1}/(1 - q_x^{x_a-1}) \). To obey Eq. (19), we require that \( x_a \geq 1 \). If \( x_a = 1 \), a two-dimensional ideal boson gas is in the state of BEC.

![FIG. 1. A plot of the reduced chemical potential \( x_a \) versus the number of atoms \( N \) in a two-dimensional ideal boson gas. The plot is drawn for \( T = 1, 15, 50, 100 \) nK. We choose the axial angular frequency \( \omega_y/2\pi = 10.0 \) Hz and the anisotropic parameter \( \omega_y/\omega_x = 0.2 \).](https://mc06.manuscriptcentral.com/cjp-pubs)

We need to make a numerical calculation and hence we set \( \omega_y/2\pi = 10.0 \) Hz, which is accessible to an actual experiment.
For comparison, one can introduce a two-dimensional condensation temperature $T_{2D}$ by the definition $k_B T_{2D} = \hbar \omega_x$. Thereby we obtain $T_{2D} = 0.4799$ nK. In the quantum statistical mechanics, the temperature of a two-dimensional boson gas is meaningful only when the number of atoms is sufficiently large. According to Eq. (19), in Fig. 1 we show the variation of the reduced chemical potential $x_a$ with the number of atoms $N$ for various $T$. We choose the anisotropic parameter $\omega_y/\omega_x = 0.2$. It is found that at $N = 10^4$, $x_a = 1$ for $T \leq 80$ nK. For a fixed $N$, $x_a$ is a monotonically increasing function of temperature $T$. These findings give an insight that for a fixed $N$, there is a critical temperature $T_c$, below which $x_a = 1$. According to Eq. (19), in Fig. 3 we show the variation of the reduced chemical potential $x_a$ with the anisotropic parameter $\omega_y/\omega_x$ for various $T$. The boson number is fixed at $N = 10^4$. It is interesting to note that for all $N$ and $T$, $x_a = 1$ at $\omega_y/\omega_x \rightarrow 0$. For a fixed $N$ and $T$, $x_a$ is a monotonically increasing function of anisotropic parameter $\omega_y/\omega_x$. These findings give an insight that for a fixed $T$ and $N$, there is a critical anisotropic parameter $\rho_c$, below which $x_a = 1$.

![Fig. 2](https://mc06.manuscriptcentral.com/cjp-pubs)

**FIG. 2.** A plot of the reduced chemical potential $x_a$ versus the temperature $T$ in a two-dimensional ideal boson gas. The plot is drawn for $N = 50, 200, 1000, 10000$. We choose the axial angular frequency $\omega_x/2\pi = 10.0$ Hz and the anisotropic parameter $\omega_y/\omega_x = 0.2$.

![Fig. 3](https://mc06.manuscriptcentral.com/cjp-pubs)

**FIG. 3.** A plot of the reduced chemical potential $x_a$ versus the anisotropic parameter $\omega_y/\omega_x$ in a two-dimensional ideal boson gas. The plot is drawn for $T = 50, 55, 60, 65$ nK and $N = 10^4$. We choose the axial angular frequency $\omega_x/2\pi = 10.0$ Hz.

![Fig. 4](https://mc06.manuscriptcentral.com/cjp-pubs)

**FIG. 4.** A plot of the condensate fraction $N_0/N$ versus the temperature $T$ in a two-dimensional boson gas. The plot is drawn for $N = 50, 200, 1000, 10000$. We choose the axial angular frequency $\omega_x/2\pi = 10.0$ Hz and the anisotropic parameter $\omega_y/\omega_x = 0.2$.

![Fig. 5](https://mc06.manuscriptcentral.com/cjp-pubs)

**FIG. 5.** A plot of the condensate fraction $N_0/N$ versus the number of atoms $N$ in a two-dimensional boson gas. The plot is drawn for $T = 1, 20, 50, 100$ nK. We choose the axial angular frequency $\omega_x/2\pi = 10.0$ Hz and the anisotropic parameter $\omega_y/\omega_x = 0.2$. 

[24].
In the next step we need to calculate the condensate fraction $N_0/N$, which is given by

$$N_0/N = \frac{q_x^{c-1}}{N(1-q_x^{c-1})}. \quad (21)$$

To obtain the condensate fraction, one must substitute the reduced chemical potential into Eq. (21). When the axial angular frequency $\omega_x$ is fixed, $N_0/N$ is a function of temperature $T$, particle number $N$ and anisotropic parameter $\omega_y/\omega_x$. When calculating $N_0/N$, we must combine Eq. (19) with Eq. (21). According to Eq. (21), in Fig. 4 we show the variation of the condensate fraction $N_0/N$ with the temperature $T$ for various $N$. We choose the anisotropic parameter $\omega_y/\omega_x = 0.2$. Figure 4 shows that for a finite number of two-dimensional trapped bosons, there is no exact transition temperature $T_c$. However, one can define an approximate transition temperature $T_c$. For a fixed $N$, $N_0/N$ decreases smoothly to zero when $T$ is near $T_c$. The appearance of phase transitions becomes clearer and clearer as the boson number becomes very large. According to Eq. (21), in Fig. 5 we show the variation of the condensate fraction $N_0/N$ with the number of atoms $N$ for various $T$. We choose the anisotropic parameter $\omega_y/\omega_x = 0.2$. It is found that at $T = 1$ nK, $N_0/N \approx 1$ for all $N$. In Fig. 5, one can define a critical number of atoms $N_c$. For a fixed $T$, $N_0/N$ increases smoothly from zero when $N$ is near $N_c$. According to Eq. (21), in Fig. 6 we show the variation of the condensate fraction $N_0/N$ with the anisotropic parameter $\omega_y/\omega_x$ for various $T$. The boson number is fixed at $N = 10^4$. When $N = 10^4$ and $T = 12$ nK, a two-dimensional atomic gas is always in the state of BEC no matter how large the anisotropic parameter $\omega_y/\omega_x$ is. Let $p = \omega_y/\omega_x$, where $p \leq 1$. When $N = 10^4$ and $T \geq 42$ nK, there is a critical anisotropic parameter $p_c$, below which a two-dimensional atomic gas is in the state of BEC and above which $N_0/N = 0$. A nice feature of the exact results in Eqs. (19) and (21) is that they are valid for arbitrary $T$, $N$ and $p$.

IV. QUANTUM STATISTICAL DESCRIPTION OF IDEAL BOSONS IN THE THERMODYNAMIC LIMIT

In what follows our task is to investigate the thermodynamic limit when $N \to \infty$. To complete this task, let us rewrite Eq. (19) in the form,

$$N_0 + H_{q_x}(x_a) = N. \quad (22)$$

The previous study has revealed that when $N \to \infty$, $x_a = 1$. The critical temperature $T_c$ can now be obtained by letting $N_0 = 0$ and $x_a = 1$ in Eq. (22). This manipulation leads to the following definition for the critical temperature,

$$H_{q_x}(1) = N, \quad (23)$$

where $q_{xc} = \exp(-\hbar \omega_x/k_B T_c)$. The function $H_{q_x}(1)$ can be rewritten as

$$h(q_{xc}) = H_{q_x}(1) = \frac{\ln(1 - q_{xc}) + \psi_{q_x}(1)}{\ln q_{xc}} + \frac{k_c \ln(1 - q_{xc})}{\ln q_{xc}} + \sum_{k=1}^{q_{xc}} \psi_{\bar{q}_x}(k\omega_y/\omega_x). \quad (24)$$

The critical temperature $T_c$ given by Eqs. (23) and (24) is an implicit function of particle number $N$ and anisotropic parameter $\omega_y/\omega_x$. It is impossible to provide an analytical formula for the dependence of $T_c$ on the anisotropy. Particularly, when $\omega_x = \omega_y = \omega$, $H_{q_x}(1)$ has the rigorous analytical expression:

$$h(q_{xc}) = H_{q_x}(1) = \frac{\ln(1 - q_{xc}) + \psi_{q_x}(1)}{\ln q_{xc}} + \frac{\psi_{\bar{q}_x}(1)}{\ln q_{xc}}, \quad (25)$$

where $q_{xc} = \exp(-\hbar \omega_x/k_B T_c)$ and $\psi_{\bar{q}_x}(x) = d[\psi_{q_x}(x)]/dx$ is the $q$-trigamma function.

If we take the limit as $N \to \infty$, we can derive the solution of Eq. (22) as

$$x_a = \begin{cases} 1, & T \leq T_c, \\ \text{the root of } H_{q_x}(x_a) = N, & T > T_c. \end{cases} \quad (26)$$

With the aid of Eq. (23), from Eq. (22) one can find that the condensate fraction of Bose atoms is given by

$$\frac{N_0}{N} = \begin{cases} 1 - \frac{h(q_{xc})}{h(q_{xc})}, & T \leq T_c, \\ 0, & T > T_c. \end{cases} \quad (27)$$

where $h(q_{xc})$ and $h(q_{xc})$ are given by Eq. (24) and $h(q_{xc})/h(q_{xc})$ represents the vapor fraction. The condensate fraction $N_0/N$ given by Eq. (27) is a monotonically decreasing function of temperature $T$. $N_0/N = 1$ at $T = 0$ K and $N_0/N = 0$ at $T = T_c$. For the BEC problem of ideal bosons in a two-dimensional anisotropic harmonic trap, the expression of a critical temperature under the quasiclassical approximation can be obtained as [25]:

$$T_c = \frac{\hbar}{\pi k_B} \sqrt{\frac{6N\omega_y \omega_x}{}}. \quad (28)$$
The rigorous two-dimensional critical temperature will be compared with the approximate two-dimensional critical temperature.

In the case of $N \to \infty$, the transition temperature and the condensate fraction are acquired by solving Eqs. (23) and (27). However, we must point out that Eqs. (23) and (27) are valid for arbitrary $T$ and large $N$ ($N \geq 10^4$). When $10^4 \leq N < 10^5$, Eqs. (23) and (27) are tenable too. Practically, we first employ Eq. (23) to determine the transition temperature $T_c$ and then utilize Eq. (27) to determine the condensate fraction $N_0/N$. Qualitatively, the phase transition of ideal bosons in a two-dimensional anisotropic harmonic trap is more vivid than that of ideal bosons in a two-dimensional isotropic harmonic trap. According to Eq. (23), in Fig. 7 we show the variation of the transition temperature $T_c$ with the number of atoms $N$ for various $\omega_y/\omega_x$. For a fixed $\omega_y/\omega_x$, the transition temperature $T_c$ is a monotonically increasing function of particle number $N$. Note that at $\omega_y/\omega_x = 1.0$, the curve of $T_c$ versus $N$ is described by Eqs. (23) and (25). According to Eq. (28), the variation with the number of atoms $N$ of the transition temperature $T_c$ is also shown in Fig. 7 for $\omega_y/\omega_x = 0.1$. When $N \to 10^5$, the approximate two-dimensional critical temperature is much smaller than the rigorous two-dimensional critical temperature. According to Eq. (23), in Fig. 8 we show the variation of transition temperature $T_c$ with the anisotropic parameter $\omega_y/\omega_x$ for various $N$. For a fixed $N$, the transition temperature $T_c$ is a monotonically decreasing function of anisotropic parameter $\omega_y/\omega_x$. According to Eq. (28), the variation with
the anisotropic parameter $\omega_y/\omega_x$ of the transition temperature $T_c$ is also shown in Fig. 8 for $N = 10^5$. The approximate two-dimensional critical temperature is always much smaller than the rigorous two-dimensional critical temperature. As shown in Figs. (7) and (8), the transition temperature $T_c$ of an anisotropic two-dimensional harmonic trap is always larger than that of an isotropic two-dimensional harmonic trap.

According to Eq. (27), in Fig. 9 we show the variation of the condensate fraction $N_0/N$ with the temperature $T$ for various $N$. The anisotropic parameter is fixed at $\omega_y/\omega_x = 0.2$. From Eq. (23), we find that at $\omega_y/\omega_x = 0.2$ and at $N = 10^5$, $10^7$, $10^9$, and $7.5 \times 10^9$, $T_c = 0.84 \times 10^2, 3.41 \times 10^2, 1.81 \times 10^3,$ and $9.13 \times 10^2$ K, respectively. Figure 9 demonstrates a clear behavior of phase transitions of BEC in a highly anisotropic two-dimensional harmonic trap. From Fig. 9, one sees that for a fixed $\omega_y/\omega_x$ and $T < T_c$, the condensate fraction $N_0/N$ is a monotonically increasing function of the number of atoms $N$. According to Eq. (26), in Fig. 10 we show the variation of the reduced chemical potential $\chi_0$ with the temperature $T$ for various $N$. The anisotropic parameter is fixed at $\omega_y/\omega_x = 0.2$. For a fixed $N$ and $\omega_y/\omega_x$, at $T > T_c$, the reduced chemical potential $\chi_0$ ascends very abruptly with the temperature $T$. Once the interaction between atoms is concerned, the thermodynamic quantities of an interacting system are changed from an ideal system. A repulsive mutual interaction between atoms will lower the critical temperature. In a weakly interacting Bose gas at zero temperature, most of the atoms will still stay in the ground state, and only a few of them will be kicked out of the condensate by the interactions. The number of noncondensed atoms is usually called the depletion. In particular, it is worth noting that the effect of particle interaction could significantly alter the BEC statistics compared to that predicted by an ideal gas model [26, 27].

V. DISCUSSION

The most important finding in the present paper is that we can analytically solve the BEC problem of ideal bosons in a two-dimensional anisotropic harmonic trap. We next point out that the analytical solution is associated with an analytical function, which relates to a series expansion of $q$-digamma functions. The $q$-digamma function is a very important function in the q-analog theory in mathematics and has extensive applications in science and technology. By an elementary substitution we introduce the reduced chemical potential to replace the fugacity. In the problem of BEC of harmonically trapped two-dimensional bosons, the fugacity $z$ is not a good physical quantity, but the reduced chemical potential $\chi_0$ is a good physical quantity. We have applied the $q$-digamma function into the BEC problem of harmonically trapped one-dimensional bosons [10, 28–31]. Further, we have applied the $q$-digamma function into the BEC problem of a two-dimensional isotropic harmonic trap [32–34]. By contrast, the analytic solution of the BEC problem of harmonically trapped one-dimensional bosons is associated with a single $q$-digamma function. This contrast illustrates that the analytic solution of harmonically trapped two-dimensional bosons is more complicated than that of harmonically trapped one-dimensional bosons. The analytical solution of a two-dimensional anisotropic harmonic trap is more complicated than that of a two-dimensional isotropic harmonic trap. Since the scientists observed the BEC of ultracold dilute atomic gases in three-dimensional harmonic traps in 1995, one has believed that the problem of BEC in three-dimensional harmonic traps may be solved analytically. We shall apply the $q$-digamma function into the BEC problem of harmonically trapped three-dimensional bosons.

It is interesting to note that the $q$-digamma function is a function of temperature, particle number and anisotropic parameter. This leads to the following results: (1) the critical temperature $T_c$ is a function of particle number $N$ and anisotropic parameter $\omega_y/\omega_x$. (2) the condensate fraction is a function of temperature $T$, particle number $N$ and anisotropic parameter $\omega_y/\omega_x$. The analytical solution describes fully the experimental results in the BEC of Bose atoms in a two-dimensional anisotropic harmonic trap [16]. We derive the analytic expressions of the critical temperature and the condensate fraction in the thermodynamic limit. Although the graphs presented in the present paper could be computed without mentioning the $q$-digamma function, these graphs can demonstrate many interesting details that cannot be obtained by other methods.

In summary, we have proposed an analytic solution to the BEC problem of ideal bosons in a two-dimensional anisotropic harmonic trap. It is found that the number of noncondensed bosons is characterized by an analytic function, which relates to a series expansion of $q$-digamma functions in mathematics. By an elementary substitution we introduce the reduced chemical potential to replace the fugacity. The accurate analytic solution of an ideal boson gas in a two-dimensional anisotropic harmonic trap can be verified in the present-day physics laboratories.

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