HARMONIC DOMAIN MODELING OF LAMINATED IRON CORE

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Abstract - The paper presents a new approach for the steady state modeling of the laminated saturating iron core. The modeling is in the harmonic domain, i.e. in terms of phasors for the different harmonics due to the nonlinear magnetization characteristic.

The procedure is based on a polynomial approximation of the magnetization characteristic. In the harmonic domain the computations are then performed based on the recognition that time domain multiplications can simply be replaced by convolutions. A lamination is discretized and Maxwell's equations are written in terms of discrete local field quantities, B , H , and E . The end result is a matrix equation solved by linearization. It yields the harmonic domain components of B and H over the cross-section of the lamination.

The new harmonic domain computational method can be used in practical problems, such as ferroresonance, for obtaining more accurate results than with the presently available simple procedures.

INTRODUCTION

The iron core of transformers and other electromagnetic devices is laminated so that at 60 Hz (or, in general, at power frequency) skin effect is insignificant. Due to saturation however, harmonics are generated in either the flux or the magnetic field intensity or in both, and for these the spatial distribution is not uniform. At higher excitation frequencies there is an even more pronounced skin effect which can not be calculated analytically because of the nonlinear nature of the phenomena. The purpose of this paper is to present a numerical computational method for the periodic (generally non-sinusoidal) steady state representation of the iron core taking into account both saturation and skin effect. For transients, a similar study has been presented in Ref.[1], while Ref.[2] examines the problem of saturating laminations using an idealized, rectangular magnetization characteristic. The latter is particularly interesting for its results regarding iron losses. The idea of harmonic domain modeling appears already in references [3] to [6] and an approach close to the one applied for laminations in this paper is first presented in references [7] and [8].

A simplified representation of the iron core, which in steady state ignores the skin effect, may not be adequate for the analysis of some phenomena, such as ferroresonance. Indeed, at a harmonic frequency, the magnetizing current is significantly increased by skin effect and its accurate knowledge is important for the analysis of a resonance phenomenon. Another instance when the accurate steady state representation of the iron core is important is the initialization needed for the calculation of transients, when the iron core is represented in detail¹, as in new EMTP developments for the modeling of transformers. (EMTP is an acronym for the widely used Electro-Magnetic Transients Program.)

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The basic idea underlying the computations in this paper is simple: in the harmonic domain, the equivalent operation to a time domain multiplication is a convolution. The procedure is naturally expanded to exponentiation and evaluation of polynomials and of rational polynomials. Since the harmonic domain equivalent of a differentiation is a multiplication, even ODEs (Ordinary Differential Equations) are solved algebraically. The paper uses a special numerical package developed for harmonic domain computations.

The procedure of the paper is the following: The magnetization characteristic is first approximated by a polynomial; then the laminations are discretized and the relevant Maxwell equations are written in discretized form; finally, they are linearized in terms of incremental harmonic domain variables and solved iteratively for B , H , and E .

The next section presents the general theory of harmonic domain modeling and the following one shows its application to the study of the electromagnetic phenomena in laminations.

GENERAL APPROACH OF HARMONIC DOMAIN MODELING

Theory

Polynomial Evaluation

According to the Weierstrass Approximation Theorem⁹, any continuous function defined on a closed, bounded interval can be approximated with arbitrarily small error by an (algebraic) polynomial of sufficiently high degree n . We will assume in the following that a satisfactory approximation has been obtained for the nonlinear function of interest:

$$y = f(x) = \sum_{q=0}^n b_q x^q \quad (1)$$

where

$$x = x(\theta) = \sum_{h=-\infty}^{+\infty} x_h e^{jh\theta} \quad (2)$$

is a periodic function of $\theta = \omega t$ and x_h is a coefficient, a phasor in the harmonic domain, corresponding to the harmonic h , and $x_{-h} = x_h^*$ (conjugate of x_h). The harmonic domain representation of $x(\theta)$ is the vector

$$\mathbf{x} = [\dots x_h \dots]^T \quad (3)$$

First we calculate x^q . We do this recursively by writing

$$x^{q-1} = x(\theta)^{q-1} = \sum_{h=-\infty}^{+\infty} x_h^{(q-1)} e^{jh\theta} \quad (4)$$

and

$$x^q = x(\theta)^q = \sum_{h=-\infty}^{+\infty} x_h^{(q)} e^{jh\theta} \quad (5)$$

where $(q-1)$ and (q) are superscripts, not exponents. Multiplying equations (2) and (4), and comparing the coefficients with those in eqn.(5), we obtain

$$x_h^{(q)} = \sum_{i=-\infty}^{+\infty} x_i x_{h-i}^{(q-1)} \quad (6a)$$

Equation (6a) represents the convolution

$$\mathbf{x}^q = \mathbf{x} \mathbf{x}^{q-1} \quad (6b)$$

of the elements of the harmonic domain vectors $\mathbf{x}$ and $\mathbf{x}^{q-1}$ to obtain the elements of the vector

$$\mathbf{x}^q = [\dots x_h^{(q)} \dots]^T \quad (3a)$$

A more general form of convolutions (6) is

$$x_h^{(q)} = \sum_{i=-\infty}^{+\infty} x_i^{(r)} x_{h-i}^{(q-r)} \quad (7a)$$

or

$$\mathbf{x}^q = \mathbf{x}^r \mathbf{x}^{q-r} \quad (7b)$$

For illustration we consider the special case $x(\theta) = 2\cos\theta$ for eqn.(2),

or

$$x(\theta) = e^{-j\theta} + e^{j\theta} \quad (2a)$$

In (2a) the only non-zero coefficients x_h are equal to 1. Using convolutions, we obtain the non-zero coefficients $x_h^{(q)}$ of $\mathbf{x}^q$, for $q=0, 1, 2, \dots$,

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 1 & 1 & \\ & & & & 1 & 2 & 1 \\ & & & & 1 & 3 & 3 & 1 \\ & & & & 1 & 4 & 6 & 4 & 1 \\ & & & & 1 & 5 & 10 & 10 & 5 & 1 \\ & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\ & & & & \dots & & & & & & & \end{array}$$

We recognize that these coefficients are those of the Pascal triangle. It is easy to check that the elements of row no. q (starting with $q=0$) are obtained by convolving the elements of rows no. r and $q-r$, for instance, for $q=7$ and $r=3$, we have

$$21 = 1 \times 6 + 3 \times 4 + 3 \times 1$$

Once the coefficients $x_h^{(q)}$ in eqn.(5) are known, we can substitute (5) into eqn.(1) and obtain

$$y(\theta) = \sum_{h=-\infty}^{+\infty} y_h e^{jh\theta} \quad (8)$$

where

$$y_h = \sum_{q=0}^n b_q x_h^{(q)} \quad (9a)$$

or, in vectorial form,

$$\mathbf{y} = \sum_{q=0}^n b_q \mathbf{x}^q \quad (9b)$$

We note that eqn.(9a) is formally analogous to eqn.(1) but, since it is in the harmonic domain, it has to be evaluated by sequential convolutions. Computationally the convolutions consist of a relatively small number of multiplications and additions due to the limited number of harmonics which are normally considered. Only half of the elements of a harmonic domain vector have to be computed by convolutions, since the other half are their conjugates.

The Jacobian

For the purpose of linearization, we consider the derivative of eqn.(1)

$$\mathbf{y}' = \sum_{q=1}^n q b_q \mathbf{x}^{q-1} \quad (10)$$

It can be written as

$$y'(\theta) = \sum_{h=-\infty}^{+\infty} y'_h e^{jh\theta} \quad (11)$$

with the coefficients

$$y'_h = \sum_{q=1}^n q b_q x_h^{(q-1)} \quad (12a)$$

or, in vectorial form,

$$\mathbf{y}' = \sum_{q=1}^n q b_q \mathbf{x}^{q-1} \quad (12b)$$

We use eqn.(11) in the incremental relation

$$\Delta y(\theta) = y'(\theta) \Delta x(\theta) \quad (13)$$

where

$$\Delta x(\theta) = \sum_{h=-\infty}^{+\infty} \Delta x_h e^{jh\theta} \quad (14a)$$

and

$$\Delta y(\theta) = \sum_{h=-\infty}^{+\infty} \Delta y_h e^{jh\theta} \quad (14b)$$

Substituting (11) and (14) into (13) and comparing coefficients, we obtain, for h from $-\infty$ to $+\infty$,

$$\Delta y_h = \sum_{i=-\infty}^{+\infty} y'_h x_{h-i} \Delta x_i \quad (15a)$$

or, in terms of harmonic domain vectors,

$$\Delta \mathbf{y} = \mathbf{y}' \Delta \mathbf{x} = \mathbf{T}(\mathbf{y}') \Delta \mathbf{x} \quad (15b)$$

where $\mathbf{T}(\mathbf{y}')$ is the Toeplitz matrix of $\mathbf{y}'$, i.e. a matrix with the same elements on all diagonals¹⁰, namely,

$$\dots y'_3 \ y'_2 \ y'_1 \ y'_0 \ y'_{-1} \ y'_{-2} \ y'_{-3} \ \dots \quad (16)$$

These are the elements of the vector $\mathbf{y}'$ of eqn.(12b), in reversed order.

We can interpret the Toeplitz matrix $\mathbf{T}(\mathbf{y}')$ as the Jacobian matrix of the harmonic domain nonlinear equation (9b). If $\mathbf{x}$ is a voltage vector and $\mathbf{y}$ a current vector in the harmonic domain, then $\mathbf{T}$ is the incremental harmonic admittance matrix, $\mathbf{Y}_{\text{harm}}$; conversely, $\mathbf{T}$ is the incremental harmonic impedance matrix, $\mathbf{Z}_{\text{harm}}$. Correspondingly, nonlinear functions of voltage or current can be written in the harmonic domain in the following linearized forms, around a given input value,

$$\mathbf{i} = \mathbf{f}(\mathbf{v}_0) + \mathbf{Y}_{\text{harm}} \Delta \mathbf{v} = \mathbf{Y}_{\text{harm}} \mathbf{v} + \mathbf{i}_N \quad (17a)$$

$$\mathbf{v} = \mathbf{f}(\mathbf{i}_0) + \mathbf{Z}_{\text{harm}} \Delta \mathbf{i} = \mathbf{Z}_{\text{harm}} \mathbf{i} + \mathbf{v}_T \quad (17b)$$

Equations (17) represent harmonic domain Norton and Thevenin equivalents, respectively, and can be used for interfacing with the external system or for building complex components.

Implicit Polynomial Functions

We consider a polynomial approximation of the nonlinear function

$$p(x, y, z, u) = \sum_{i=0}^n b_i x^{j(i)} y^{k(i)} z^{l(i)} u^{m(i)} \quad (18)$$

Let us evaluate the product

$$w = xy \quad (19)$$

where

$$x = x(\theta) = \sum_{k=-\infty}^{+\infty} x_k e^{jk\theta} \quad (20a)$$

$$y = y(\theta) = \sum_{l=-\infty}^{+\infty} y_l e^{jl\theta} \quad (20b)$$

$$w = w(\theta) = \sum_{m=-\infty}^{+\infty} w_m e^{jm\theta} \quad (20c)$$

Substituting (20) into (19) and comparing the coefficients yields

$$w_m = \sum_k x_k y_{m-k} \quad (21)$$

Defining the harmonic domain vectors as in eqn.(9), the convolution (21) becomes

$$\mathbf{w} = \mathbf{x} \mathbf{y} \quad (22)$$

We note here that the equivalence of multiplication and convolution in the time and frequency domain is well known but it is most often used for multiplications in the frequency domain. Here the multiplication (19) is in the time domain and the corresponding convolution (21) or (22) in the harmonic frequency domain.

Clearly, operation (22) is commutative. In the particular case when $y = x$, it becomes a self-convolution

$$w = x x = x^2 \quad (23)$$

This is a particular case of the repeated self-convolution of the exponential form x^q .

Since in eqn.(22) both x and y are vectors, it is clear that the operations can not be directly interpreted in terms of conventional matrix operations. However, the corresponding matrix equation can be written as

$$w = T(x)y \quad (24a)$$

or

$$w = T(y)x \quad (24b)$$

where T denotes a Toeplitz matrix, as defined in the previous section.

From the above derivations related to multiplications and exponentiations it results that formally the polynomial function (18) remains similar when written in the harmonic domain:

$$p(x, y, z, u) = \sum_{i=0}^n b_i x^{j(i)} y^{k(i)} z^{l(i)} u^{m(i)} \quad (25)$$

Its interpretation and evaluation has to be done, however, in terms of convolutions.

When a factor x^q in a polynomial expression has to be evaluated for larger values of q it is convenient to express the integer q as a binary number and use this for guidance for the sequence of convolutions to be performed. For instance, $q = 19 = 2^4 + 2 + 1$ we calculate the following sequence of convolutions:

$$w_1 = x x \quad (= x^2)$$

$$w_2 = w_1 w_1 \quad (= x^4)$$

$$w_3 = w_2 w_2 \quad (= x^8)$$

$$w_4 = w_3 w_3 \quad (= x^{16})$$

$$w_5 = x w_1 \quad (= x^3)$$

$$w_6 = w_4 w_5 \quad (= x^{19})$$

This example shows that the number of convolutions is $(1...2) \times \log_2 q$. In most practical cases this is a small number.

When a polynomial expression is evaluated, one will take advantage of the above exponentiation procedure and of recursive evaluation methods, such as Horner's rule⁹.

Dynamic Elements

Let us assume that in eqn.(18)

$$z = \dot{x} \quad (26)$$

Substitution of (20a) into (26) yields

$$\dot{x} = \dot{x}(\theta) = \sum_{k=-\infty}^{+\infty} j\omega k x_k e^{jk\theta} \quad (27)$$

Comparing the coefficients of z of eqn.(20c) with those of eqn.(27), we can write

$$z = \dot{x} = [\cdots j\omega k x_k \cdots]^T = \text{diag}(j\omega k) x \quad (28)$$

This equation shows that the evaluation of a polynomial expression, corresponding to a dynamic system representation, is done in the harmonic domain in the same way as for simple algebraic variables, since $\dot{x}$ is itself calculated by an algebraic operation.

Rational Polynomials

Often a nonlinear function

$$y = f(x) \quad (29)$$

is more efficiently approximated by a rational polynomial

$$y = \frac{\sum b_q x^q}{\sum c_r x^r} \quad (30)$$

than by a polynomial (eqn. (1)). Even for such cases, a direct harmonic domain representation is possible.

We write (30) in the harmonic domain as

$$wy = z \quad (31a)$$

or

$$T_w y = z \quad (31b)$$

where $T_w = T(w)$ is the Toeplitz matrix (see eqn. (15b)) of

$$w = \sum c_r x^r \quad (31c)$$

and

$$z = \sum b_q x^q \quad (31d)$$

Then, from (31b)

$$y = T_w^{-1} z \quad (32)$$

This equation is the harmonic domain equivalent of the rational polynomial (30). The evaluation of (32) implies the calculation of w and z from (31c) and (31d) and the solution of the matrix equation (31b).

Numerical Software

In order to facilitate the computations in the harmonic domain, a set of subroutines has been written. These perform simple and sequential convolutions, evaluate polynomials, form Toeplitz matrices, evaluate rational functions by solving equations with Toeplitz matrices, and form and solve linear and nonlinear ODEs and state equations. The software package consists of the following routines.

- 1 Polynomial evaluation: $\sum b_k x^k y^j z^m \cdots$ (POLY)
 - 1.1 Harmonic convolution and self-convolution (CONV)
 - 1.2 Binomial decomposition of exponent (BINOM)
 - 1.3 Evaluation of polynomial (EVPOL)
- 2 Rational polynomial evaluation (RPOLY)
- 3 Implicit ODE evaluation (IODE)
 - 3.1 Derivative: $z = \dot{x}$ (DERIV)
 - 3.2 POLY with $z = \dot{x}$ (POLYZ)
 - 3.3 RPOLY with $z = \dot{x}$ (RPOLYZ)
- 4 Newton-Raphson linearization (NR)
 - 4.1 Single variable (NRS)
 - 4.2 Multi-variable (NRM)
- 5 Linear state equations (LINSE)
 - 5.1 Solve for x (SX)
 - 5.2 Solve for y (SY)
 - 5.3 Solve state equations (SXY)
- 6 Nonlinear state equations (NLSE)
 - 6.1 Linearize (NRSE)
 - 6.2 Solve linearized form (NRSXY)

HARMONIC DOMAIN MODELING OF LAMINATIONS

Polynomial Approximation of Magnetization Characteristic

It is crucial for the method of this paper to obtain a good polynomial approximation of the magnetization characteristic $H=f(B)$. Hysteresis is not considered in this study. Since direct polynomial fitting is known to be ill-conditioned for higher order polynomials, we have initially used Chebyshev approximation^{9,11}. Good approximations have been obtained with polynomials of order 11 or 13. However, it is a characteristic for orthogonal polynomials of order q (not just for Chebyshev but for all, e.g. Legendre, Laguerre, Hermite, etc. which only differ from each other by their weighting functions) that they have q real roots which makes them oscillatory. Because of this orthogonal polynomial approximation produces wriggling curves which, even if barely noticeable by the bare eye, is still not quite acceptable from a physical point of view. Therefore, we used for fitting a polynomial with only three terms,

$$H = a_1 B + a_p B^p + a_q B^q \quad (33)$$

The second derivative of eqn.(33) changes sign when

$$a_p p(p-1) + a_q q(q-1)B^{p-q} = 0 \quad (33a)$$

This can be avoided by choosing p not much smaller than q (both odd). Then both p and q will result positive if q is large enough. We have this way obtained good, smooth fits, with relative errors of less than 2% with the polynomial (33b), where H is in A/m and B in tesla.

$$H = 51.8025B + 0.2181B^{17} + 0.1353B^{21} \quad (33b)$$

This fit was obtained with five sample points all equally weighted and placed as follows: one on the linear part, three on the knee portion of the characteristic, and one in the saturated region. The procedure was a linear least squares fit with the exponents 17 and 21 chosen from a number of possible combinations for minimal errors. The approximation was made for a magnetization curve obtained by measurements on an Ontario Hydro transformer, with flux densities up to 1.7 T.

We note here that at lower values of B magnetization curves have in fact an inflection point. If one wishes to have it included in the approximation, the requirement of having both a_p and a_q positive should be relaxed.

Discretization of Maxwell's Equations

Maxwell's equations for the plane geometry of a lamination are

$$\frac{\partial H}{\partial x} = \sigma E \quad (34a)$$

$$\frac{\partial E}{\partial x} = \dot{B} \quad (34b)$$

These can be combined into

$$\frac{\partial^2 H}{\partial x^2} = \sigma \dot{B} \quad (35)$$

We discretize the laminations in $2n$ layers which we call sublaminations. This defines the discrete points $0, 1, \dots, n$, point 0 being in the middle and point n at the surface of the lamination. Sublamination k is between points $k-1$ and k .

For input and output the variables of interest are B_n and E_n , at the surface of the lamination. In many applications E_n may be directly specified; it is proportional to the time derivative of the total flux in the lamination. Therefore, we discretize equations (34) for the surface layer, using the trapezoidal rule, and after eliminating E_{n-1} we obtain

$$4(H_n - H_{n-1}) + c(\dot{B}_n + \dot{B}_{n-1}) - 4c_o E_n = 0 \quad (36)$$

where

$$c = \sigma(\Delta x)^2 \quad \text{and} \quad c_o = \sigma(\Delta x) \quad (36a)$$

For internal points we use a discretized form of eqn.(35). The three point central difference formula for the second derivative is

$$\frac{H_{k+1} - 2H_k + H_{k-1}}{\Delta x^2} \quad (37)$$

Since this is evaluated over two layers, the right hand side of (35) should not be simply $\sigma \dot{B}_k$. A more accurate result is obtained by discretizing equation (35) according to the Stormer-Numerov formula¹²

$$\frac{H_{k+1} - 2H_k + H_{k-1}}{\Delta x^2} = \sigma \frac{\dot{B}_{k+1} + 10\dot{B}_k + \dot{B}_{k-1}}{12} \quad (38)$$

The discretization error is of concern especially at higher frequencies when the field distribution over the cross section of the lamination becomes strongly non-uniform. It is then preferable to use multi-point discretizations rather than a larger number of layers which increases the size of the matrices in the computations. Therefore we have developed higher order multi-point formulas for the complete differential equation (35) which are more accurate than the respective multi-point difference formulas. The three-point Stormer-Numerov formula of eqn.(38) appears as a particular case and for five points the corresponding equation is

$$\frac{\sum_{i=-2}^{i=2} \alpha_{k+i} H_{k+i}}{(\Delta x)^2} = \sigma \sum_{i=-2}^{i=2} \beta_{k+i} \dot{B}_{k+i} \quad (39)$$

where

$$\alpha_0 = -\frac{53}{42} \quad \alpha_{\pm 1} = \frac{32}{63} \quad \alpha_{\pm 2} = \frac{31}{252}$$

$$\beta_0 = \frac{131}{210} \quad \beta_{\pm 1} = \frac{172}{945} \quad \beta_{\pm 2} = \frac{23}{3780}$$

The general form of a multi-point difference formula for the normalized second order differential equation

$$g(x) = \frac{d^2 y(x)}{dx^2} - z(x) = 0 \quad (40)$$

is

$$L_k = \sum_i \alpha_{k+i} y_{k+i} - h^2 \sum_i \beta_{k+i} z_{k+i} = 0 \quad (40a)$$

where h stands for Δx . The derivation of the values for the coefficients α and β is given in the Appendix.

Linearized Equations for Iterative Solution

The mathematical model of a lamination with $2n$ layers consists of $n+1$ equations, n of the form of eqn.(38) (for three point discretization) for the discretization points $0, 1, \dots, n-1$, (where point 0 corresponds to the middle of the lamination) and one equation for the surface of the lamination (point n), eqn.(36). In the practically important case when the interface of the lamination with the external system consists in having E_n as specified input, the number of unknowns in these equations is also $n+1$. If this is not the case, then an additional equation, reflecting the external relation between E_n and H_n , is added.

By replacing H_k using eqn.(33), we remain with the state variables B_k as unknowns. For internal points eqn.(38) together with (33) yields, in the harmonic domain,

$$\begin{aligned} & [a_1 \mathbf{B}_{k-1} + a_p \mathbf{B}_{k-1}^p + a_q \mathbf{B}_{k-1}^q - \frac{1}{12} c \text{diag}(j\omega h) \mathbf{B}_{k-1}] + \\ & + [-2(a_1 \mathbf{B}_k + a_p \mathbf{B}_k^p + a_q \mathbf{B}_k^q) - \frac{5}{6} c \text{diag}(j\omega h) \mathbf{B}_k] + \\ & + [a_1 \mathbf{B}_{k+1} + a_p \mathbf{B}_{k+1}^p + a_q \mathbf{B}_{k+1}^q - \frac{1}{12} c \text{diag}(j\omega h) \mathbf{B}_{k+1}] = 0 \end{aligned} \quad (41a)$$

$$k = 0, 1, \dots, n-1$$

We note that because of symmetry, $\mathbf{B}_{-k} = \mathbf{B}_k$.

Similarly, eqn.(36) becomes

$$\begin{aligned} & 4[a_1 (\mathbf{B}_n - \mathbf{B}_{n-1}) + a_p (\mathbf{B}_n^p - \mathbf{B}_{n-1}^p) + a_q (\mathbf{B}_n^q - \mathbf{B}_{n-1}^q)] + \\ & + c(\dot{\mathbf{B}}_n + \dot{\mathbf{B}}_{n-1}) - 4c_o E_n = 0 \end{aligned} \quad (41b)$$

In equations (41) the bold typed variables denote harmonic domain vectors. In a short form these equations can be written as

$$\mathbf{f}_k(\mathbf{B}) = \sum_j \mathbf{f}_{kj}(\mathbf{B}_j) = 0 \quad (42a)$$

For iterative solution, linearization of the system $\mathbf{f}=\mathbf{0}$ at step i yields

$$\mathbf{f}^{(i)} + \mathbf{J}^{(i)} \Delta \mathbf{B}^{(i)} = \mathbf{0} \quad (42b)$$

where $\mathbf{J}$ is the harmonic domain Jacobian matrix.

If we take (42a) into account, we find that the Jacobian consists of blocks

$$\mathbf{J}_{kj} = \frac{d \mathbf{f}_{kj}(\mathbf{B}_j)}{d \mathbf{B}_j} = \mathbf{f}'_{kj} \quad (43)$$

Therefore the Jacobian is easy to calculate. Its structure is band block-diagonal (block tri-diagonal in the case of the three-point discretization of eqn.(41a)). The blocks of eqn.(43) are quasi-Toeplitz since on their main diagonal the elements are not equal due to the effect of $\text{diag}(j\omega h)$ of eqn.(41a). This reflects the different dynamic behaviour of the individual harmonics.

The size of the problem (42b) depends on the number N of harmonics considered and the number $2n$ of sublaminations. The submatrices $\mathbf{J}_{kj}$ are $2N \times 2N$ and the resultant Jacobian $\mathbf{J}$ will be $2N(n+1) \times 2N(n+1)$. For example in the case of 8 layers ($n=4$) and the harmonics considered being 1, 3, ..., 25 ($N=13$) the Jacobian will be of size 130×130 . The width of the side-band of this tri-diagonal matrix is $2 \times 26 = 52$. The subsequent numerical calculations can take advantage of the structural properties of the Jacobian.

Solution Procedures

Equation (42b), or

$$\mathbf{J} \Delta \mathbf{B} = -\mathbf{f} \quad (44)$$

represents Newton's method for finding the harmonic domain solution. We note however that in actual implementation the Jacobian is not accurate since in its evaluation harmonics above a certain threshold are truncated. Therefore, even if $\mathbf{J}$ is recalculated at each iteration, the convergence will be only close to quadratical.

An alternate approach is to keep the Jacobian constant after the first or some later iteration step. We have found that keeping the Jacobian unchanged after the first iteration has slowed down the process significantly, but the Jacobian of the second iteration yielded good convergence. Still, our preference is to use the base quasi-Newton approach of eqn.(44) because it converges in at most 10 steps, at 60 Hz, with an error tolerance in the mismatch of 10^{-5} . This is the relative error in H_n , the field intensity at the lamination surface. At internal points the accuracy is 10^5 times better. This is so because eqn.(38) is more accurate than eqn.(36). Such high precision is of course of no significance in view of rather large errors in the physical data. Very good results can be obtained with less than five iterations.

Interfacing with the External System

The external system can in general be represented in the harmonic domain by its Thevenin equivalent

$$\mathbf{v} = \mathbf{v}_T - \mathbf{Z}_T \mathbf{i} \quad (45)$$

We assume that the resistance of the winding of the iron core being magnetized is included in $\mathbf{Z}_T$. Since $\mathbf{v}$ is proportional to E_n at the lamination surface and $\mathbf{i}$ to H_n , we may rewrite eqn.(45) as

$$\mathbf{E}_n = \mathbf{E}_T - \mathbf{Z}'_T \mathbf{H}_n \quad (46)$$

This equation is added to the set considered in the previous section and E_n is added to the set of unknowns.

COMPUTATIONAL RESULTS

The computations have been performed for laminations of thickness $2d = 0.33 \times 10^{-3}$ m and conductivity $\sigma = 6 \times 10^6$ S/m. We have assumed a number of $2n = 8$ sublaminations. Thus the surface field quantities are E_4 , H_4 and B_4 . The subscript for the middle of the lamination is 0. The base frequency is 60 Hz. D.c. offset and even harmonics are not considered in the computations.

Numerous tests have been performed to validate the computational methods and to obtain insight into the complex phenomena of eddy currents in saturating laminations. The most significant ones are described below.

A first check relates to the errors due to space discretization. These can be assessed by comparing the results from the discretized model with those of an analytical solution. For this purpose the magnetization curve has been assumed linear, given by the first term of equation (33b). We define an index of non-uniformity $k = B_4/B_0$. Table 1 contains the values of k for different frequencies for the analytical and numerical calculations. As expected, both the non-uniformity and the error increase with frequency. Thus, the error is negligible at 60 Hz, 0.3% at 600 Hz, and 3% at the 25th harmonic. For still higher harmonics the discretization has to be improved because of the fast increase of the errors.

Table 1. Index of Non-Uniformity

f (Hz)	Value of k		Error %
	Analytical Solution	Numerical Solution	
1	1.000034	1.000034	0.0
60	1.116267	1.116261	0.00005
600	5.876983	5.859069	0.3
900	10.229463	10.144259	0.8
1500	24.548351	23.815894	2.98
6000	1205.176	548.557	54.48

Table 2 gives the harmonics (magnitude and angle) of B_4 , B_0 , H_4 , and H_0 . The input is $E_4 = E_{4\max} \sin 2\pi f t$ and its phase is taken as reference. We define B_{eff} as the flux density related to E_4 assuming uniform flux distribution at frequency f . The input E_4 has been chosen to correspond to $B_{eff} = 1.65$ T. We note that the content of harmonics is significant (because of the strong saturation) but it decreases fast (because of the smooth magnetization characteristic – see Fig. 3a), especially in H (since the magnetization curve is strongly nonlinear). Harmonics above 15 are of order 0.1% or less.

The fast decrease of harmonic content does not imply that in the harmonic domain convolutions and in related computations higher order harmonics can be truncated at a lower level: the results in Table 2 are calculated retaining all harmonics including the 25th; truncation after harmonic 7 has resulted in significant inaccuracy.

Table 2a. Harmonics of B and H at Surface

Harmonic order	B_4		H_4	
	magnitude (T)	angle (degrees)	magnitude (A/m)	angle (degrees)
1	1.701127	-73.50	2171.2392	-89.06
3	0.227076	-172.63	1718.2243	-89.98
5	0.002492	-174.64	1173.2427	-90.04
7	0.047620	28.25	658.7390	-90.02
9	0.013967	-85.13	298.2124	-89.90
11	0.011327	-162.92	106.7387	-89.93
13	0.010233	126.17	30.4977	-90.90
15	0.007128	-19.82	7.4789	-88.92

Table 2b. Harmonics of B and H at Center

Harmonic order	B_0		H_0	
	magnitude (T)	angle (degrees)	magnitude (A/m)	angle (degrees)
1	1.650676	-98.53	2167.2966	-90.09
3	0.108284	-14.93	1723.1702	-89.99
5	0.015088	16.57	1174.0276	-90.03
7	0.029571	150.47	658.8862	-90.01
9	0.007237	-125.46	297.8930	-89.92
11	0.012053	-40.49	108.2880	-89.98
13	0.005370	70.99	30.8082	-90.59
15	0.005903	-149.54	6.6034	-89.50

Fig.1 represents the time variation for a half period of B and H at different discretization points in the lamination. The input was the same as in the previous cases, i.e. E_4 sinusoidal, at $f = 60$ Hz but with different values for B_{eff} , ranging from strong saturation ($B_{eff} = 1.7$ T), to normal ($B_{eff} = 1.65$ T), to low ($B_{eff} = 1.3$ T), to negligible saturation ($B_{eff} = 1.0$ T). We see how the flux wave propagates to the center of the lamination.

It is interesting to note that due to saturation the peaks are practically the same and appear simultaneously in all layers.

(a) Strong saturation ($B_{eff} = 1.7$ T)

(b) Normal saturation ($B_{eff} = 1.65$ T)

(c) Low saturation ($B_{eff} = 1.3$ T)

(d) Negligible saturation ($B_{eff} = 1.0$ T)

Fig.1 $B(t)$ and $H(t)$ at different discretization points obtained with $E_4 = E_{4max} \sin 377t$ ($f = 60$ Hz)

Fig.2 gives similar results as Fig.1, except that the frequency is $f = 300$ Hz. At this frequency, the convergence of the Newton process was slower because of the more pronounced nonuniformity of the field distribution and the related increased significance of truncating higher order harmonics. Only two saturation levels are shown here, as they are fully representative for intermediate cases. The wavy curve shapes are due to the fact that the 25th harmonic is of 7500 Hz and, as discussed earlier, at such high frequencies the four layer discretization (per half lamination) is not satisfactory.

(a) Strong saturation ($B_{eff} = 1.7$ T)

(b) Low saturation ($B_{eff} = 1.3$ T)

Fig.2 $B(t)$ and $H(t)$ at different discretization points obtained with $E_4 = E_{4max} \sin 1885t$ ($f = 300$ Hz)

Fig.3 Amplitude spectrum of H_4 , for $E_4 = E_{4max} \sin 2\pi ft$

Table 3 gives eddy current losses for 60 Hz and 300 Hz and two levels of saturation. These compare well with practical data; see Ref.[13], page 45.

Table 3. Eddy Current Losses (W/kg) Function of Frequency and of B_{eff}

f (Hz)	$B_{eff} = 1.65$ T	$B_{eff} = 1.7$ T
60	1.434	1.525
300	40.125	44.218

Fig.3 gives the amplitude spectrum of H_4 for sinusoidal input E_4 corresponding to two different levels of saturation, $B_{eff} = 1.7$ T and $B_{eff} = 1.65$ T. The latter are consistent with the corresponding column of Table 2. The plots connect the discrete values calculated for the individual harmonics. It is seen that with saturation the harmonic content increases almost in proportion with the fundamental.

Fig.4 Space distribution of the harmonics of B

Fig.4 represents the space distribution of the harmonics of B . The calculated values correspond to the discretization points from the middle to the surface of the lamination. We note that skin effect does not necessarily mean larger values of flux density at the surface. This is strongly apparent in the case of the fifth harmonic. In many cases the spatial variation of the harmonics is quite pronounced.

CONCLUSIONS

A novel, harmonic domain, analysis method has been developed and used to examine the periodic steady state behaviour of saturating iron laminations. It is based on direct harmonic domain computations using convolutions for the evaluation of polynomials which represent the nonlinear magnetization characteristic. The procedure can be applied to various practical problems as it is based on a general purpose numerical package for harmonic domain analysis.

A number of interesting results have been shown as part of the many numerical experiments performed for the analysis of fundamental phenomena in saturated laminations. It is worthwhile to note the particular spatial distribution of the different harmonics.

The new harmonic domain computational method could be used in practical problems, such as ferroresonance, for obtaining more accurate results than with the presently available simple procedures.

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APPENDIX

We intend to represent equation (40) at $x=x_k$ by the discrete linear expression (40a). Since this expression is only an approximation of the original second order differential equation, we will choose the coefficients α_{k+i} and β_{k+i} so that L_k approaches zero as closely as possible. First however we note that equation (40a) is homogeneous in the parameters α_{k+i} , β_{k+i} and we impose the normalizing condition

$$\sum_i \beta_{k+i} = 1 \quad (\text{A.1})$$

We now expand y_{k+i} and z_{k+i} by Taylor series around y_k and z_k , respectively:

$$y_{k+i} = y_k + y'_k i h + \frac{1}{2} y''_k i^2 h^2 + \dots + \frac{1}{m!} y^{(m)}_k i^m h^m + \dots \quad (\text{A.2a})$$

$$z_{k+i} = z_k + z'_k i h + \frac{1}{2} z''_k i^2 h^2 + \dots + \frac{1}{m!} z^{(m)}_k i^m h^m + \dots \quad (\text{A.2b})$$

Note that in the expansion (A.2a) of y_{k+i} , the differential equation (40) has been taken into account.

We substitute (A.2) into (40a) and set equal to zero the maximum possible number of expressions, which are coefficients of powers of h , to minimize the truncation error. This yields

for the coefficient of $y_k h^0$:

$$\sum_i \alpha_{k+i} = 0 \quad (\text{A.3.0})$$

for the coefficient of $y'_k h^1$:

$$\sum_i i \alpha_{k+i} = 0 \quad (\text{A.3.1})$$

for the coefficient of $z_k h^2$:

$$\frac{1}{2} \sum_i i^2 \alpha_{k+i} - \sum_i \beta_{k+i} = 0 \quad (\text{A.3.2})$$

for the coefficient of $z'_k h^3$:

$$\frac{1}{6} \sum_i i^3 \alpha_{k+i} - \sum_i i \beta_{k+i} = 0 \quad (\text{A.3.3})$$

for the coefficient of $z''_k h^4$:

$$\frac{1}{24} \sum_i i^4 \alpha_{k+i} - \frac{1}{2} \sum_i i^2 \beta_{k+i} = 0 \quad (\text{A.3.4})$$

for the coefficient of $z_k^{(3)} h^5$:

$$\frac{1}{120} \sum_i i^5 \alpha_{k+i} - \frac{1}{6} \sum_i i^3 \beta_{k+i} = 0 \quad (\text{A.3.5})$$

and so on.

The general form of equations (A.3), for the coefficient of $z_k^{(n-2)} h^n$ ($n > 1$), is

$$\frac{1}{n!} \sum_i i^n \alpha_{k+i} - \frac{1}{(n-2)!} \sum_i i^{(n-2)} \beta_{k+i} = 0 \quad (\text{A.3.n})$$

For an m -point formula, i will take m values and we will have $2m$ parameters α_{k+i} , β_{k+i} . Taking equation (A.1) into account, $2m-1$ equations of the sequence (A.3) can be satisfied, so that the truncation error in L_k of eqn.(40a) will be of order h^{2m-1} . If m is odd, it turns out that one additional equation of (A.3) is satisfied. This makes the odd order formulas more attractive; their error is of order h^{2m} . For instance, for a 3-point formula the error is of order h^6 , or for a 5-point formula it is of order h^{10} which is certainly quite remarkable. The actual expressions for the error terms can be calculated, once the parameters α_{k+i} , β_{k+i} are known, from

eqn.(A.3.n). Thus, the truncation error of the 3-point formula is of order $\frac{1}{240} z_k^{(4)} h^6$, and of the 5-point formula of order $10^{-5} h^{10}$.

It is interesting to note that shifting the points i does not change the formula and the error. This may appear as paradoxical but it can be proved and demonstrated by example. Thus, for both sets $i = -1, 0, +1$ and $i = 0, 1, 2$ the formula and the error remain unchanged. In other words, the Störmer-Numerov equation (38) represents the differential equation (40) with the same truncation error in any point of the one-dimensional grid. This remark holds also for higher order discretization formulas.

The following equations exemplify the application of (A.3) to the 3-point case, with $i = -1, 0, +1$. We obtain

for the coefficient of $y_k h^0$:

$$\alpha_{-1} + \alpha_0 + \alpha_1 = 0 \quad (\text{A.4.0})$$

for the coefficient of $y'_k h^1$:

$$-\alpha_{-1} + \alpha_1 = 0 \quad (\text{A.4.1})$$

for the coefficient of $z_k h^2$:

$$\frac{1}{2} (\alpha_{-1} + \alpha_1) - (\beta_{-1} + \beta_0 + \beta_1) = 0 \quad (\text{A.4.2})$$

for the coefficient of $z'_k h^3$:

$$\frac{1}{6} (-\alpha_{-1} + \alpha_1) - (-\beta_{-1} + \beta_1) = 0 \quad (\text{A.4.3})$$

for the coefficient of $z''_k h^4$:

$$\frac{1}{24} (\alpha_{-1} + \alpha_1) - \frac{1}{2} (\beta_{-1} + \beta_1) = 0 \quad (\text{A.4.4})$$

To these five equations we add the normalization equation

$$\beta_{-1} + \beta_0 + \beta_1 = 0 \quad (\text{A.5})$$

Solving these, we obtain

$$\alpha_{-1} = \alpha_1 = 1, \quad \alpha_0 = -2 \quad (\text{A.6a})$$

$$\beta_{-1} = \beta_1 = \frac{1}{12}, \quad \beta_0 = \frac{10}{12} \quad (\text{A.6b})$$

which are the parameters of the Störmer-Numerov formula (38).

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Discussion

Alexander E. Emanuel (Worcester Polytechnic Institute, Worcester, MA): Drs. Semlyen and Rajakovic presented a most elegant, and at the same time, effective approach for the determination of the time variation of B , H , E , in the laminations of magnetic cores. This method can be easily applied in design analysis of solid-state ballast for fluorescent lamps or eventually for the evaluation of losses in the iron cores of transformers and cores used in switched power supplies.

Compared with other techniques, the method presented today leads to computer programs and software which seem to require less preparation than methods based on finite difference and finite element algorithms, and at the same time yields more accurate results than the approximative methods based on empirical equations of the B/H characteristic.

A few questions remain to be answered.

- 1) How will the authors take into consideration the airgaps?
- 2) How does this method apply when the excitation source is nonsinusoidal?

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Closure

Adam Semlyen and Nikola Rajakovic: We would like to thank Dr. Emanuel for his remarks and in particular for suggesting areas of application for the harmonic analysis method presented in our paper. Our answers to the the discussor's questions are:

1. The effect of an air gap is to distort the fields in the lamination and requires therefore special treatment. However, this end effect becomes insignificant at a few milli-meters away from the air gap. An equivalent reluctance for the air gap has still to be considered in the representation of the complete iron core.
2. There is no restriction in the harmonic domain analysis method in relation to the applied voltage. It can in general be a Thevenin or Norton equivalent for all harmonics, i.e. a linear relation between E and H . Purely sinusoidal voltage or current sources appear as particular cases.

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