Some Frequency Domain Aspects of Wave Propagation on Nonuniform Lines

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Abstract—Nonuniform line (NL) is the designation of a transmission line segment where the conductors are not parallel or where the sag is significant. In addition, transformers and also parts of metallic towers can be viewed as NLs.

In this paper, the propagation characteristics of NLs are examined. They appear as generalizations of concepts from uniform transmission lines (ULs). It was found that an n -conductor NL has n pairs of modes and the forward and backward waves of a mode have the same propagation constants. However, the modeshapes defined by the voltages and currents of a given mode are different. For illustration, the variation with frequency of these propagation characteristics has been presented for the wide-span river crossing of an overhead transmission line.

Index Terms—Frequency domain, nonuniform lines, overhead lines, propagation analysis.

I. INTRODUCTION

IN studies of electromagnetic transients, particular components called *nonuniform lines* or *nonuniform line segments* (NLs), are occasionally encountered. These could be portions of transmission lines where the conductors are not parallel, as in the connection between an overhead line and a cable, or where the sag is significant, for instance, at the crossing of rivers, valleys, or straits, or they could be parts of structures such as metallic towers, or even two-winding transformers. For single-phase lossless NLs, a methodology for the calculation of electromagnetic transients based on traveling waves was described in [1]. There also exist proposals for analytical solutions to the modeling of particular cases of NLs, notably lines with inductance varying as e^{kx} (exponential lines) or x^n [2], [3]. These are restricted to single-phase, lossless situations and require some fitting in order to approximate actual NLs (for instance, the tower used as an example in [1]).

Presently, the usual practical way for handling such components is to subdivide them into sufficiently short segments so that each could then be modeled as a line with parallel conductors [4], [5] or as an exponential line [3], [6]. The great advantage of this approach is that it permits the inclusion of nonlinear branches needed, for example, in the modeling of corona. On the other hand, subdividing a line into many segments results in heavier computational load. Under idealized, lossless conditions, the only difficulty that could appear may be related to the

required small time-step [7]. However, when the frequency dependence of the parameters is modeled, it is difficult to satisfy the requirements for accuracy at both the high frequencies, related to the short subdivisions, and also at much lower frequencies characterizing the simulated transient (for more details, see [4]).

Because of the above difficulties, the traditional representation of uniform transmission lines has been generalized in this paper for the NL case. It was found that, just as in the case of a uniform line (UL), an n -conductor NL has n pairs of modes and the forward and backward waves of a mode have equal propagation constants. However, the modeshapes defined by the voltages and currents of a given mode are different in the two directions. The model is completely general since no particular geometry (such as exponential) or losslessness or frequency independence of parameters are assumed. The required initial data, single- or multi-phase, are in the frequency domain: these are either the values of the NL parameters, Z and Y , along the line segment, or an admittance matrix Y_{bus} obtained most likely from measurements. However, this paper does not address the related, very important but difficult, problem of the initial calculation of the NL parameters (impedances and admittances). Aspects of this topic pertaining to transmission line segments are treated in great detail in [4] (and its reference [5]) and in [8]. Both of these contributions are based on the simplified representation of ground return effects via the concept of complex depth [9], used also in this paper.

The analysis presented in this study uses the A , B , C , and D -constants obtained from the integration of the general frequency domain ordinary differential equations (ODEs) of transmission lines. Some preliminary suggestions regarding this methodology have previously been made in [2], [4], and [5]. In the general case, which may be single- or multi-phase, lossless or with frequency dependent parameters, the A , B , C , and D -constants are calculated by numerical integration. Alternatively, as mentioned, data from a measured Y_{bus} -matrix can be used.

Section II is the core of the paper; it shows how the propagation of waves can be described in the frequency domain by the eigenanalysis of the A , B , C , D -matrix. The results are illustrated in Section III for the case of a wide river crossing.

II. PROPAGATION CHARACTERISTICS

A. Terminal Matrix Relations

In the following, a procedure is presented based on the NL globally modeled from its terminals. This terminal approach is a

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Fig. 1. Notations and reference directions.

generalization of the traditional modeling of ULs with constant or frequency dependent parameters. It is of course only possible in the absence of nonlinearities.

For notations and reference directions, see Fig. 1. The terminal relations we consider have one of the two following forms:

$$\begin{bmatrix} v_S \\ i_S \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_R \\ i_R \end{bmatrix}$$

or

$$\begin{bmatrix} i_S \\ -i_R \end{bmatrix} = \begin{bmatrix} Y_{SS} & Y_{SR} \\ Y_{RS} & Y_{RR} \end{bmatrix} \begin{bmatrix} v_S \\ v_R \end{bmatrix} \quad (1)$$

where the matrices

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad Y_{bus} = \begin{bmatrix} Y_{SS} & Y_{SR} \\ Y_{RS} & Y_{RR} \end{bmatrix} \quad (2)$$

have their elements related as follows:

$$Y_{SS} = DB^{-1}, \quad Y_{RR} = B^{-1}A, \quad Y_{SR} = C - DB^{-1}A, \quad Y_{RS} = -B^{-1} \quad (3a)$$

$$A = -Y_{RS}^{-1}Y_{RR}, \quad B = -Y_{RS}^{-1}, \quad C = Y_{SR} - Y_{SS}Y_{RS}^{-1}Y_{RR}, \quad D = -Y_{SS}Y_{RS}^{-1}. \quad (3b)$$

We note in passing that if we interchange the sending with the receiving end (and accordingly also change the sign of the currents in Fig. 1), then instead of P in (2) we obtain

$$P' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}^{-1}. \quad (2')$$

Since the NL is a passive circuit, Y_{bus} is a symmetrical matrix, i.e.

$$Y_{SS} = Y_{SS}^T, \quad Y_{RR} = Y_{RR}^T, \quad Y_{RS} = Y_{SR}^T. \quad (4)$$

From these conditions, we obtain the following two constraints:

$$AD^T - BC^T = I, \quad A^T D - C^T B = I \quad (5a)$$

where I is the identity matrix. In the single-phase case, when the A, B, C, D -elements are scalars, constraints (5a) reduce to

$$AD - BC = 1. \quad (5b)$$

Clearly, the familiar condition (5b) holds not just for ULs but for single-phase NLs as well. However, because the bi-directional (longitudinally nonsymmetrical) nature of the NLs, we have, in general, $A \neq D$. (A few details for ULs are given in Appendix A.)

In the following, both forms in (1) are needed for further derivations but only one form is assumed to be available based on initial frequency domain data. Specifically, the form with the matrix P is obtained directly from the line parameters $Z(x)$ and $Y(x)$ by direct numerical integration, as given in Appendix B, and Y_{bus} is assumed to be known from measurements.

B. Propagation

The P matrix of (2) may serve to provide a model supporting traveling waves, i.e., an equivalent transmission line. The essential characteristic of a wave on a UL is that its voltage and current components keep their initial proportions during propagation. Thus, the individual traveling waves represents “modes” of the input and output vectors. We will have forward and backward waves, as shown in Fig. 1, n in each direction of an n -phase line. If λ is the factor relating the output to the input vector, then the first equation in (1) becomes

$$\lambda \begin{bmatrix} v_R \\ i_R \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_R \\ i_R \end{bmatrix}. \quad (6)$$

We thus see that a wave pattern or modeshape is an eigenvector of the P matrix in (6) and the corresponding eigenvalue λ represents the propagation (transfer) function over the length l of the line.

In the following, we first focus on the simpler case of single-phase lines. Then, we generalize to multi-phase lines.

Taking (5b) into account, we get the characteristic equation of the P matrix from (2)

$$\lambda^2 - (A + D)\lambda + 1 = 0. \quad (7)$$

Since in (7), λ depends only on the average value

$$A_{ave} = \frac{A + D}{2} \quad (8)$$

we could replace the A, B, C, D -matrix by an A_{ave}, B, C, A_{ave} -matrix that corresponds to an equivalent UL. Moreover, if λ is a root of the characteristic equation (7), then so is $1/\lambda$ and we can write

$$\lambda_{F,B} = e^{\pm \gamma l} \quad (9)$$

where $\lambda_{F,B}$ corresponds to forward and backward waves. Since in (6) λ multiplies phasors that stand for time variation as $\exp(j\omega t)$, the “+” sign in (9) represents a forward wave (F), according to the reference directions in Fig. 1 and the “−” sign represents a backward wave (B). Their propagation speed and attenuation is the same in both directions.

In the case of a multi-phase NL, the eigenanalysis of P is based on the equation

$$\begin{bmatrix} A - \lambda I & B \\ C & D - \lambda I \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (10)$$

Elimination of i using the first equation in (10) gives

$$[C - (D - \lambda I)B^{-1}(A - \lambda I)]v = 0 \quad (11)$$

or, taking (3a) and (5a) into account

$$[Y_{RS}\lambda^2 + (Y_{SS} + Y_{RR})\lambda + Y_{SR}]v = 0. \quad (12)$$

An eigenvalue λ renders singular the matrix in (12). Clearly, if λ is an eigenvalue (real or complex), then so is $1/\lambda$ (divide

by λ^2 and *transpose*), as in the scalar case, and again waves can be associated to the eigenvalues. If λ corresponds to a forward wave, then λ^{-1} is for its backward counterpart. The corresponding vectors v are the right and the left null-vectors of the matrix in (12) and thus they are *not* equal. A similar property exists for the vectors i . Thus, the eigenvectors in (10) related to a pair of forward and backward waves are in general different.

Next, we discuss the other aspects of propagation: the modeshapes and the related characteristic admittances.

C. Mode Shapes

The following discussion is based on (10), related to the eigenanalysis of the matrix P of (2). We note that (10) shows that an eigenvector (or modal waveshape vector) w of the A , B , C , D -matrix P consists of both a voltage and a current vector. The first n elements of a waveshape vector correspond to wave (or modal) voltages, the next n elements correspond to wave currents.

Let Λ and W be the resulting matrices of eigenvalues and eigenvectors

$$\Lambda = \begin{bmatrix} \Lambda_F & \\ & \Lambda_B \end{bmatrix}, \quad \Lambda_B = \Lambda_F^{-1}, \quad W = \begin{bmatrix} W_{F_v} & W_{B_v} \\ W_{F_i} & W_{B_i} \end{bmatrix}. \quad (13)$$

Accordingly, the first equation in (1) becomes

$$\begin{aligned} \begin{bmatrix} v_S \\ i_S \end{bmatrix} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_R \\ i_R \end{bmatrix} \\ &= \begin{bmatrix} W_{F_v} & W_{B_v} \\ W_{F_i} & W_{B_i} \end{bmatrix} \begin{bmatrix} \Lambda_F & \\ & \Lambda_B \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} W_{F_v} & W_{B_v} \\ W_{F_i} & W_{B_i} \end{bmatrix}^{-1} \begin{bmatrix} v_R \\ i_R \end{bmatrix}. \end{aligned} \quad (14)$$

This suggests the use of a modal transformation into modal vectors u such that

$$u_S = \Lambda u_R \quad \text{or} \quad \begin{bmatrix} u_{F,S} \\ u_{B,S} \end{bmatrix} = \begin{bmatrix} \Lambda_F & \\ & \Lambda_B \end{bmatrix} \begin{bmatrix} u_{F,R} \\ u_{B,R} \end{bmatrix}. \quad (15)$$

The following notations are introduced:

$$\begin{aligned} \begin{bmatrix} v \\ i \end{bmatrix} &= z = Wu \\ &= \begin{bmatrix} \left(W_{F_v} \right) & \left(W_{B_v} \right) \\ \left(W_{F_i} \right) & \left(W_{B_i} \right) \end{bmatrix} \begin{bmatrix} u_F \\ u_B \end{bmatrix} \\ &= [W_F \quad W_B] \begin{bmatrix} u_F \\ u_B \end{bmatrix} = z_F + z_B. \end{aligned} \quad (16)$$

The vector z designates the set of voltages and currents on the n conductors. They are treated together in our analysis. The notation W was chosen to convey the idea that the modal transformation $z = Wu$ in (16) is a decomposition of z into waves u . The part z_F consists of waves u_F that propagate forward, while z_B consists of backward wave components u_B . Thus, u_F and u_B designate the forward and the backward waves, respectively. The waveshapes or modeshapes are given by the $2n$ columns of W . The magnitude of each wave is given by the elements of the vector u . This latter vector is the modal equivalent of z . The first n modes (or waves) propagate forward according to the corresponding elements of Λ_F , while the next n modes propagate backward according to the corresponding elements of Λ_B . The

Fig. 2. Longitudinal profile of river crossing.

waveshapes of the forward waves are given by the columns of W_F and, vice versa, the waveshapes of the backward waves are given by the columns of W_B . It is important to emphasize that, in the case of NLs, in contrast to ULs, the waveshapes W_F and W_B are, in general, *not equal*.

The submatrices in W of (13) can be used to define the characteristic admittance matrices

$$Y_{C,F} = W_{F_i} W_{F_v}^{-1} \quad \text{and} \quad Y_{C,B} = W_{B_i} W_{B_v}^{-1}. \quad (17)$$

Again, the characteristic admittances $Y_{C,F}$ and $Y_{C,B}$, corresponding to forward and backward waves, are in general *not equal*.

For the purpose of clarity and illustration, Appendix D shows the theoretical results of Section II in terms of numerical values for a single frequency on the example of the NL of a river crossing (given in Appendix C).

III. ILLUSTRATIVE EXAMPLE

The new concepts developed in the paper are illustrated below (in addition to the numerical results for a single frequency given in Appendix D) by showing the main variables characteristic for propagation as a function of frequency, in order to assess the differences with respect to a UL. The line parameters and the phase domain frequency dependent matrices $Z(x)$ and $Y(x)$ are calculated using simple (but highly accurate) expressions based on the concept of complex penetration depth, as described in [6] and [9].

The different propagation characteristics derived in the previous section will be illustrated on hand of a wide-span, unsymmetrical, 3-phase river crossing of a high-voltage transmission line. Its longitudinal profile, from the sending end S to the receiving end R , is shown in Fig. 2(a). For details, see Appendix C.

Figs. 3–5 show, for the NL river crossing of Fig. 2(a), how the variables discussed previously vary with frequency.

Fig. 3 gives the absolute values $|\lambda_i|$ of the modal propagation functions. In the first column, the horizontal scale is logarithmic, as usual. We see that up to nearly 250 kHz the propagation functions are smooth, similarly to those of ULs. At higher frequencies, however, spikes appear for the first two modes where the effects of the nonuniformity are more pronounced: mode-1 is the ground mode where the sag matters most, and mode-2 is the β -mode where there is still some influence of the ground return. These spikes can be seen more clearly in the second column where the frequency scale is linear. They are regularly spaced and represent a superimposed oscillation in the time domain with base frequency corresponding to the first spike at

Fig. 3. Profile (a): Modal propagation constants $|\lambda_i|$.

Fig. 4. Profile (a): Modal characteristic admittances Y_{ci} .

~ 250 kHz and harmonics determined by the other spikes (at 500 kHz, 750 kHz,...). Clearly, this oscillatory effect reflects a resonance due to the coincidence of the 600 m length of the NL and the 250 kHz half-wave length at light velocity.

Fig. 4 shows the magnitudes of the modal characteristic impedances Y_{ci} . The resonance effect discussed above appears here in the same way as in the propagation function.

Fig. 5 shows the differences in magnitude and angle between the modal characteristic admittances $Y_{C,F}$ and $-Y_{C,B}^*$ of the forward and backward waves. We see that these differences are negligible at all frequencies except at the spikes of resonance, where they become quite significant, especially for the ground mode.

Therefore, the following practical conclusion can be made. For transients where resonant frequencies related to the length

Fig. 5. Profile (a): Differences in magnitude $|z|$ and angle $\langle i \rangle$ of forward and backward Y_{ci} .

Fig. 6. Profile (b): Modal propagation constants $|\lambda_i|$.

of the NL are of no significance, the NL can be replaced by an equivalent UL.

Next, we examine the profiles (b) and (c) of Fig. 2. Profile (b) is even more nonuniform than profile (a). Fig. 6 shows that the spikes are now more pronounced and a similar result (as in Fig. 4) has been obtained for the spikes in the characteristic admittances. Fig. 7 shows that the differences in characteristic admittances of the forward and backward waves at the spikes are even larger than in Fig. 5. Thus, the existence of spikes and the directional differences in Y_C are a characteristic feature of NLs.

Profile (c) is nonuniform but “flippable,” i.e., longitudinally symmetrical. This does not make it a UL, but the algebraic structure of its propagation and admittance matrices (from (2)) do not provide distinguishing characteristics. Fig. 8 shows that spikes still exist but are smaller than for the true NLs with profiles (a) and (b). Of course, there are no directional differences as in Figs. 5 and 7. Fig. 9 shows that, in the case of a UL, there are no spikes at all, again as expected.

In conclusion, NLs are characterized by the existence of spikes in the functions related to propagation and directional differences in the characteristic admittances. If the NL is flippable, only the spikes persist but become much smaller. They do not exist in the case of ULs.

Fig. 7. Profile (b): Differences in magnitude and angle of forward and backward Y_{ci} .

Fig. 8. Profile (c): Modal propagation constants $|\lambda_i|$.

Fig. 9. UL: Modal propagation constants $|\lambda_i|$.

IV. CONCLUSIONS

The most important conclusion of this paper is that the propagation characteristics of traveling waves on NLs can be obtained

directly from its A, B, C, D-matrices and that the propagation constants are the same but the modeshapes and characteristic admittances are different in the two directions. Thus, an NL is a directional structure.

Correspondingly, the modal transformation matrices and the characteristic admittances are also different in the two directions and the Norton equivalents to be used in the calculation of transients are different at the two ends of the NL. The latter are the object of time domain calculations and therefore beyond the scope of this study. However the results of this paper could be used for obtaining the time domain information needed for the inclusion of NLs in a program for simulation of transients.

A salient feature of the propagation-related functions of NLs in the frequency domain are spikes superimposed on otherwise smooth characteristics. These appear at frequencies harmonically related to a base resonance frequency determined by the length of the NL. In the time domain, these spikes translate into normally weak, attenuated modulations of the step or impulse responses corresponding to the propagation function and the characteristic admittance. These will usually be of high frequency and often without much significance.

APPENDIX A UNIFORM LINES

We have, in addition to (4)

$$Y_{RR} = Y_{SS} = Y_{self}, \quad Y_{RS} = Y_{SR} = Y_{mutual} \quad (\text{A.1})$$

(both symmetrical) so that we get the additional relations

$$\begin{aligned} A &= D^T, B = B^T \\ C &= C^T \quad (A = D \text{ for a single-phase line}). \end{aligned} \quad (\text{A.2})$$

Thus, (5a) and (5b) become, respectively

$$A^2 - BC = I \quad (\text{or } D^2 - CB = I) \quad \text{and} \quad A^2 - BC = 1. \quad (\text{A.3})$$

We note that the designation of “uniform” does not necessarily imply exactly parallel conductors. For example, in our global, terminal perspective, a line segment with symmetrical conductor sag is “flippable” with respect to its two terminals and thus it appears as uniform and it has a UL equivalent.

APPENDIX B DERIVATION OF THE P MATRIX

The derivation of the P matrix is based on the *frequency domain* (vector-phaser) differential equations

$$\frac{d}{dx} \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 & Z(x) \\ Y(x) & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}, \quad \begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ i_0 \end{bmatrix} \quad (\text{B.1})$$

where $v, i \in C^n$ ($n = 1$ for single-phase lines), and x is the longitudinal space parameter (defined in the direction opposite to that for the current, as shown in Fig. 1). We seek at $x = l$ a solution of the form

$$\begin{bmatrix} v(l) \\ i(l) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v(0) \\ i(0) \end{bmatrix}. \quad (\text{B.2})$$

In general, the ODEs (B.1) do not have an analytical solution. We can, however, integrate them numerically (at any given frequency) in order to obtain the A, B, C, D-matrix from (B.2).

For this, we note that in the case of a single-phase line its two columns are equal to the left-hand side vector if the input is taken sequentially as the two columns of the 2×2 identity matrix. Therefore, we simply integrate the ODE (B.1), using any efficient numerical method with the two columns of the 2×2 identity matrix I as inputs. In the case of a 3-phase line, the A, B, C, D -matrix is of dimension 6×6 and we need six consecutive integrations for its column-by-column identification. This amounts to solving for $x = l$ the initial value problem

$$\begin{aligned} \frac{d}{dx} \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} &= \begin{bmatrix} 0 & Z(x) \\ Y(x) & 0 \end{bmatrix} \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} \\ \begin{bmatrix} A(0) & B(0) \\ C(0) & D(0) \end{bmatrix} &= I_{2n \times 2n}. \end{aligned} \quad (\text{B.3})$$

We note that this procedure is completely general in a practical sense, since the numerical integration is possible even if some discontinuities exist in $Z(x)$ and $Y(x)$. In general, these parameters are complex and thus, the frequency dependence of the NL parameters is fully preserved; there is no need to use a simplified, lossless model. Of equal importance is the fact that once the A, B, C, D -matrix is calculated, we have a global representation of the NL. Therefore, in the calculation of an electromagnetic transient, we deal with a single element as opposed to a large number of segments needed in alternative procedures.

APPENDIX C

DETAILS ABOUT THE RIVER CROSSING

The line segment has a horizontal span of $l = 600$ m and its height is much higher at end S than at end R . We place a coordinate system with its origin at the deepest point of the catenary so that the equation of the longitudinal profile is

$$y = q \left(\cosh \left(\frac{x}{q} \right) - 1 \right) \cong \frac{x^2}{2q} + \frac{x^4}{24q^3} \cong \frac{x^2}{2q} \quad (\text{C.1})$$

where q is a scaling parameter.

We chose $q = 626.3$ m (which yields the realistic sag of 8 m for a horizontal span of 200 m). The left end of the segment is at $x_S = -100$ m, the right end is at $x_R = 500$ m. For these abscissae, the ordinates are $y_S = 8$ m and $y_R = 210.413$ m. There is indeed a significant vertical difference between the two ends.

The water surface is assumed to be by another 20 m below the lowest point of the conductors and the return path conductivity was assumed 0.1 S/m. The line is of horizontal configuration with 10 m distance between neighboring conductors. The data for the latter was in the range of usual practice.

For comparison with this NL, we also consider a (longitudinally) symmetrical geometry (in the following: SL, for symmetrical line segment) of the same span, $l = 600$ m, but with $x_S = -300$ m and $x_R = 300$ m, so that the sag is $y_S = y_R = 73.235$ m.

APPENDIX D

PROPAGATION AT A SINGLE FREQUENCY

- The first step is the *calculation of A, B, C, D* —submatrices of matrix P in (1), using the numerical integration of the ODE (B.3). The computations were performed for logarithmically spaced frequency points from 10 Hz to 1 MHz. Trapezoidal integration proved very effective, with only five discretization steps yielding errors below 1% even at higher frequencies and much better at lower frequencies. The computing time for obtaining the line parameters and the A, B, C, D -matrices with MATLAB on a (rather slow) DX2-66 MHz PC for 100 frequencies was 33 s. This is about twice the time needed to calculate the line parameters alone for all the incremental steps in which the span is subdivided for the numerical integration of (B.3), since the latter is performed “on the fly” as the calculation of the parameters proceeds from right to left. Below are the A, B, C, D -matrices at $f = 100$ kHz, shown in (D.1a)–(D.1d) at the bottom of the page.

It can be readily checked that these matrices satisfy (5a). (Note that the transpose used is non-Hermitian.) We also note that $A \neq D^T$, since the NL is unsymmetrical.

For the SL, we get quite different matrices from those in (D.1), but now $A = D^T$, as expected and all the other relations in (A.2) and (A.3) are also satisfied.

- At 100 kHz, the quarter wavelength is 750 m and the 600 m line could be compared to a 1000-km line at 60 Hz (80% of a quarter wavelength). Thus, significant *Ferranti effect* (voltage increase at the open end) could be expected. The voltage amplification for a uniform lossless line of this length

$$A = \begin{bmatrix} 0.3479 + 0.0041i & 0.0345 + 0.0030i & 0.0363 + 0.0029i \\ 0.0382 + 0.0033i & 0.3433 + 0.0037i & 0.0382 + 0.0033i \\ 0.0363 + 0.0029i & 0.0345 + 0.0030i & 0.3479 + 0.0041i \end{bmatrix} \quad (\text{D.1a})$$

$$B = 100 \times \begin{bmatrix} 0.0216 + 4.9560i & 0.0188 + 1.3979i & 0.0173 + 1.0159i \\ 0.0187 + 1.3980i & 0.0215 + 4.9561i & 0.0187 + 1.3980i \\ 0.0173 + 1.0159i & 0.0188 + 1.3979i & 0.0216 + 4.9560i \end{bmatrix} \quad (\text{D.1b})$$

$$C = \frac{1}{100} \times \begin{bmatrix} -0.0002 + 0.2040i & -0.0001 - 0.0476i & -0.0001 - 0.0258i \\ -0.0001 - 0.0476i & -0.0002 + 0.2119i & -0.0001 - 0.0476i \\ -0.0001 - 0.0258i & -0.0001 - 0.0476i & -0.0002 + 0.2040i \end{bmatrix} \quad (\text{D.1c})$$

$$D = \begin{bmatrix} 0.2470 + 0.0022i & -0.0600 + 0.0016i & -0.0572 + 0.0015i \\ -0.0543 + 0.0014i & 0.2534 + 0.0020i & -0.0543 + 0.0014i \\ -0.0572 + 0.0015i & -0.0600 + 0.0016i & 0.2470 + 0.0022i \end{bmatrix} \quad (\text{D.1d})$$

is $1/\cos(0.8 \times \pi/2) = 1/0.3090 = 3.2361$, but it is (surprisingly at first) even higher for a lossy line (see (D.2b) below).

In the case of the NL examined here, if the voltages at the open receiving end are 1, 1, 1, the sending end voltages calculated with the data in (D.1) are

$$v_S = 0.4186 + 0.0100i, 0.4197 + 0.0103i, 0.4186 + 0.0100i \quad (\text{D.2a})$$

but only

$$v_S = 0.2884 + 0.0097i, 0.2880 + 0.0099i, 0.2884 + 0.0097i \quad (\text{D.2b})$$

for the SL. The reduced amplification of (D.2a) in the NL can be explained by the capacitance being smaller at the right and, at the same time, the inductance being also smaller at the left, compared to the SL case. If we calculate the A' , B' , C' , D' matrices of (2') for the NL, we obtain for the span open at the left end, a voltage at the right end much lower than expected for Ferranti effect on the SL

$$v_R = 0.1355 + 0.0052i, 0.1335 + 0.0052i, 0.1355 + 0.0052i \quad (\text{D.2c})$$

corresponding to the higher capacitance at the left and also higher inductance at the source side at the right of the line segment.

For a set of line mode voltages, e.g., 1, 0, -1, the line parameters of the NL are, of course, more uniform and thus the Ferranti effect is less unsymmetrical with respect to the two line ends

$$v_S = 0.3116 + 0.0013i, 0, -0.3116 - 0.0013i \quad (\text{D.3a})$$

$$v_R = 0.3042 + 0.0006i, 0, -0.3042 - 0.0006i. \quad (\text{D.3b})$$

Since the effect of the ground return is now small, these values are close to 0.309, indicated above for a lossless line.

- Next, we focus on aspects of *wave propagation*, with numerical results shown for 100 kHz. First, we calculate, as indicated in (10), the *eigenvalues* of the A, B, C, D -matrix P of (2) and of P' from (2')

$$\lambda_F = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.2791 + 0.9686i \\ 0.3082 + 0.9524i \\ 0.3086 + 0.9518i \end{bmatrix}$$

$$\lambda_B = \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = \begin{bmatrix} 0.2747 - 0.9533i \\ 0.3076 - 0.9505i \\ 0.3083 - 0.9507i \end{bmatrix}. \quad (\text{D.4})$$

As expected from the remark about (12), the eigenvalues corresponding to the forward and backward waves, λ_F and λ_B come in reciprocal pairs. For instance, in (D.4) $\lambda_4 = 1/\lambda_1$.

The modal propagation constants are obtained from (9)

$$\gamma_{F,B} = \begin{bmatrix} \gamma_{1,4} \\ \gamma_{2,5} \\ \gamma_{3,6} \end{bmatrix} = \begin{bmatrix} \pm(0.0133 + 2.1504i) \times 10^{-3} \\ \pm(0.0017 + 2.0963i) \times 10^{-3} \\ \pm(0.0010 + 2.0954i) \times 10^{-3} \end{bmatrix}. \quad (\text{D.5})$$

It can be seen that, over the length of $l = 600$ m, the resultant rotation will be about 1.26 rad, i.e., $\sim 80\%$ of the quarter wavelength, as expected, and there will be an attenuation of about 0.8% for the ground mode [the first in (D.5)] and much smaller for the two line modes.

- Related to the eigenvalues and to the modal propagation constants obtained above are the corresponding *eigenvectors* or *modeshapes*. Still at 100 kHz, we calculate the matrix W of eigenvectors of the A, B, C, D -matrix P of (2) for which we already have the eigenvalues listed in (D.4). In (13), W is partitioned into four submatrices related to forward and backward waves and to voltages and currents. Below are, for exemplification, its columns #1 and #4, associated to the eigenvalues λ_1 and λ_4 [which correspond to equal propagation constants—except for their sign “−,” $\gamma_{1,4}$, given in (D.5)]

$$w_1 = \begin{bmatrix} 0.5741 - 0.0102i \\ 0.5835 - 0.0114i \\ 0.5741 - 0.0102i \\ (0.7586 + 0.1038i) \times 10^{-3} \\ (0.7021 + 0.0921i) \times 10^{-3} \\ (0.7586 + 0.1038i) \times 10^{-3} \end{bmatrix}$$

$$w_4 = \begin{bmatrix} 0.1405 + 0.5570i \\ 0.1431 + 0.5654i \\ 0.1405 + 0.5570i \\ (-0.2915 - 0.7130i) \times 10^{-3} \\ (-0.2695 - 0.6588i) \times 10^{-3} \\ (-0.2915 - 0.7130i) \times 10^{-3} \end{bmatrix}. \quad (\text{D.6})$$

Even though the propagation constant for this mode is the same $\gamma_{1,4} = \pm(0.0133 + 2.1504i) \times 10^{-3}$ of (D.5), the wave-shapes in the forward direction (w_1) and the backward direction (w_4) are different. (However, we shall soon see that this difference is small.) The first three values in the vectors in (D.6) give the modal phase voltages (phasors) while the next three values give the corresponding currents. They all form a single mode and these modal voltages and currents are rigidly related to each other. Of course, this relation defines the *characteristic admittance matrices* $Y_{C,F}$ and $Y_{C,B}$ of (17), associated to the submatrices of W in (13). They are different in the two directions, as in (D.7a) and (D.7b), shown at the bottom of the page, but $Y_{C,F} \cong -Y_{C,B}^*$, where $*$ denotes the conjugate. This means that if in the forward wave (in our example, “uphill” in terms of the increasing local

$$Y_{C,F} = 1000 \times \begin{bmatrix} 2.1274 + 0.0753i & -0.5150 + 0.0630i & -0.2864 + 0.0640i \\ -0.5150 + 0.0630i & 2.2137 + 0.0592i & -0.5150 + 0.0630i \\ -0.2864 + 0.0640i & -0.5150 + 0.0630i & 2.1274 + 0.0753i \end{bmatrix} \quad (\text{D.7a})$$

$$-Y_{C,B} = 1000 \times \begin{bmatrix} 2.1315 - 0.0681i & -0.5119 - 0.0593i & -0.2840 - 0.0605i \\ -0.5119 - 0.0593i & 2.2168 - 0.0532i & -0.5119 - 0.0593i \\ -0.2840 - 0.0605i & -0.5119 - 0.0593i & 2.1315 - 0.0681i \end{bmatrix} \quad (\text{D.7b})$$

characteristic impedance along the NL) the current leads the voltage, then in the backward wave (i.e., “downhill”), it will be lagging.

The difference between $Y_{C,F}$ and $-Y_{C,B}^*$ is less than 1% (in norm). However, if we do not take the conjugate, the difference is 13.6%. Thus, we do not have a single characteristic admittance for the NL as for a UL.

In order to compare the two modes in (D.6), we normalize w_4 (essentially by a rotation) so that its first element becomes equal to that of w_1 . While these two modes are not exactly equal, they differ (in norm) by less than 1%. By inspection, (D.6) shows that this is a ground mode. If we now compare w_2 with w_5 and w_3 with w_6 , we find that they also differ by less than 1%. As to the modeshape, w_2 and w_5 appear to be β -modes (1, 0, -1), while w_3 and w_6 are α -modes (1, -2, 1). These modeshapes correctly reflect the line geometry.

The explanation of the differences between modes and characteristic admittances (where we take the conjugate for the backward wave) that seem surprisingly small, is the fact that they are obtained from the A, B, C, D -matrix P of (2) which is the result of an integration process along the line. This averages out the nonuniformity of the NL. Specifically, the characteristic admittances in the forward and backward directions, $Y_{C,F}$ and $Y_{C,B}$, are to be distinguished from the local characteristic admittances at the sending and receiving ends of the NL.

- To make the analysis in Section II-C more concrete, and to further clarify the notations, we now consider an arbitrary set of phase voltages and currents, e.g., $z = \{1, 1, 1; 0.01, 0.02, 0.01\}$. Then, the modal decomposition of (16) yields (with W already calculated)

$$u = \begin{bmatrix} 9.8972 + 0.0030i \\ 0.0000 + 0.0000i \\ 1.2728 + 0.9961i \\ -1.8241 + 7.9601i \\ 0.0000 - 0.0000i \\ 0.7883 + 1.4301i \end{bmatrix}. \quad (\text{D.8})$$

The entries in u above are modal quantities: each gives the participation of an eigenvector (a column of W) in the given vector z . The first three values of u form u_F and the next three form u_B . With these, the decomposition $z = z_F + z_B$, in (16), into forward and backward components, becomes

$$z = \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0.01 \\ 0.02 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 5.0573 - 0.1016i \\ 7.1277 - 0.1039i \\ 5.0573 - 0.1016i \\ 0.0057 + 0.0010i \\ 0.0106 + 0.0009i \\ 0.0057 + 0.0010i \end{bmatrix} + \begin{bmatrix} -4.0573 + 0.1016i \\ -6.1277 + 0.1039i \\ -4.0573 + 0.1016i \\ 0.0043 - 0.0010i \\ 0.0094 - 0.0009i \\ 0.0043 - 0.0010i \end{bmatrix}. \quad (\text{D.9})$$

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